Partial Order Sets

Partially Ordered Sets

- Partial Order

  A relation R on a set A is called a partial order if R is ***reflexive, antisymmetric and transitive***. The set A together with the partial order R is called a partially ordered set, or simply a poset, denoted by (A, R)

For instance,

1. Let A be a collection of subsets of a set S. The relation ⊆ of set inclusion is a partial order on A, so (A, ⊆) is a poset.

2. Let Z^+ be the set of positive integers. The usual relation ≤ is a partial order on Z^+, as is "≥"

Partially Ordered Sets

- Example

  Let R be a partial order on a set A, and let R^{-1} be the inverse relation of R. Then R^{-1} is also a partial order.

Proof:

(a)Reflexive $\Delta \subseteq R \leftrightarrow \Delta = \Delta^{-1} \subseteq R^{-1}$

(b)Antisymmetric $R \cap R^{-1} \subseteq \Delta \leftrightarrow R^{-1} \cap R \subseteq \Delta$

(c)Transitive $R^2 \subseteq R \leftrightarrow (R^{-1})^2 \subseteq R^{-1}$

Thus, R^{-1} is also a partial order.

The poset (A, R^{-1}) is called the **dual** of the poset (A, R).

whenever $(A, \leq)$ is a poset, we use "$\geq$" for the partial order $\leq^{-1}$

Partially Ordered Sets

- Comparable

If (A, $\leq$) is a poset, elements a and b of A are comparable if

$$a \leq b \text{ or } b \leq a$$

In some poset, e.g. the relation of divisibility (a R b iff a | b), some pairs of elements are not comparable

$$2 \nmid 7 \text{ and } 7 \nmid 2$$

Note: if every pair of elements in a poset A is comparable, we say that A is **linear ordered** set, and the partial order is called a **linear order**. We also say that A is a **chain**.

Partially Ordered Sets

- Theorem 1

  If $(A, \le)$ and $(B, \le)$ are posets, then $(A \times B, \le)$ is a poset, with partial order $\le$ defined by

  $$(a, b) \le (a', b') \quad \text{if } a \le a' \text{ in } A \text{ and } b \le b' \text{ in } B$$

Note: the $\le$ is used to denote three different partial orders.

Proof:

(a) Reflexive

support (a, b) in $A \times B$, then

$(a, b) \le (a, b)$ since $a \le a$ in A and $b \le b$ in B ($(A, \le)$ and $(B, \le)$ are posets)

Partially Ordered Sets

(b) Antisymmetry

support $(a, b) \leq (a', b')$ and $(a', b') \leq (a, b)$, then

$a \leq a'$ and $a' \leq a$ in A ; $b \leq b'$ and $b' \leq b$ in B

since A and B are posets, $a=a'$, $b=b'$ (antisymmetry property in A and B, respectively), which means that $(a, b)=(a', b')$ and thus $\leq$ satisfies the antisymmetry property in $A \times B$

(c) Transitive

support $(a, b) \leq (a', b')$ and $(a', b') \leq (a'', b'')$, then

$a \leq a'$ and $a' \leq a''$ in A ; $b \leq b'$ and $b' \leq b''$ in B ,

since A and B are posets, $a \leq a''$ and $b \leq b''$ (transitive property in A and B, respectively), which means that

$$(a, b) \leq (a'', b'')$$

Partially Ordered Sets

- Product partial order

  The partial order $\leq$ defined on the Cartesian product $A \times B$ is called the Product partial order

- The symbol $<$

  If $(A, \leq)$ is a poset, we say $a<b$ if $a \leq b$ but $a \neq b$

- Lexicographic (dictionary) order

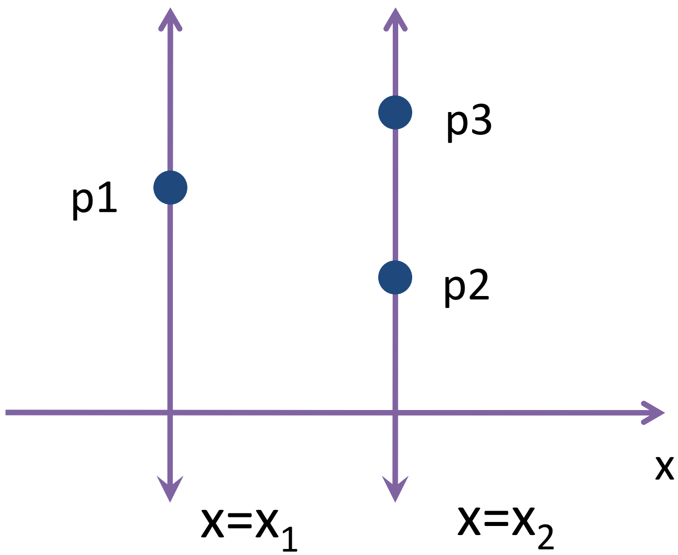
  Another useful partial order on $A \times B$, denoted by $\prec$, is defined as $(a, b) \prec (a', b')$ if $a<a'$ or $a=a'$ and $b \leq b'$

  why $\prec$ is a partial order?

Partially Ordered Sets

- Example

Let A=R, with the usual order $\leq$. Then the plane $R^2=R\times R$ may be given lexicographic order



P1=(x1,y1)

P2=(x2,y2) P3=(x2,y3)

P1 $\prec$ P2

P1 $\prec$ P3 (x1 $\leq$ x2, x1 $\neq$x2)

P2 $\prec$ P3 (x2=x2, y2 $\leq$ y3)

Partially Ordered Sets

- Lexicographic ordering is easily extended to Cartesian products $A_1 \times A_2 \dots \times A_n$ as follows:

  $(a_1, a_2, \ldots, a_n) \prec (a'_1, a'_2, \ldots, a'_n)$ if and only if

  $a_1 < a'_1$ or

  $a_1 = a'_1$ and $a_2 < a'_2$ or

  $a_1 = a'_1$ and $a_2 = a'_2$ and $a_3 < a'_3$ or …

  $a_1 = a'_1$ and $a_2 = a'_2$ … , $a_{n-1} = a'_{n-1}$, $a_n \leq a'_n$

  Why?

Partially Ordered Sets

- Theorem 2

  The digraph of a partial order has no cycle of length larger than 1

Proof: support that the digraph of the partial order $\leq$ on the set A contains a cycle of length n>=2. Then there exist **distinct elements** $a_1, a_2, \ldots, a_n$ in A such that

$$a_1 \leq a_2,\ a_2 \leq a_3, \ldots,\ a_{n-1} \leq a_n,\ \ a_n \leq a_1$$

by the transitivity of the partial order, used n-1 times,

$$a_1 \leq a_n$$

by antisymmetry, $a_n \leq a_1$ and $a_1 \leq a_n$ then $a_1 = a_n$

(Contradiction)

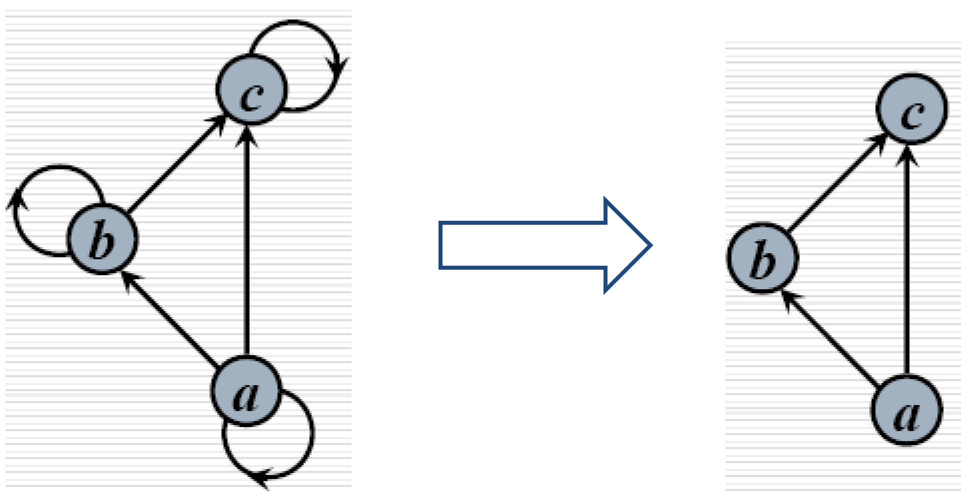
Partially Ordered Sets

- Hasse Diagrams

  Just a reduced version of the diagram of the partial order of the poset.

  a) Reflexive

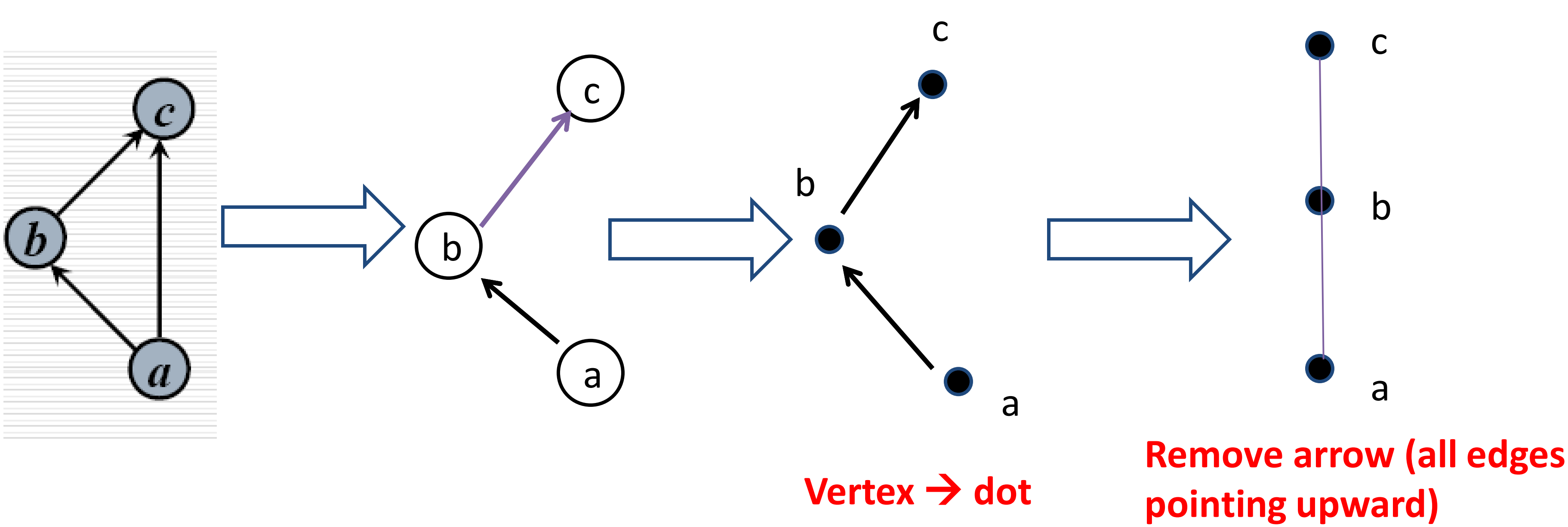
  Every vertex has a cycle of length 1 **(delete all cycles)**



Partially Ordered Sets

- Transitive

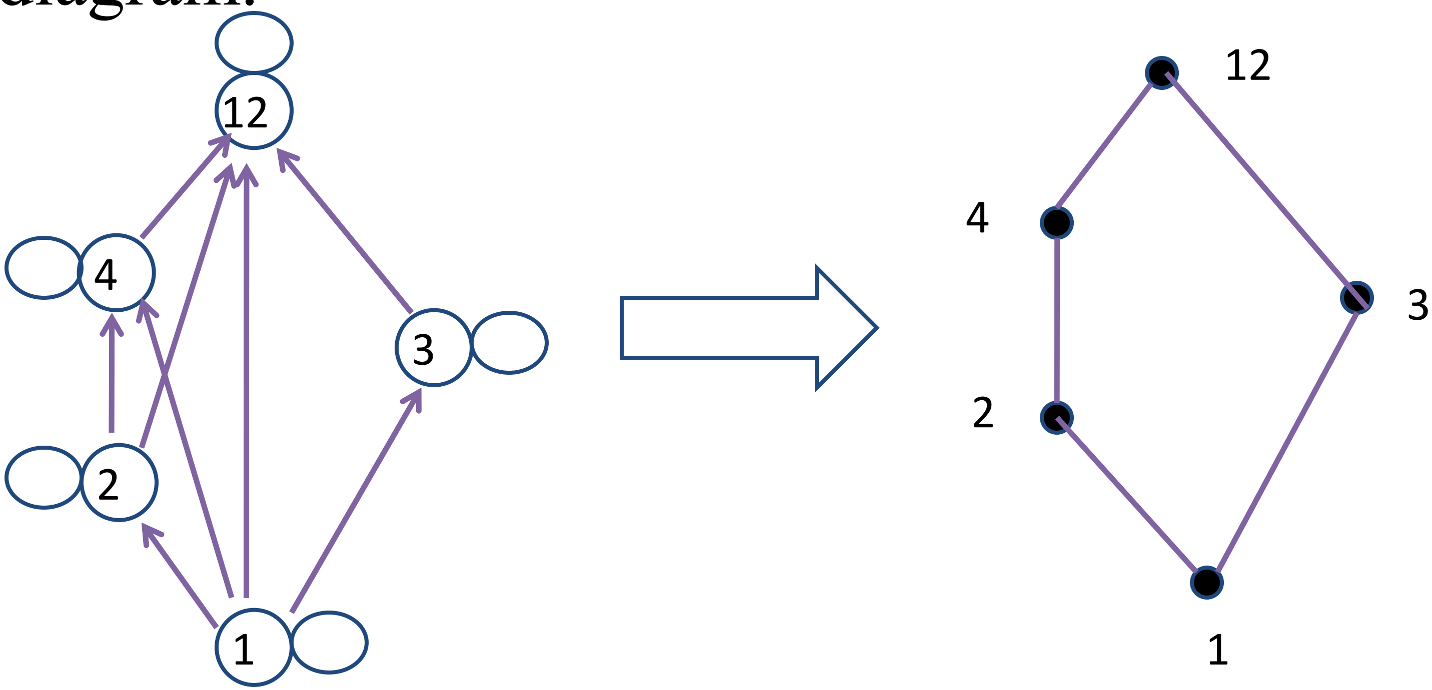
  a ≤ b, and b ≤c, then a ≤c (delete the edge from a to c)



Partially Ordered Sets

- Example

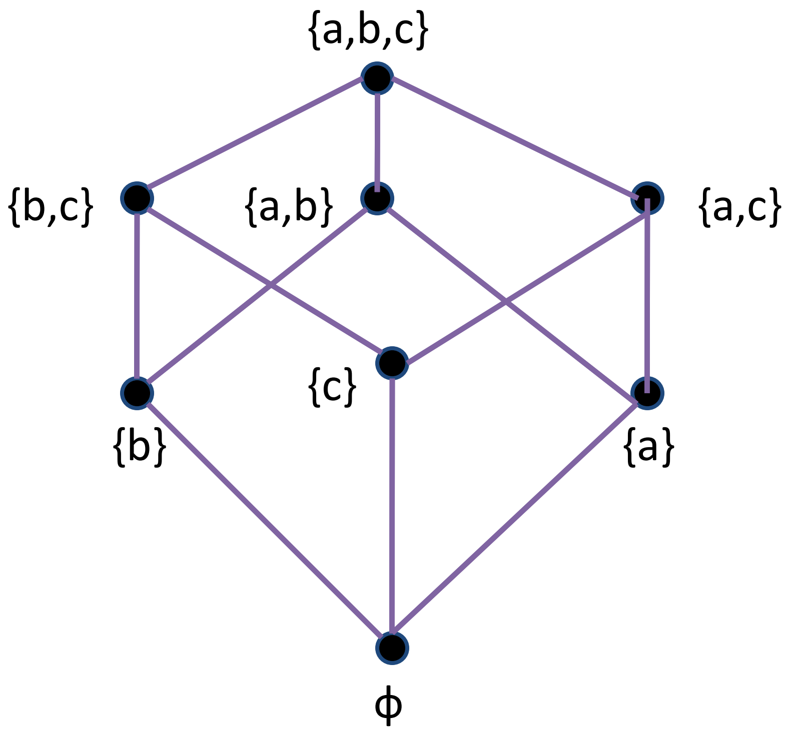
Let A={1,2,3,4,12}. Consider the partial order of divisibility on A. Draw the corresponding Hasse diagram.



Partially Ordered Sets

- Example

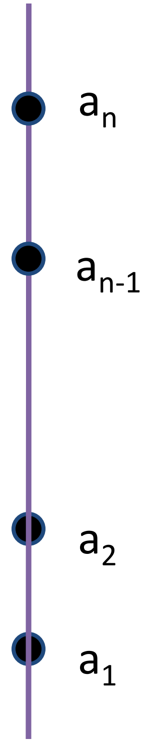
  Let S={a,b,c} and A=P(S). Draw the Hasse diagram of the poset A with the partial order ⊆



Partially Ordered Sets

- The Hasse diagram of a finite linearly order set

  Let $A=\{a_1,a_2,\ldots,a_n\}$ be a finite set, and $a_i \leq a_j$ if $i \leq j$



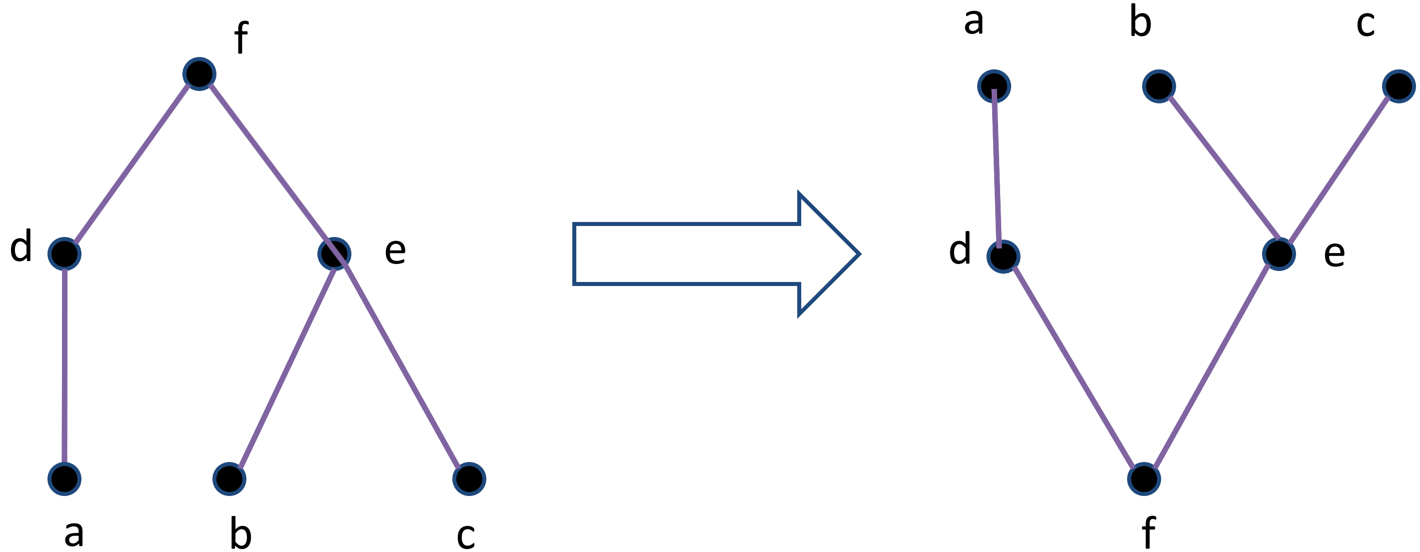
Partially Ordered Sets

- Example

Fig. a shows the Hasse diagram of a poset (A, $\leq$), where

$$A=\{a, b, c, d, e, f\}$$

Fig. b shows the Hasse diagram of the dual poset (A, $\geq$)



Partially Ordered Sets

- Topological Sorting

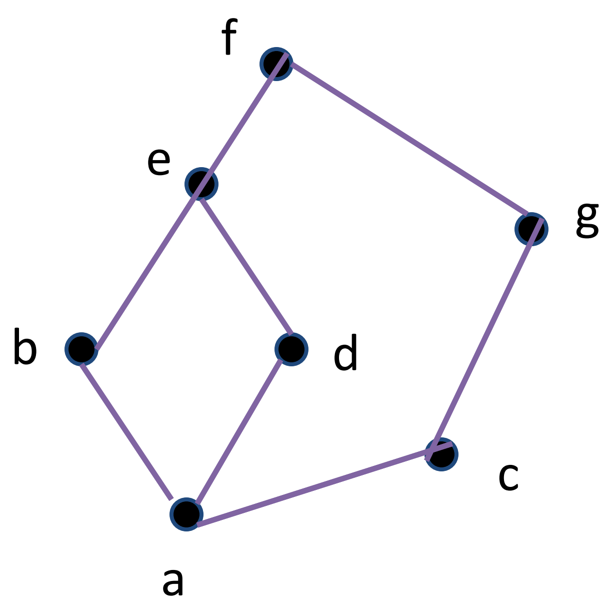
  If A is a poset with partial order $\leq$, we sometimes need to find a linear order $\prec$ for the set A that will merely be an extension of the given partial order in the sense that if $a \leq b$, then $a \prec b$. The process of constructing a linear order such as $\prec$ is called topological sorting.

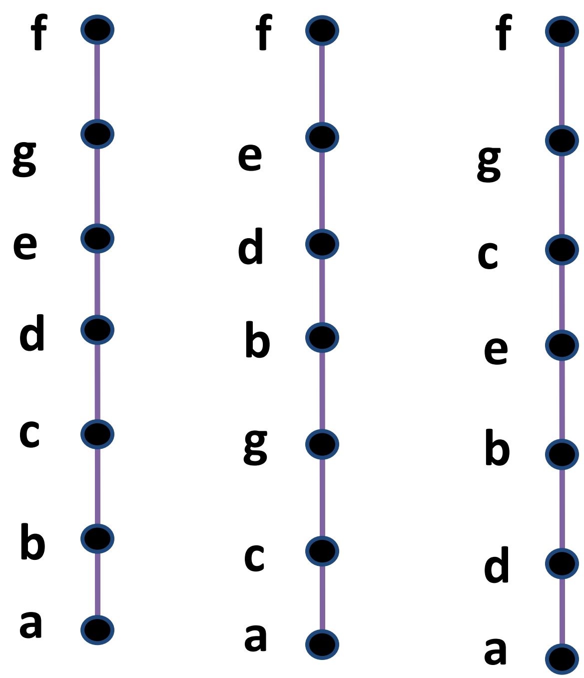
  (refer to p.229 for the details of the algorithm)

Partially Ordered Sets

- Example

Give a topological sorting for the poset whose Hasse diagram as follows





Usually, the topological sorting is not unique.

Partially Ordered Sets

- Isomorphism

  Let (A, $\leq$) and (A', $\leq$') be posets and let f: A$\rightarrow$A' be a one to one correspondence between A and A'. The function f is called an isomorphism from (A, $\leq$) to (A', $\leq$') if, for any a and b in A,

$$a \leq b \text{ if and only if } f(a) \leq' f(b)$$

  If f: A$\rightarrow$A' is an isomorphism, we say that (A, $\leq$) and (A', $\leq$') are isomorphic posets.

Partially Ordered Sets

- Example

Let A be the set Z^+ of positive integers, and let $\leq$ be the usual partial order on A. Let A' be the set of positive even integers, and let $\leq'$ be the usual partial order on A'. The function f: A→A' given by

$$f(a) = 2\ a$$

is an isomorphism form $(A, \leq)$ to $(A', \leq')$

Proof: First, it is very to show that f is one to one, everywhere defined and onto (one to one correspondence).

Finally, if a and b are elements of A, then it is clear that $a \leq b$ if and only if $2a \leq' 2b$. Thus f is an isomorphism.

Partially Ordered Sets

- Theorem 3

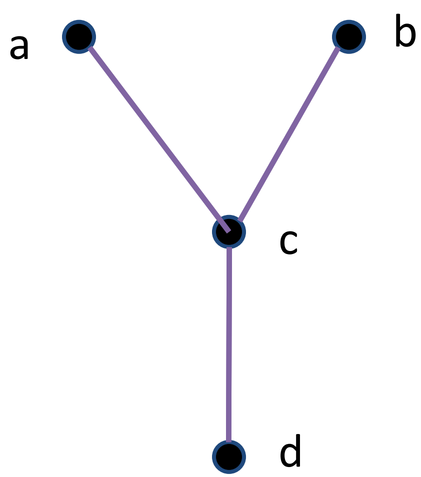
Suppose that f : A→A' is an isomorphism from a poset (A, ≤) to a poset (A', ≤'). Suppose also that B is a subset of A, and B'=f(B) is the corresponding subset of A'. The following principle must hold.

If the elements of B have any property relating to one another or to other elements of A, and if this property can be defined entirely in terms of the relation ≤, then the elements of B' must possess exactly the same property, defined in terms of ≤'.

Partially Ordered Sets

- Example

Let (A, $\leq$) be the poset whose Hasse diagram is shown below, and suppose that f is an isomorphism from (A, $\leq$) to some other poset (A', $\leq$'). Note d $\leq$ x for any x in A, then the corresponding element f(d) in A' must satisfy the property f(d) $\leq$y for all y in A'.



As another example, $a \not\leq b$ and $b \not\leq a$.
Such a pair is called incomparable in A, then f(a) and f(b) are also incomparable in A'

Partially Ordered Sets

Let (A, ≤) and (A', ≤ ') be finite posets, let f: A→A' be a one-to-one correspondence, and let H be any Hasse diagram of (A, ≤). Then

If f is an isomorphism and each label a of H is replaced by f(a), then H will become a Hasse diagram for (A, ≤ ')

Conversely

If H becomes a Hasse diagram for (A', ≤ '), whenever each label a is replaced by f(a), then f is an isomorphism.

Two finite isomorphic posets have the same Hasse diagrams

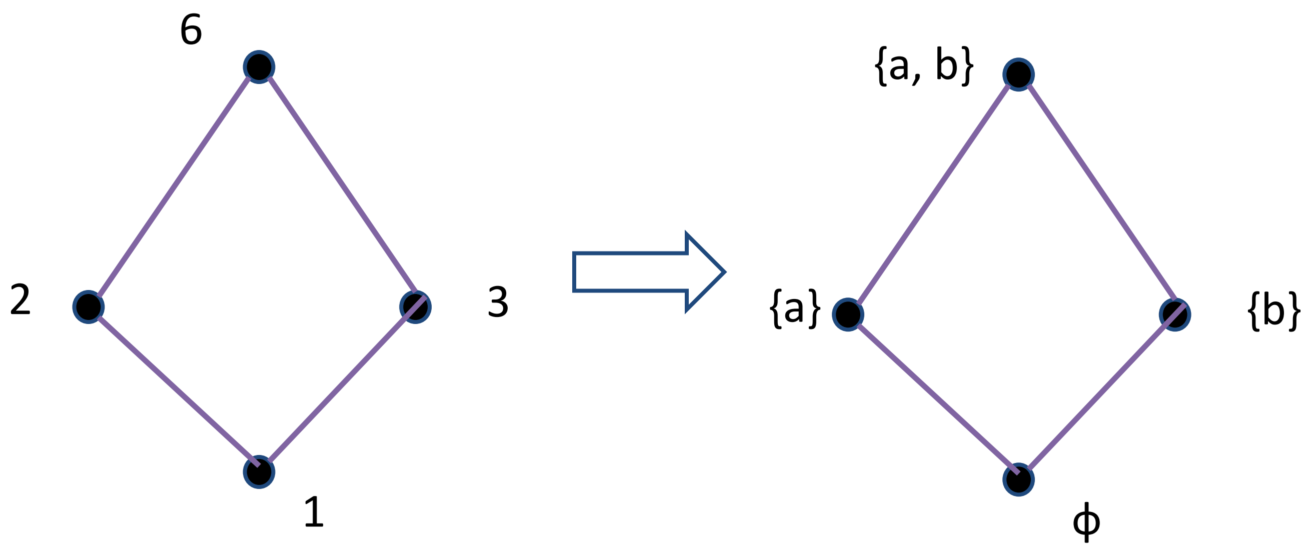
Partially Ordered Sets

- Example

  Let A={1,2,3,6} and let ≤ be the relation | .

  Let A'= {ϕ, {a}, {b}, {a, b}} and let ≤' be set containment, ⊆.

  If f(1)= ϕ, f(2)={a}, f(3)={b}, f(6)={a, b}, then f is an isomorphism. They have the same Hasse diagrams.



Partially Ordered **Sets**

Extremal Elements of Partially Ordered Sets

Consider a poset $(A, \leq)$

- Maximal Element

  An element a in A is called a maximal element of A if there is no element c in A such that $a<c$.

- Minimal Element

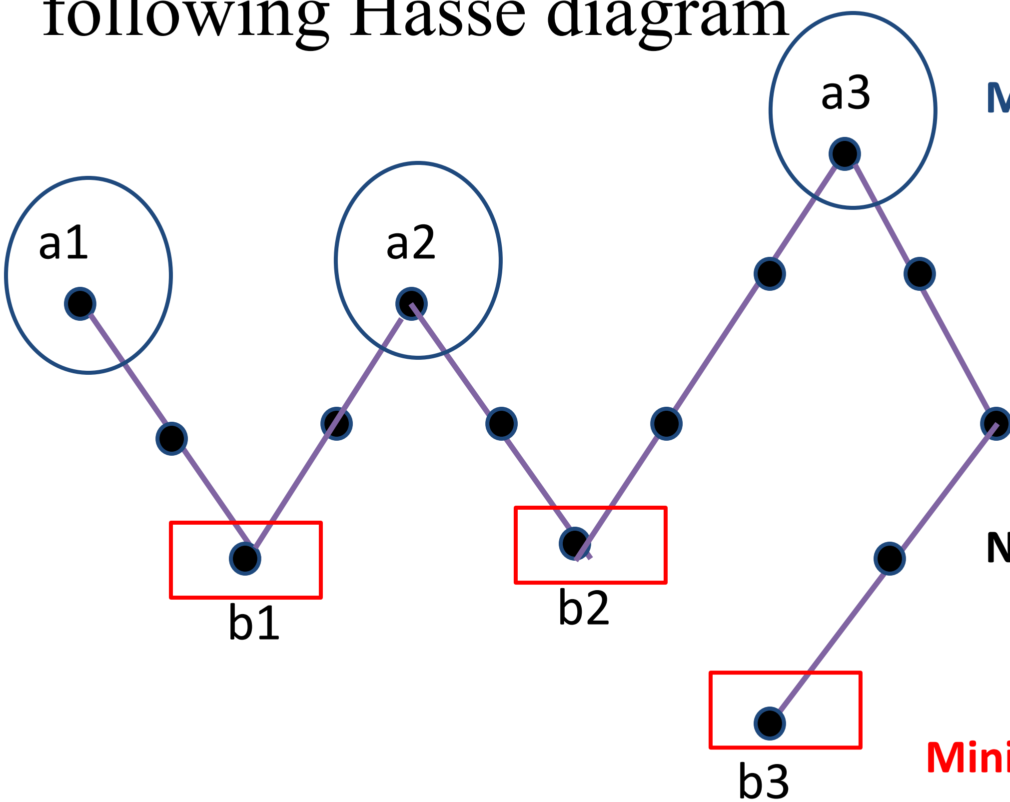
  An element b in A is called a minimal element of A if there is no element c in A such that $c<b$.

an element a in A is a maximal (minimal) element of $(A, \geq)$ if and only if a is a minimal (maximal) element of $(A, \leq)$

Extremal Elements of Partially Ordered Sets

- Example 1

Find the maximal and minimal elements in the following Hasse diagram



Maximal elements

Note: a1, a2, a3 are incomparable
b1, b2, b3 are incomparable

Minimal element

Extremal Elements of Partially Ordered Sets

- Example 2

  Let A be the poset of nonnegative real number with the usual partial order $\leq$. Then 0 is a minimal element of A. There are no maximal elements of A

- Example 3

  The poset Z with the usual partial order $\leq$ has no maximal elements and has no minimal elements

Extremal Elements of Partially Ordered Sets

- Theorem 1

  Let A be a **finite nonempty** poset with partial order $\leq$. Then A has at least one maximal element and at least one minimal element.

Proof: Let a be any element of A. If a is not maximal, we can find an element a1 in A such that $a<a_1$. If a_1 is not maximal, we can find an element a_2 in A such that $a_1<a_2$. This argument can not be continued indefinitely, since A is a finite set. Thus we eventually obtain the finite chain

$$a <a_1<a_2<\ldots<a_{k-1}<a_k$$

which cannot be extended. Hence we cannot have $a_k < b$ for any b in A, so A_k is a maximal element of (A, $\leq$).

Extremal Elements of Partially Ordered Sets

- Algorithm

For finding a topological sorting of a finite poset (A $\leq$).

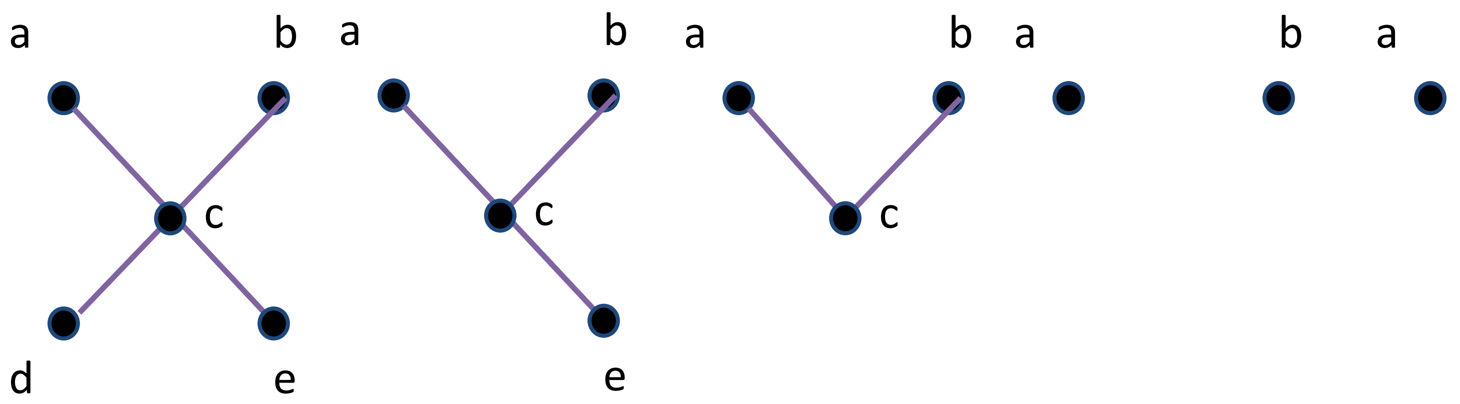
Step 1: Choose a minimal element a of A

Step 2: Make a the next entry of SORT and replace A with A-{a}

Step 3: Repeat step 1 and 2 until A={ }.

Extremal Elements of Partially Ordered Sets

- Example 4



SORT: d e c b a

Extremal Elements of Partially Ordered Sets

- Greatest element

An element a in A is called a greatest element of A if

$$x \leq a \text{ for all } x \text{ in } A.$$

- Least element

An element a in A is called a least element of A if

$$a \leq x \text{ for all } x \text{ in } A.$$

Note: an element a of $(A, \leq)$ is a greatest (or least) element if and only if it is a least (or greatest) element of $(A, \geq)$

Extremal Elements of Partially Ordered Sets

- Example 5

  Let A be the poset of nonnegative real number with the usual partial order $\leq$. Then 0 is a least element of A. There are no greatest elements of A

- Example 7

  The poset Z with usual partial order has neither a least nor a greatest element.

Extremal Elements of Partially Ordered Sets

- Theorem 2

  A poset has at most one greatest element and at most one least element.

Proof: Support that a and b are greatest elements of a poset A. since b is a greatest element, we have $a \leq b$;

since a is a greatest element, we have $b \leq a$; thus

$a=b$ by the antisymmetry property. so, if a poset has a greatest element, it only has one such element.

This is true for all posets, the dual poset $(A, \geq)$ has at most one greatest element, so $(A, \leq)$ also has at most one least element.

Extremal Elements of Partially Ordered Sets

- Unit element

  The greatest element of a poset, if it exists, is denoted by I and is often called the unit element.

- Zero element

  The least element of a poset, if it exists, is denoted by 0 and is often called the zero element.

Q: does unit/zero element exist for a finite nonempty poset?

Extremal Elements of Partially Ordered Sets

Consider a poset A and a subset B of A

- Upper bound

  An element a in A is called an upper bound of B if $b \leq a$ for all b in B

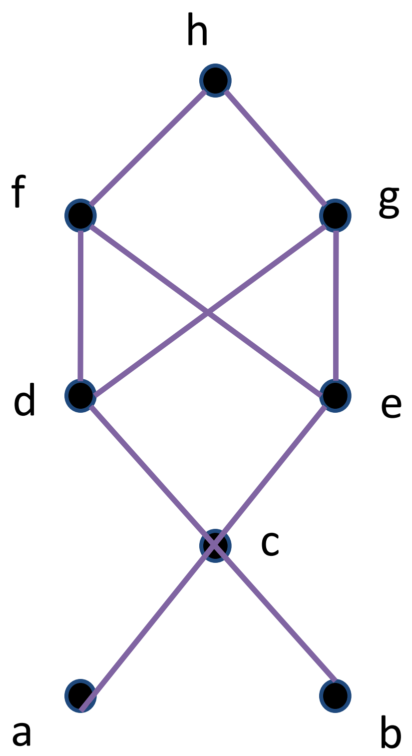
- Lower bound

  An element a in A is called a lower bound of B if $a \leq b$ for all b in B

Extremal Elements of Partially Ordered Sets

- Example 8

Find all upper and lower bounds of the following subset of A: (a) $B_1=\{a, b\}$; $B_2=\{c, d, e\}$



B_1 has no lower bounds; The upper bounds of B_1 are c, d, e, f, g and h

The lower bounds of B_2 are c, a and b
The upper bounds of B_2 are f, g and h

Extremal Elements of Partially Ordered Sets

Let A be a poset and B a subset of A,

- Least upper bound

  An element a in A is called a least upper bound of B, denoted by (LUB(B)), if a is an upper bound of B and $a \leq a'$, whenever a' is an upper bound of B.

- Greatest lower bound

  An element a in A is called a greatest lower bound of B, denoted by (GLB(B)), if a is a lower bound of B and $a' \leq a$, whenever a' is a lower bound of B.

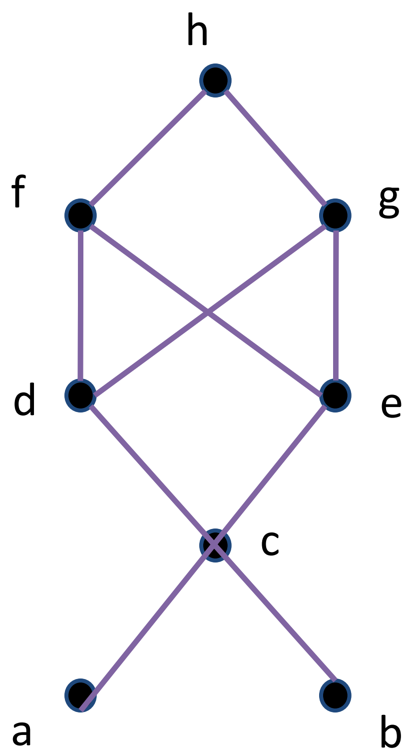
Some properties of dual of poset

- The upper bounds in $(A, \leq)$ correspond to lower bounds in $(A, \geq)$ (for the same set of elements)
- The lower bounds in $(A, \leq)$ correspond to upper bounds in $(A, \geq)$ (for the same set of elements)
- Similar statements hold for greatest lower bounds and least upper bounds.

Extremal Elements of Partially Ordered Sets

- Example 9

  Find all least upper bounds and all greatest lower bounds of (a) $B_1=\{a, b\}$ (b) $B_2=\{c, d, e\}$



(a) Since B1 has no lower bounds, it has no greatest lower bounds; However,

LUB(B1)=c

(b)Since the lower bounds of B2 are c, a and b, we find that GLB(B2)=c

The upper bounds of B2 are f, g and h. Since f and g are not comparable, we conclude that B2 has no least upper bound.

Extremal Elements of Partially Ordered Sets

- Theorem 3

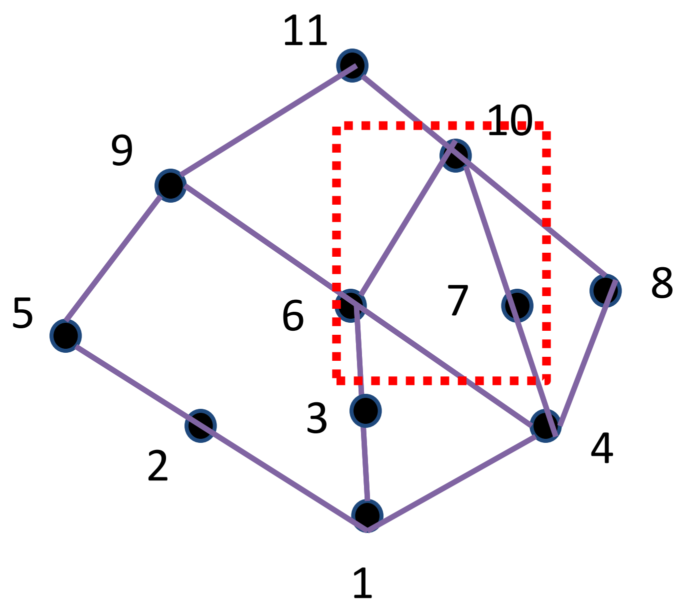
  Let (A, $\leq$) be a poset. Then a subset B of A has at most one LUB and at most one GLB

  Please refer to the proof of Theorem 2.

Extremal Elements of Partially Ordered Sets

- Example 10

  Let A={1,2,3,…,11} be the poset whose Hasse diagram is shown below. Find the LUB and GLB of B={6,7,10}, if they exist.



The upper bounds of B are 10, 11, and LUB(B) is 10 **(the first vertex that can be Reached from {6,7,10} by upward paths)**

The lower bounds of B are 1,4, and GLB(B) is 4 **(the first vertex that can be Reached from {6,7,10} by downward paths)**

Extremal Elements of Partially Ordered Sets

- Theorem 4

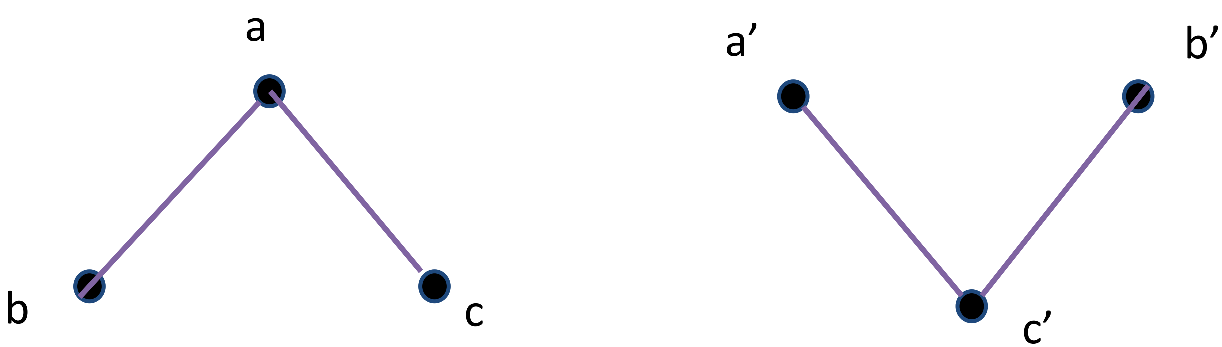
  Suppose that (A, ≤) and (A, ≤') are isomorphic posets under the isomorphic f: A→A'

1. If a is a maximal (minimal) element of (A, ≤), then f(a) is a maximal (minimal) element of (A', ≤')
2. If a is the greatest (least) element of (A, ≤), then f(a) is the greatest (least) element of (A',≤')
3. If a is an upper (lower, least upper, greatest lower) bound of a subset B of A, then f(a) is an upper (lower, least upper, greatest lower) bound for subset f(B) of A'
4. If every subset of (A, ≤) has a LUB (GLB), then every subset of (A', ≤') has a LUB (GLB)

Extremal Elements of Partially Ordered Sets

- Example 11

Show that the posets (A, ≤) and (A', ≤'), whose Hasse diagrams are shown below are not isomorphic.



(A, ≤) has a greatest element a, while (A', ≤') does not have a greatest element

Extremal Elements of Partially Ordered Sets

- Homework

  Ex. 2, Ex. 18, Ex. 22, Ex. 28, Ex. 34, Ex. 37