Combinatorics

Combinatorics

Count the number of ways to put things together into various combinations.

e.g. If a password is 6, 7, or 8 characters long; a character is an uppercase letters or a digit, and the password is required to include at least one digit - how many passwords can there be?

Or, how many graphs are there on ***N*** nodes? How many of those are 3-colorable?

Many uses in discrete math (because of all the discrete strucures), including e.g. probability theory (next topic).

E.g., what is the probability that a randomly generated graph is 3-colorable? How can we figure that out?

First, two most basic rules:

- Sum rule
- Product rule

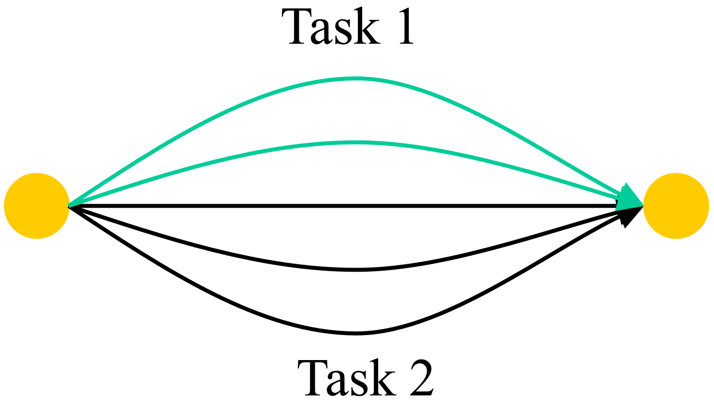
Sum Rule

Let us consider two tasks:

- *m* is the number of ways to do **task 1**
- *n* is the number of ways to do **task 2**
- Tasks are <u>independent</u> of each other, i.e.,
  - Performing **task 1** does not accomplish **task 2** and vice versa.

Sum rule: the number of ways that "**either** task 1 **or** task 2 can be done, but **not both**", is $m + n$.

Generalizes to multiple tasks ...



Example

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects respectively. How many possible projects are there to choose from?

$$23+15+19$$

Ok… not to worry. things will get more exciting... ☺

Sum rule example

How many strings of 4 decimal digits, have exactly three digits that are 9s?

- The string can have:
  - The non-9 as the first digit
  - OR the non-9 as the second digit
  - OR the non-9 as the third digit
  - OR the non-9 as the fourth digit
  - Thus, we use the sum rule
- For each of those cases, there are 9 possibilities for the non-9 digit (any number other than 9)
- Thus, the answer is 9+9+9+9 = 36

Set Theoretic Version

If A is the set of ways to do task 1, and B the set of ways to do task 2, and if A and B are disjoint, then:

"the ways to do either task 1 or 2 are

$A \cup B$, and $|\boldsymbol{A} \cup \boldsymbol{B}| = |\boldsymbol{A}| + |\boldsymbol{B}|$"

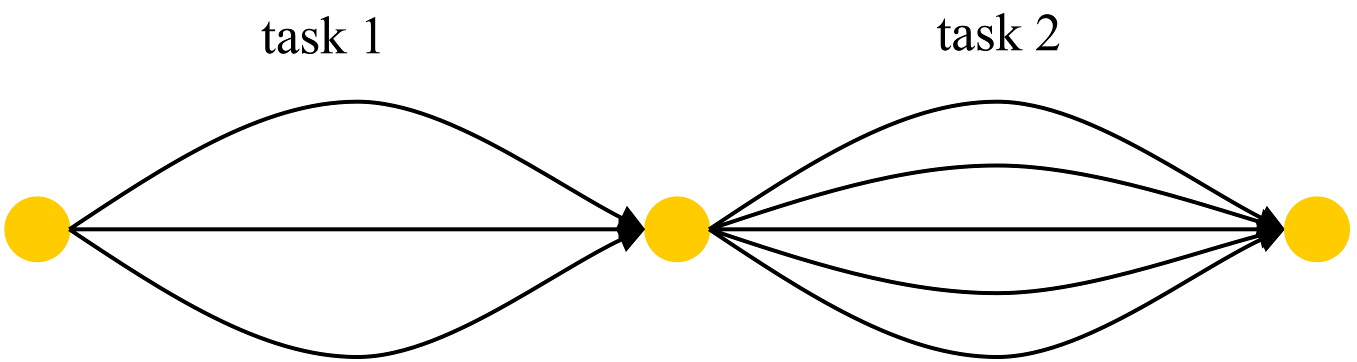
Product Rule

Let us consider two tasks:

- *m* is the number of ways to do **task 1**
- *n* is the number of ways to do **task 2**
- Tasks are <u>independent</u> of each other, i.e.,
  - Performing task 1does not accomplish task 2 and vice versa.

<u>Product rule</u>: the number of ways that "**both** tasks 1 and 2 can be done" in *mn*.

Generalizes to multiple tasks ...



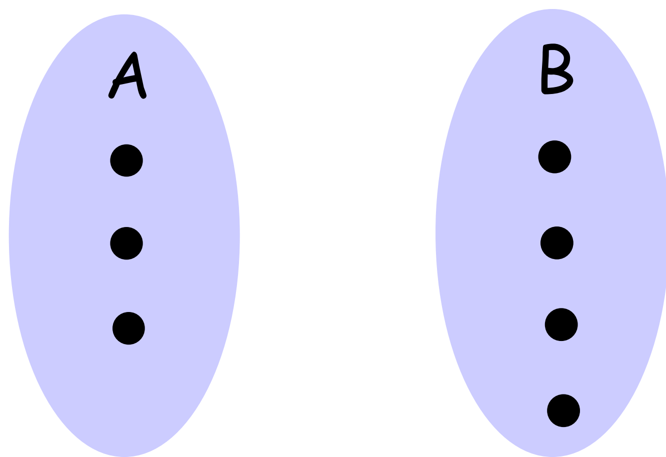
Product rule example

- There are 18 math majors and 325 CS majors
- How many ways are there to pick one math major and one CS major?

Total is 18 * 325 = 5850

Product Rule

How many functions are there from set A to set B?



So, how many Boolean functions on n vars?

$$2^{2^n}$$

To define each function we have to make 3 choices, one for each element of A. Each has 4 options (to select an element from B).

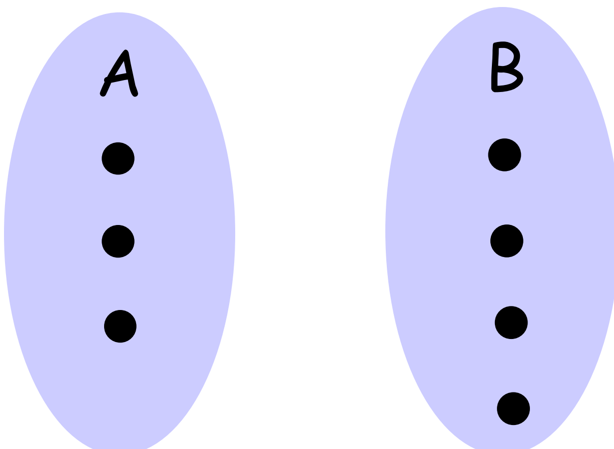
4 ___ 4 ___ 4 ___

How many ways can each choice be made?

$$4^3 = 64 = |B|^{|A|}$$

is called P(n,r) for r-permutations (here P(4,3) --- “3 unique choices out of 4 objects”, order matters)

How many one-to-one functions are there from set A to set B?



Ex: S={1,2,3}. Ordered arrangement 3,1,2 is called a permutation.
There are n! of those (product rule).
3,2 is a r-permutation (r=2).
There are n!/(n-r!) of those.

To define each function we have to make 3 choices, one for each element of A.

How many ways can each choice be made?

4 3 2

24 = 4! / (4-3)!

Hmm. What if |A| = 4?

Product rule example

How many strings of 4 decimal digits, do not contain the same digit twice?

- We want to chose a digit, then another that is not the same, then another…
  - First digit: 10 possibilities
  - Second digit: 9 possibilities (all but first digit)
  - Third digit: 8 possibilities
  - Fourth digit: 7 possibilities
- Total = 10*9*8*7 = 5040

How many strings of 4 decimal digits, end with an even digit?

- First three digits have 10 possibilities
- Last digit has 5 possibilities
- Total = 10*10*10*5 = 5000

Set Theoretic Version

If A is the set of ways to do task 1, and B the set of ways to do task 2, and if A and B are disjoint, then

The ways to do both task 1 and 2 can be represented as $A \times B$,

and $|A \times B| = |A| \cdot |B|$

More complex counting problems

Combining the product rule and the sum rule.

Thus we can solve more interesting and complex problems.

Count the number of ways to put things together into various combinations.

E.g. If a password is 6, 7, or 8 characters long; a character is an uppercase letters or a digit, and the password is required to include at least one digit. How many passwords can there be?

Let P – total number of possible passwords

P_i – total number of passwords of length i, i = 6,7,8

$P = P_6 + P_7 + P_8$ (sum rule)

P_i – **computing it directly is tricky** (hmm…) -

"popular" counting trick: let's calculate all of them, including those with no digits and then subtract the ones with no digits.

$P_i = 36^i - 26^i$

$P = 36^6 - 26^6 + 36^7 - 26^7 + 36^8 - 26^8 = 2,684,483,063,360$

IP Address Example (Internet Protocol v. 4)

An address is a string of 32 bits – it begins with a network id (netid), followed by a host number (hostid), which identifies a computer as a member of a particular network.

Main computer addresses are in one of 3 types:

- *Class A (largest networks):* address contains a 0 followed by 7-bit "netid" ≠ 17, and a 24-bit "hostid"
- *Class B (medium networks):* address contains a 10 followed by a 14-bit netid and a 16-bit hostid.
- *Class C (smallest networks):* address contains a 110 has 21-bit netid and an 8-bit hostid.

Netids all 1s are **not allowed**. Hostids that are all 0s or all 1s are **not allowed**.

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0				netid			hostid	
Class B	1	0				netid		hostid	
Class C	1	1	0				netid		hostid

How many valid IP addresses are there?

Example Using Both Rules: IP address solution

(# addrs) = (# class A) + (# class B) + (# class C)
(by sum rule)

class A = (# valid netids)·(# valid hostids)
(by product rule)

(# valid class A netids) = $2^7 - 1 = 127$.
(# valid class A hostids) = $2^{24} - 2 = 16{,}777{,}214$.

Continuing in this fashion we find the answer is:
3,737,091,842 (3.7 billion IP addresses)

Wedding pictures example

Consider a wedding picture of 6 people

- There are 10 people, including the bride and groom

How many possibilities are there if the bride must be in the picture?

- Product rule: place the bride AND then place the rest of the party
- First place the bride
  - She can be in one of 6 positions
- Next, place the other five people via the product rule
  - There are 9 people to choose for the second person, 8 for the third, etc.
  - Total = 9*8*7*6*5 = 15120
- Product rule yields 6 * 15120 = 90,720 possibilities

Q.: Are we counting same subsets of folks in different positions?

Yes! (note bride is treated “differently”; has to be in; draw diagram)

Wedding pictures example

Consider a wedding picture of 6 people

- There are 10 people, including the bride and groom

How many possibilities are there if the bride and groom must both be in the picture

- Product rule: place the bride/groom AND then place the rest of the party
- First place the bride and groom
  - She can be in one of 6 positions
  - He can be in one 5 remaining positions
  - Total of 30 possibilities
- Next, place the other four people via the product rule
  - There are 8 people to choose for the third person, 7 for the fourth, etc.
  - Total = 8*7*6*5 = 1680
- Product rule yields 30 * 1680 = 50,400 possibilities

Wedding pictures example

Consider a wedding picture of 6 people

- There are 10 people, including the bride and groom

How many possibilities are there if only one of the bride and groom are in the picture?

- Sum rule: place only the bride
  - Product rule: place the bride AND then place the rest of the party
  - First place the bride
    - She can be in one of 6 positions
  - Next, place the other five people via the product rule
    - There are 8 people to choose for the second person, 7 for the third, etc.
      - » We can't choose the groom!
    - Total = 8*7*6*5*4 = 6720
  - Product rule yields 6 * 6720 = 40,320 possibilities
- OR place only the groom
  - Same possibilities as for bride: 40,320
- Sum rule yields 40,320 + 40,320 = 80,640 possibilities

(hmm… quickly, how many?)

Wedding pictures example

Consider a wedding picture of 6 people

- There are 10 people, including the bride and groom

Alternative means to get the answer

How many possibilities are there if only one of the bride and groom are in the picture?

- Total ways to place the bride (with or without groom): 90,720
  - See before.
- Total ways for both the bride and groom: 50,400
  - See before.
- Total ways to place ONLY the bride: 90,720 – 50,400 = 40,320
- Same number for the groom
- Total = 40,320 + 40,320 = 80,640

The inclusion-exclusion principle
(seen briefly when we did sets)

When counting the possibilities, we can't include a given outcome more than once.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

- E.g. Let A_1 have 5 elements, A_2 have 3 elements, and 1 element be both in A_1 and A_2
- Total in the union is 5+3-1 = 7, not 8

Inclusion-exclusion example

How may bit strings of length eight start with 1 or end with 00?

Count bit strings that start with 1

- Rest of bits can be anything: $2^7 = 128$
- This is $|A_1|$

Count bit strings that end with 00

- Rest of bits can be anything: $2^6 = 64$
- This is $|A_2|$

Count bit strings that both start with 1 and end with 00

- Rest of the bits can be anything: $2^5 = 32$
- This is $|A_1 \cap A_2|$

Use formula $|A_1 \text{ U } A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Total is $128 + 64 – 32 = 160$

How many bit strings of length 10 contain either 5 consecutive 0s or 5 consecutive 1s?

Consider 5 consecutive 0s first

Sum rule: the 5 consecutive 0's can start at position 1, 2, 3, 4, 5, or 6

- Starting at position 1
  - Remaining 5 bits can be anything: $2^5 = 32$
- Starting at position 2
  - First bit must be a 1
    - Otherwise, we are including possibilities from the previous case!
  - Remaining bits can be anything: $2^4 = 16$
- Starting at position 3
  - Second bit must be a 1 (same reason as above)
  - First bit and last 3 bits can be anything: $2^4 = 16$
- Starting at positions 4 and 5 and 6
  - Same as starting at positions 2 or 3: 16 each
- Total = 32 + 16 + 16 + 16 + 16 + 16 = 112

The 5 consecutive 1's follow the same pattern, and have 112 possibilities

There are two cases counted twice (that we thus need to exclude):

0000011111 and 1111100000

Total = 112 + 112 – 2 = 222

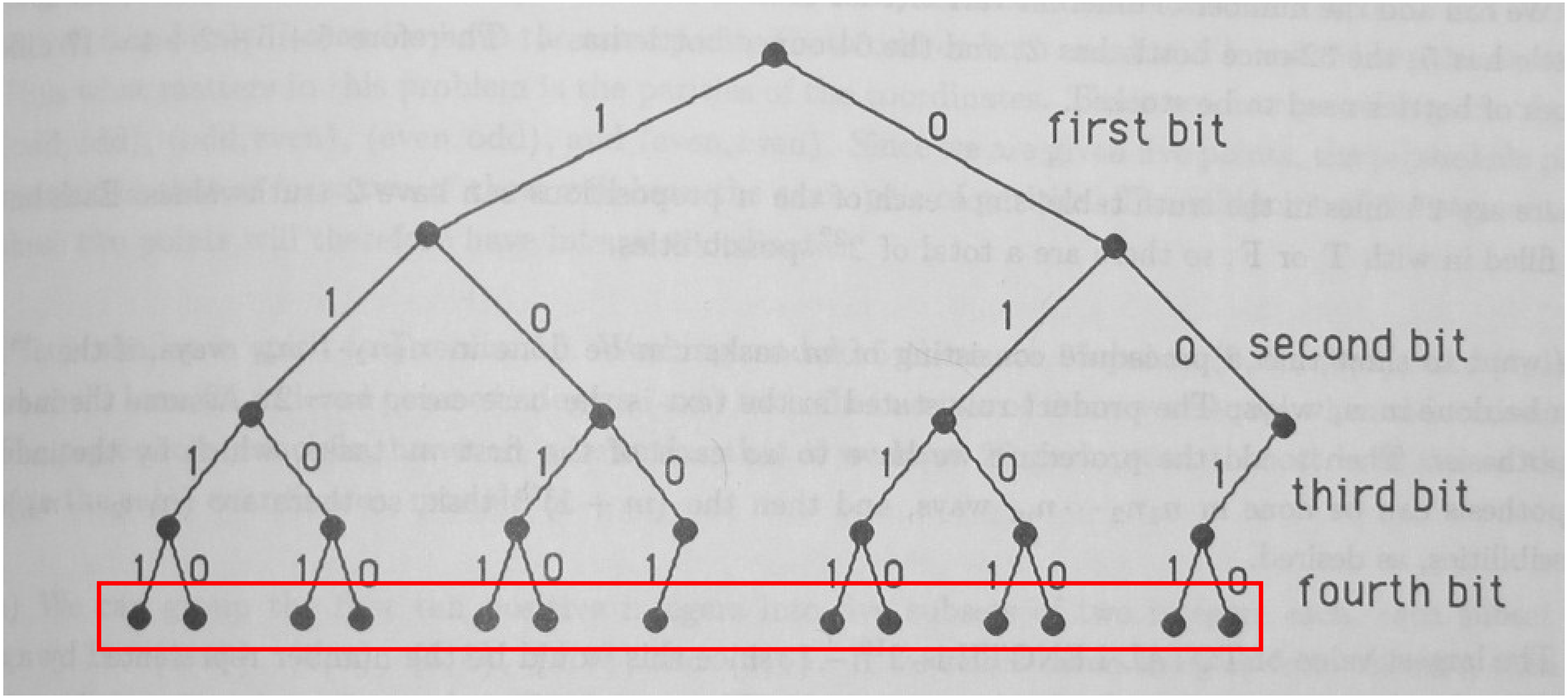
Tree diagrams

We can use tree diagrams to enumerate the possible choices.

Once the tree is laid out, the result is the number of (valid) leaves.

Tree diagrams example

Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s



Pigeonhole Principle

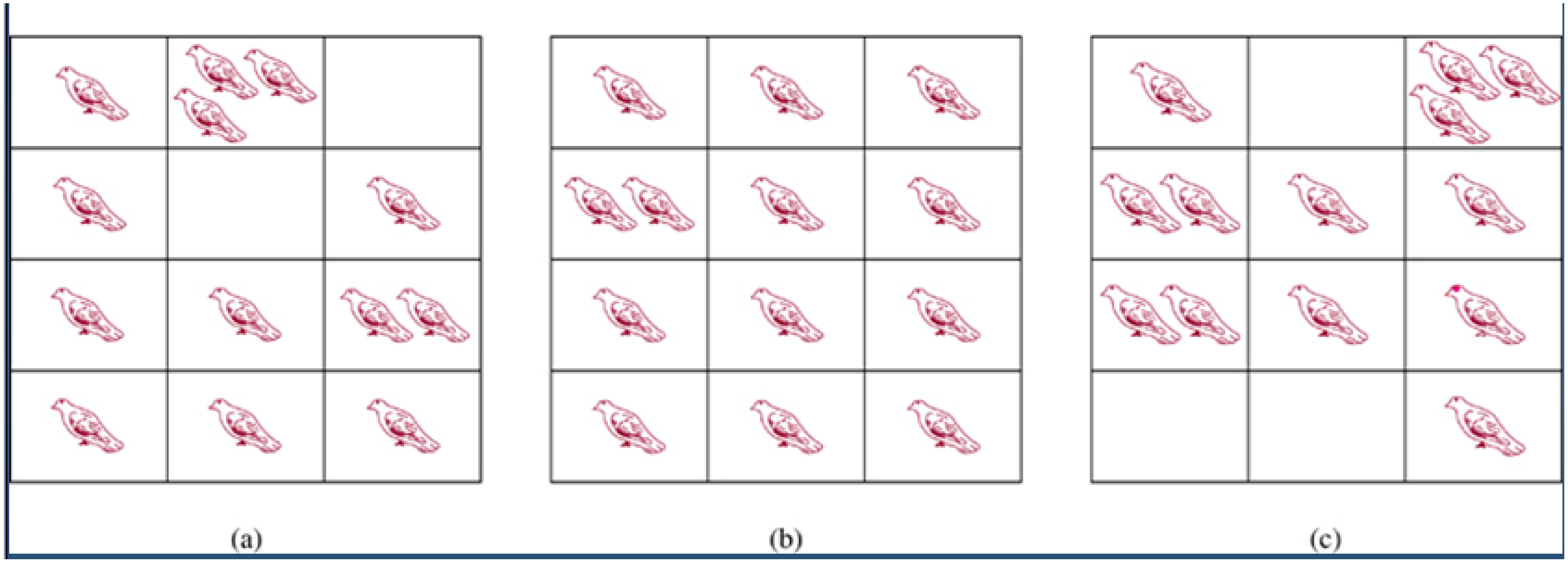
If k+1 objects are assigned to k places, then **at least** 1 place must be assigned ≥2 objects.

Proof: (by contradiction)

Suppose none of the k places contains more than one object. Then the total number of objects would be at most k. This is a contradiction, since there are k + 1 objects. QED ☺

In terms of the assignment function:

If $f: A \rightarrow B$ and $|A| \geq |B|+1$, then some element of B has ≥2 pre-images under f. I.e., f **is not** one-to-one.



More pigeons than pigeonholes

Example

How many students must be in class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

102

So, if a million students take a national test with say 100 questions, many must have the same score (in expectation 10,000). So, would need at least a million questions to get a chance of a unique score for everyone.

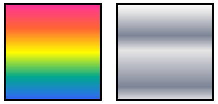
Simple Example

It's dark; you know that in your drawer there are:

10+12

But you can't see a thing. How many socks should you get to guarantee a correct pair?
What does it have to do with the pigeon hole principle?

A.: 3



1 hole per color

There must be at least two people in New York city with exactly the same number of hairs on their heads. Why?

Typical head of hair has around 150,000 hairs. So, it is reasonable to assume that no one has more than 1,000,000 hairs on their head (m = 1 million holes).

There are more than 1,000,000 people in NYC (n is bigger than 1 million objects). If we assign a pigeonhole for each number of hairs on a head, and assign people to the pigeonhole with their number of hairs on it, there must be at least two people with the same number of hairs on their heads.

Useful stuff to know… ☺

Generalized Pigeonhole Principle

If $N \geq k+1$ objects are assigned to k places, then **at least** one place must be assigned at **least** $\lceil N/k \rceil$ objects.

E.g., there are $N = 280$ people in a party. There are $k = 52$ weeks in the year.

– Therefore, there must be at **least** 1 week during which **at least** $\lceil 280/52 \rceil = \lceil 5.38 \rceil = 6$ students in the party have a birthday.

Proof of G.P.P.

By <u>contradiction</u>. Suppose **every** place has $< \lceil N/k \rceil$ objects,
thus $\leq \lceil N/k \rceil - 1$.

Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = k\left(\frac{N}{k}\right) = N$$

So, there are less than N objects, which contradicts our assumption of N objects!

QED

G.P.P. Example

Given: There are 280 people in the party. Without knowing anybody's birthday, what is the **largest** value of n for which we can prove that at **least** n people must have been born in the same month?

Answer:

$$\lceil 280/12 \rceil = \lceil 23.3 \rceil = 24$$

A bowl contains 10 red and 10 yellow balls
How many balls must be selected to **ensure** 3 balls of the same color?

- One solution: consider the "worst" case
  - Consider 2 balls of each color
  - You can't take another ball without hitting 3
  - Thus, the answer is 5
- Via generalized pigeonhole principle
  - How many balls are required if there are 2 colors, and one color must have 3 balls?
  - How many pigeons are required if there are 2 pigeon holes, and one must have 3 pigeons?
  - number of boxes: $k = 2$
  - We want $\lceil N/k \rceil = 3$
  - What is the minimum N?
  - $N = 5$

A bowl contains 10 red and 10 yellow balls

How many balls must be selected to **ensure** 3 yellow balls?

- Consider the "worst" case
  - Consider 10 red balls and 2 yellow balls
  - You can't take another ball without hitting 3 yellow balls
  - Thus, the answer is 13

PH principles can be pop up in "all kinds of places"…

Consider 5 distinct points (x_i, y_i) with integer values, where i = 1, 2, 3, 4, 5. Show that the midpoint of at least one pair of these five points also has integer coordinates.

Thus, we are looking for the midpoint of a segment from (a,b) to (c,d)

- The midpoint is ((a+c)/2, (b+d)/2)

Note that the midpoint will be integers if a and c have the same parity: are either both even or both odd

- Same for b and d

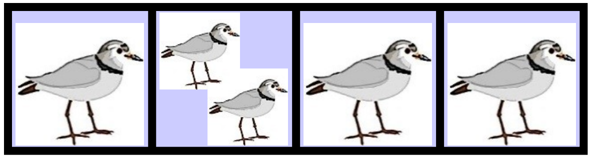
There are four parity possibilities

- (even, even), (even, odd), (odd, even), (odd, odd)

Since we have 5 points, by the pigeonhole principle, there must be two points that have the same parity possibility

- Thus, the midpoint of those two points will have integer coordinates.

QED

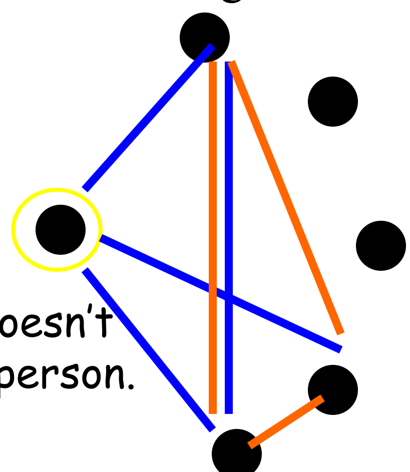


"The party problem"

Dinner party of six: Either there is a group of 3 who all know each other, or there is a group of 3 who are all strangers.

By contradiction. Assume we have a party of six where no three people all know each other and no three people are all strangers.

Consider one person.



She either knows or doesn't know each other person.

But there are 5 other people! So, she knows, or doesn't know, at least 3 others. (GPH)

Let's say she knows 3 others.

If any of those 3 know each other, we have a blue △, which means 3 people know each other. Contradicts assumption.

So they all must be strangers. But then we have three strangers. Contradicts assumption.

The case where she *doesn't* know 3 others is similar. Also, leads to constradiction.

So, such a party does not exist! QED

Party problem: Nicer in terms of graphs.

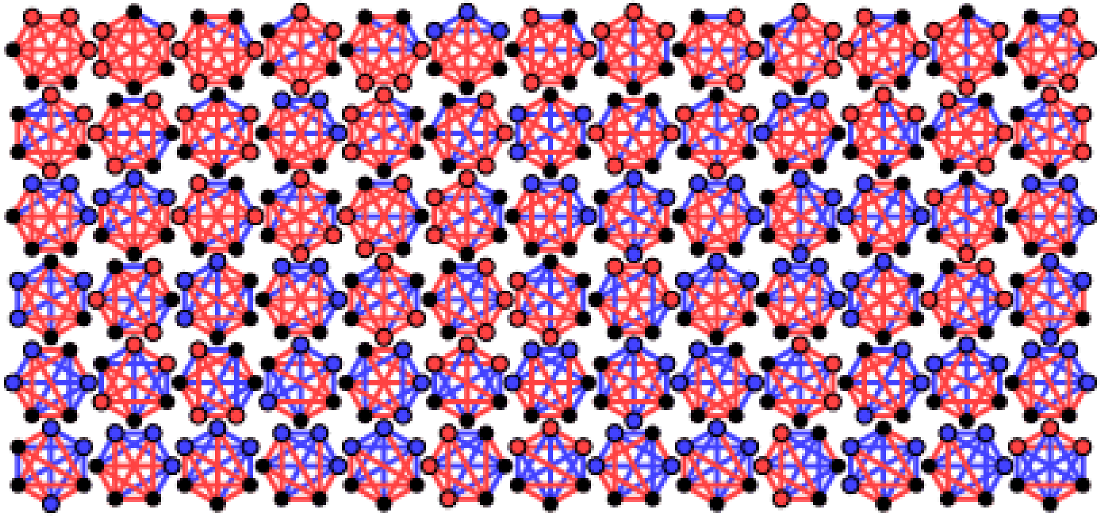
Consider the complete graph on N = 6 nodes.

Now color each edge either blue ("know each other") or red ("don't know each other").

It follows that each coloring will contain a red or a blue triangle, no matter how the graph is colored!

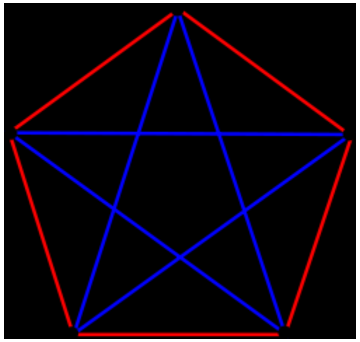
Proof handles:
2^{15} = 32,768 possible edge colorings.
A blue or red triangle is always present.

Removing "symmetries": 78 cases remain.



Example of a Ramsey theory: hidden structure in graphs!

What about a party of five?



No red or blue triangle! So, property does not hold for party of five.

Define: Let R(k,t) be the minimal n such that if the edges of the complete graph on n nodes are colored Red and Blue, then there either is a complete subgraph of k nodes with all edges Red or a complete subgraph of t nodes with all edges Blue.

R(k,t): $\mathbf{N}^2$ **->** **N** is the Ramsey function. R(k,t) is also called the Ramsey number.

What is K(3,3)? K(3,3) = 6

Ramsey proved that R(k,t) is well-defined. I.e., for any values of k and t (>= 2), when the n gets large enough, there will always be a monochromatic Red complete subgraph of size k or a Blue one of size t.

What are the values of R(k,k)?

R(2,2) = 2

R(3,3) = 6 (shown in 1955)

R(4,4) = 18 (shown in 1955)

R(4,5) = 25 (shown in 1993)

R(5,5) = ? (only recently: $43 <= R(5,5) <= 49$)

R(6,6) = ??

Problem becomes surprisingly difficult, very quickly!

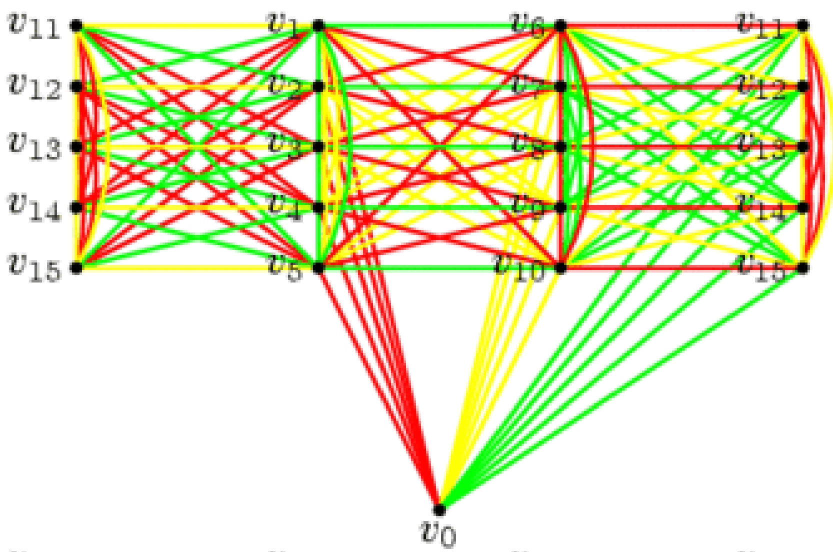
Note: N nodes, $2^{O(N^2)}$ colorings. N = 10, gives $> \sim 10^{30}$; N=30, gives $>\sim 10^{135}$

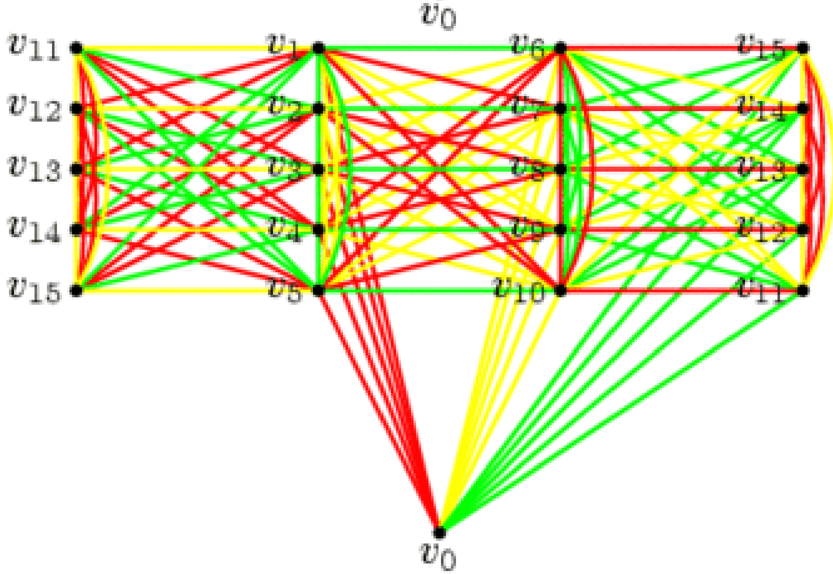
Paul Erdös** (most productive contemporary mathematician)**:

"Imagine an alien force, vastly more powerful than us landing on Earth and demanding the value of R(5, 5) or they will destroy our planet. [hmm?] In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. …
But suppose, instead, that they asked for R(6, 6), then we should attempt to destroy the aliens". ☹

Hmm. Or hire some computer scientists! ... (Selman ’07) ☺

Aside: Can extend to 3 colorings etc.





The *only two* 3-colorings of the complete graph on 16 nodes, that has no monochromatic triangles.

Permutations

A *permutation* of a set *S* of objects is an ordered arrangement of the elements of S where each element appears only once:

e.g., 1 2 3, 2 1 3, 3 1 2

There are **n!** permutations of n objects. (by product rule)

An ordered arrangement of *r* distinct elements of *S* is called an *r-permutation*.

The number of *r*-permutations of a set *S* with $n=|S|$ elements is

$$P(n,r) = n(n-1)\ldots(n-r+1) = n! \,/\, (n-r)!$$

Permutations

In a running race of 12 sprinters, each of the top 5 finishers receives a different medal. How many ways are there to award the 5 medals?

a) 60
b) 12^5
c) 12!/7!
d) 5^{12}
e) No clue

12 11 10 9 8

A.: 12!/7!

Permutations

Suppose you "have" time to listen to 10 songs on your daily jog around campus. There are 6 A tunes, 8 B tunes, and 3 C tunes to choose from.

Finally, suppose you still want 4 A, 4 B, and 2 C tunes, and the order of the groups doesn't matter, but you get dizzy and fall down if all the songs by any one group aren't played together.

How many playlists are there?

$P(6,4) \times P(8,4) \times P(3,2) \times 3!$

Combinations

The number of ways of choosing *r* elements from S (order <u>does not</u> matter).

S={1,2,3}

e.g., 1 2 , 1 3, 2 3

The number of *r*-combinations *C(n,r)* of a set with *n*=|*S*| elements is

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

= P(n,r) / r!

Note: we have $C(n,r) = C(n, n-r)$

"n choose r". Also called a "binomial coefficient".

Combination Example

How many distinct 7-card hands can be drawn from a standard 52-card deck?

– The order of cards in a hand doesn't matter.

Answer $C(52,7) = P(52,7) \,/\, P(7,7)$

$= 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \,/\, 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 52.17.10.7.2.47.23$

$= 133{,}784{,}560$

Permutations

In how many ways can 5 distinct Martians and 3 distinct Jovians stand in line, if no two Jovians stand together? Hmm…

___ M1 ___ M2 ___ M3 ___ M4 ___ M5 ___

5! X P(6,3)

= 5! x C(6,3) x 3!

Combinatorial proof

A *combinatorial proof* is a proof that uses counting arguments to prove a theorem.

- Rather than some other method such as algebraic techniques

Most of the questions in this section are phrased as, “find out how many possibilities there are if …”

- Instead, we could phrase each question as a theorem:
- “Prove there are *x* possibilities if …”
- The same answer could be modified to be a combinatorial proof to the theorem

Circular seatings

How many ways are there to sit 6 people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?

First, place the first person in the north-most chair

- Only one possibility

  (why can we restrict ourselves to only one specific person in that chair?)

Then place the other 5 people

- There are P(5,5) = 5! = 120 ways to do that

  any more issues with rotating table?

  no!

By the product rule, we get 1*120 =120

Alternative means to answer this:

There are P(6,6) = 720 ways to seat the 6 people around the table

For each seating, there are 6 “rotations” of the seating

Thus, the final answer is 720/6 = 120

Horse races

How many ways are there for 4 horses to finish if ties are allowed?

- Note that order does matter!

Solution by cases

- No ties
  - The number of permutations is P(4,4) = 4! = 24
- Two horses tie
  - There are $C(4,2) = 6$ ways to choose the two horses that tie
  - There are $P(3,3) = 6$ ways for the "groups" to finish
    - A "group" is either a single horse or the two tying horses
  - By the product rule, there are 6*6 = 36 possibilities for this case
- Two groups of two horses tie
  - There are $C(4,2) = 6$ ways to choose the two winning horses
  - The other two horses tie for second place
- Three horses tie with each other
  - There are $C(4,3) = 4$ ways to choose the two horses that tie
  - There are $P(2,2) = 2$ ways for the "groups" to finish
  - By the product rule, there are 4*2 = 8 possibilities for this case
- All four horses tie
  - There is only one combination for this
- By the sum rule, the total is 24+36+6+8+1 = 75

Binomial Coefficients

$(a + b)^4 = (a + b)(a + b)(a + b)(a + b)$

$$= \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

Binomial Theorem: Let *x* and *y* be variables, and let *n* be any nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

What is the coefficient of a^8b^9 in the expansion of $(3a + 2b)^{17}$?

What is n? 17

What is j? 9

What is x? 3a

What is y? 2b

$$\binom{17}{9}(3a)^8(2b)^9 = \binom{17}{9}3^8 2^9 a^8 b^9$$

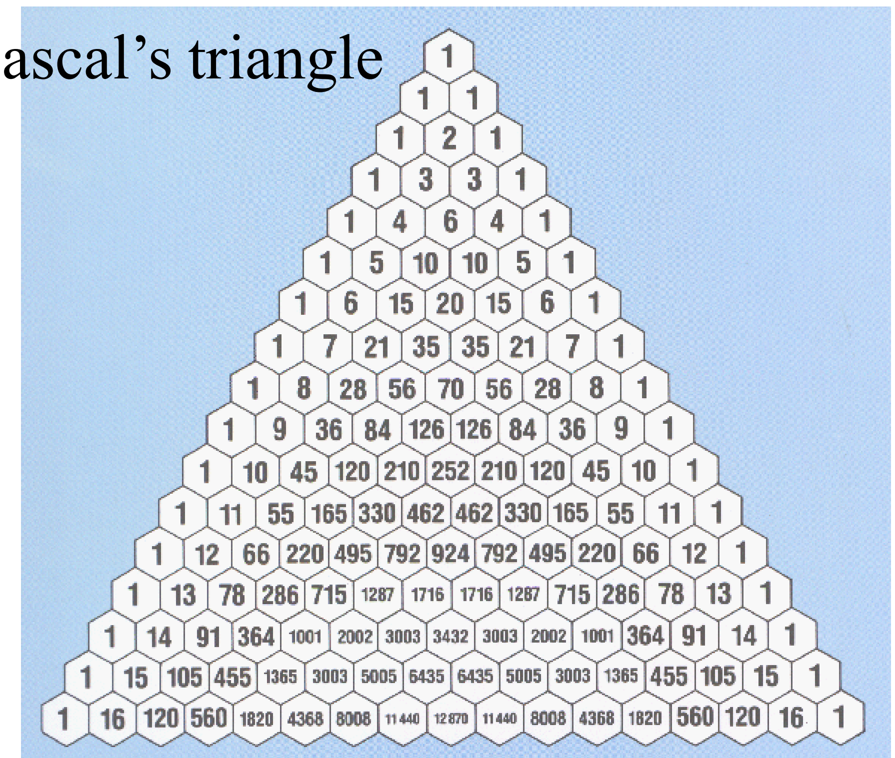
Binomial Coefficients

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Pascal's triangle



What is coefficient of a^9b^3 in $(a + b)^{12}$?

A. 36
B. 220
C. 15
D. 6
E. No clue

A.: 220

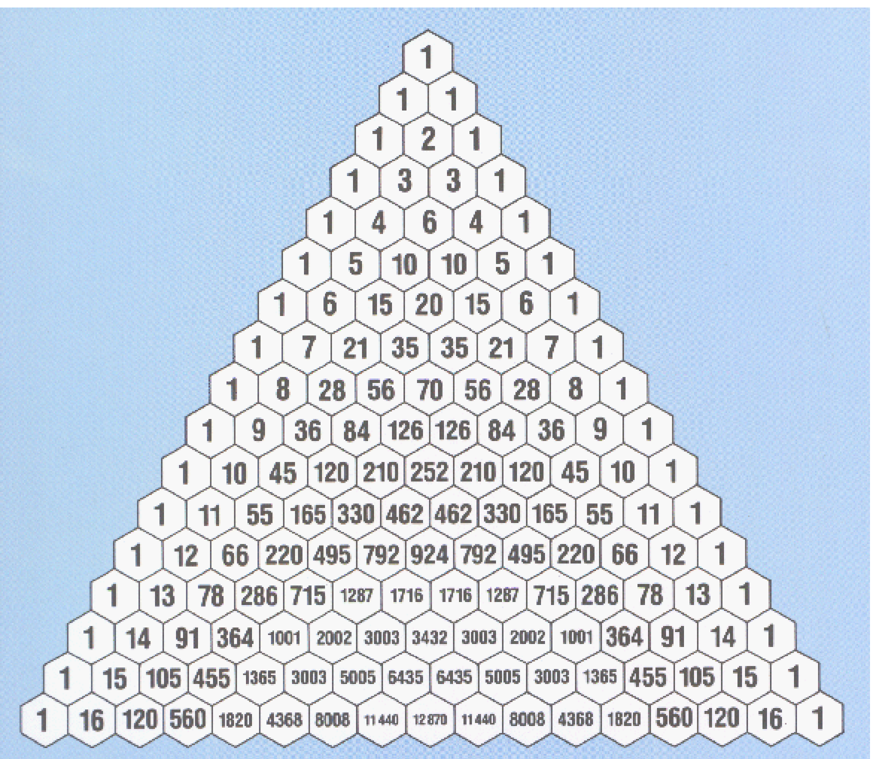
$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Binomial Coefficients

Powers of 2

Sum each row of Pascal's Triangle:

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n$$



Suppose you have a set of size n. How many subsets does it have?

2^n

How many subsets of size 0 does it have?

${}_nC_0$

How many subsets of size 1 does it have?

${}_nC_1$

How many subsets of size 2 does it have?

${}_nC_2$

Add them up we have the result.

QED

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n$$

Alternative (clever) proof? Look at binomial theorem…

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

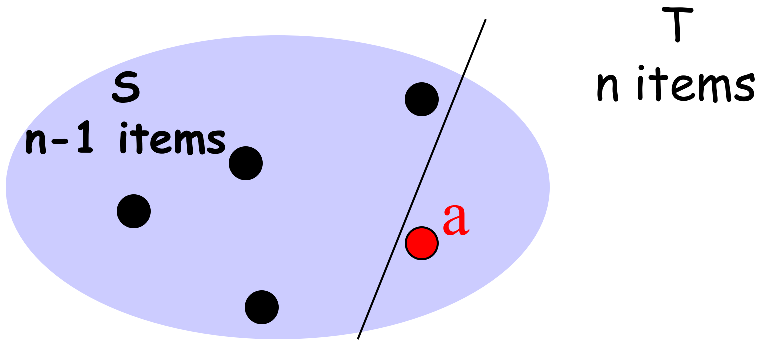
x and y are variables; can pick any numbers… hmm…

Pick x=1 and y=1 !

$$\sum_{j=0}^{n} \binom{n}{j} 1^{n-j} 1^j = (1+1)^n$$

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n$$

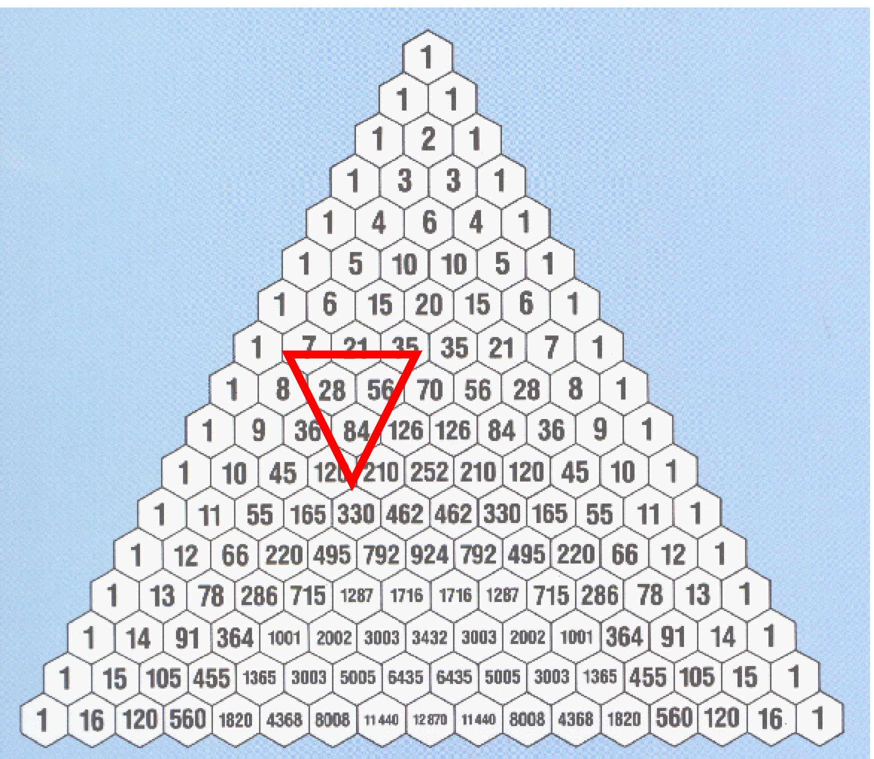
QED



Pascal's Identity

A relationship between the entries in Pascal's triangle.

$$\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}$$



Suppose T is a set, |T|=n. Let a be an element in T, and let S = T - {a}. So, |S| = n-1. Let's count the $\binom{n}{j}$ subsets of size j. Note that some of these contain a, and some don't.

How many contain a? $\binom{n-1}{j-1}$

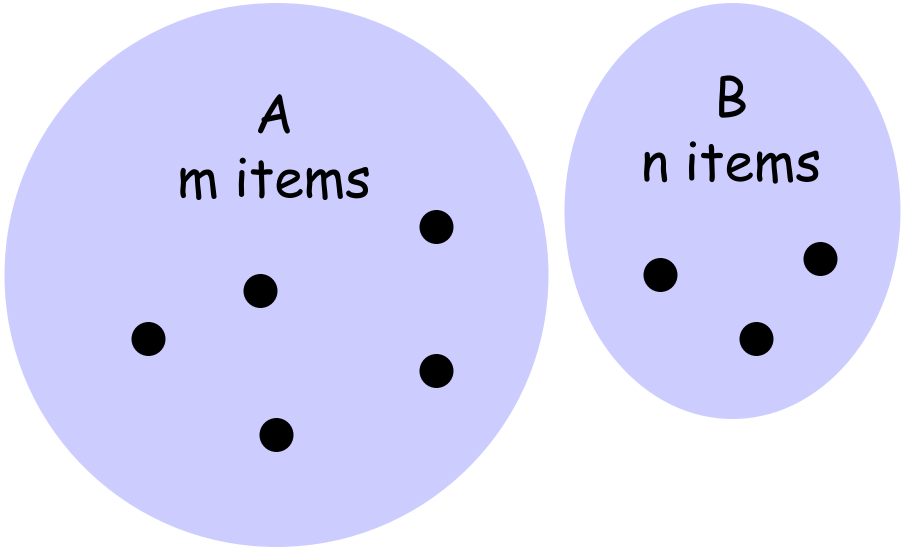
How many don't? $\binom{n-1}{j}$

Vandermonde's Identity

Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{j=0}^{r} \binom{m}{r-j}\binom{n}{j}$$

Why?



To choose r items, take some [(r-j)] from A and some [j] from B. All possible ways of doing this gives the result. (note: items should all be distinct.)

Another combinatorial identity

$$\binom{2n}{n} = \sum_{k=0}^{r} \binom{n}{k}^2$$

Follows directly from the Vandermonde's identity with m = r = n:

$$\binom{2n}{n} = \binom{n+n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

for the last step we used:

$$\binom{n}{k} = \binom{n}{n-k}$$

And another

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Proof. See thm. 4, section 5.4

Combinations with repetition

There are C(n+r-1,r), r-sized combinations from a set of n elements when repetition is allowed.

Proof. See thm. 2, section 5.5.

Example: How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 10$$

When the variables are nonnegative integers?

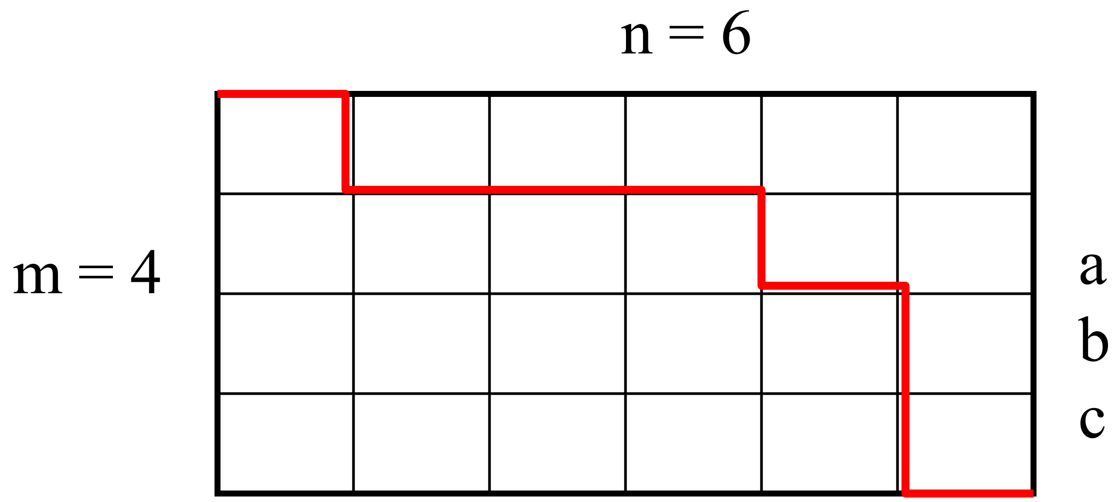
11 locations for bars. Pick 3 allowing repetitions.

C(13,3)

$$1 + 3 + 6 + 0 = 10$$

Counting paths

A turtle begins at the upper left corner of an n x m grid and meanders to the lower right corner.



Need m steps down.
n+1 positions
to go down.

How many routes could she take if she only moves right and down? Hmm…

$$\binom{n+m}{m}$$

$$\binom{n+1+m-1}{m} = \binom{n+m}{m}$$

Permutations with indistinguishable objects

How many different strings can be made from the letters in the word rat?

6

How many different strings can be made from the letters in the word egg?

3

Permutations with indistinguishable objects

How many different strings can be made from the letters in the phrase nano-nano?

Key thoughts: 8 positions, 3 kinds of letters to place.

In how many ways can we place the ns? $_8C_4$, now 4 spots are left

In how many ways can we place the as? $_4C_2$, now 2 spots are left

In how many ways can we place the os? $_2C_2$, now 0 spots are left

$$\binom{8}{4}\binom{4}{2}\binom{2}{2} = \frac{8!}{4!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{2!0!} = \frac{8!}{4!2!2!}$$

Permutations with indistinguishable objects

How many distinct permutations are there of the letters in the word APALACHICOLA?

$$\frac{12!}{4!2!2!}$$

How many if the two Ls must appear together?

$$\frac{11!}{4!2!}$$

How many if the first letter must be an A?

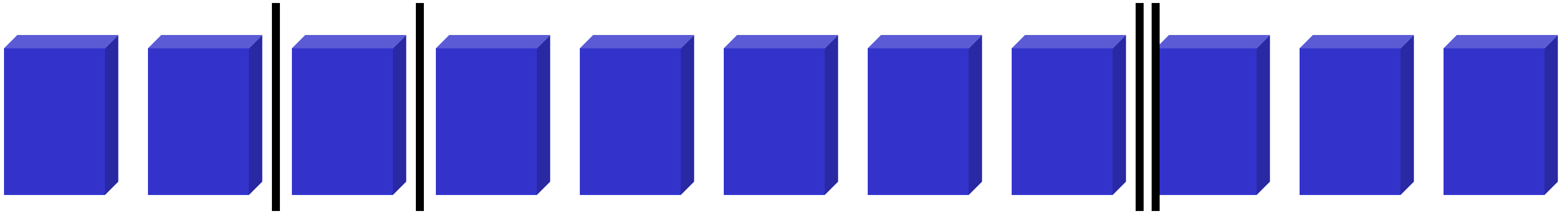
$$\frac{11!}{3!2!2!}$$

A little practice

In how many ways can 11 identical computer science books and 8 identical psychology books be distributed among 5 students?

$$\binom{15}{11} \cdot \binom{12}{8}$$

Hint: forget about the psychology books for the moment.



Hint: how can you combine your soln for the CS books with your soln for the Psych books?