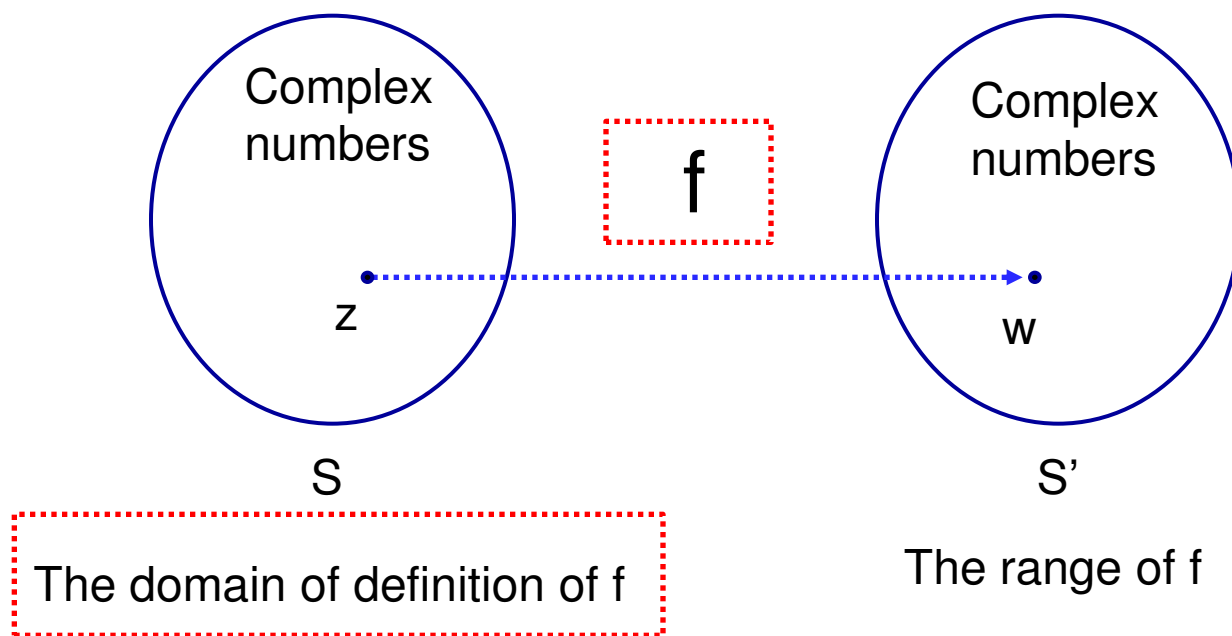

Analytic Functions

Functions of a Complex Variable

- Function of a complex variable

Let S be a set complex numbers. A function f defined on S is a rule that assigns to each z in S a complex number w .



Functions of a Complex Variable

Suppose that $w=u+iv$ is the value of a function f at $z=x+iy$, so that

$$u + iv = f(x + iy)$$

Thus each of real number u and v depends on the real variables x and y , meaning that

$$f(z) = u(x, y) + iv(x, y)$$

Similarly if the polar coordinates r and θ , instead of x and y , are used, we get

$$f(z) = u(r, \theta) + iv(r, \theta)$$

Functions of a Complex Variable

■ Example 2

If $f(z)=z^2$, then

When $v=0$, f is a real-valued function.

case #1: $z = x + iy$

$$f(z) = (x + iy)^2 = x^2 - y^2 + i2xy$$

$$\implies u(x, y) = x^2 - y^2; v(x, y) = 2xy$$

case #2: $z = re^{i\theta}$

$$f(z) = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2 \cos 2\theta + ir^2 \sin 2\theta$$

$$\implies u(r, \theta) = r^2 \cos 2\theta; v(r, \theta) = r^2 \sin 2\theta$$

Functions of a Complex Variable

- Example 3

A real-valued function is used to illustrate some important concepts later in this chapter is

$$f(z) = |z|^2 = x^2 + y^2 + i0$$

- Polynomial function

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$$

where n is zero or a positive integer and a_0, a_1, \dots, a_n are complex constants, a_n is not 0; **The domain of definition is the entire z plane**

- Rational function

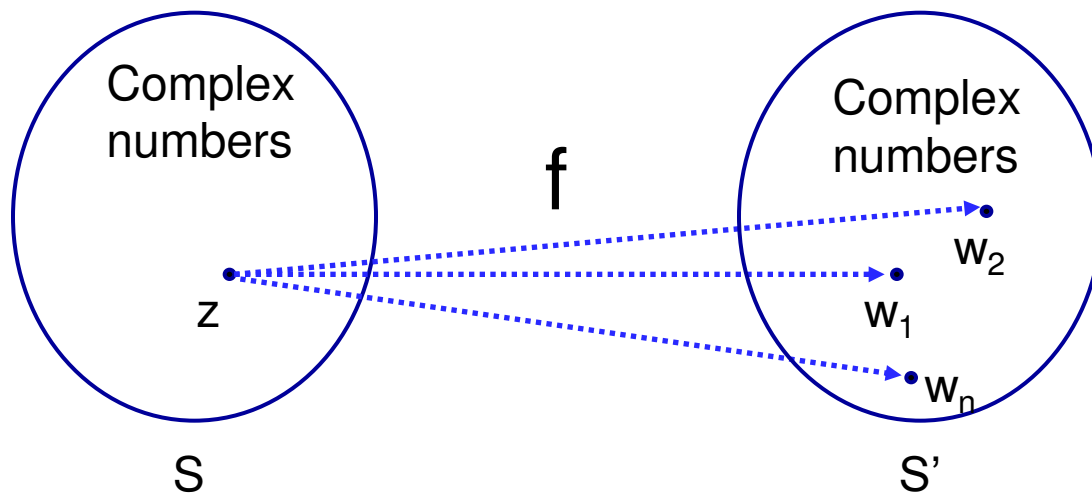
the quotients $P(z)/Q(z)$ of polynomials

The domain of definition is $Q(z) \neq 0$

Functions of a Complex Variable

- Multiple-valued function

A generalization of the concept of function is a rule that assigns more than one value to a point z in the domain of definition.



Functions of a Complex Variable

■ Example 4

Let z denote any nonzero complex number, then $z^{1/2}$ has the two values

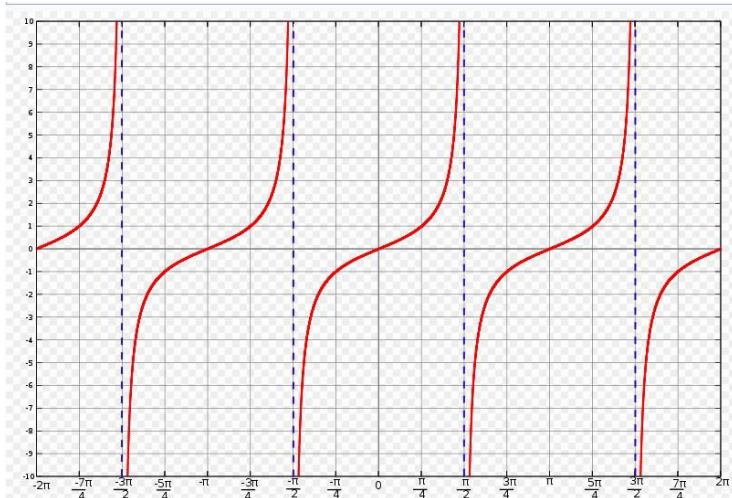
$$z^{1/2} = \pm\sqrt{r} \exp\left(i\frac{\theta}{2}\right) \quad \text{Multiple-valued function}$$

If we just choose only the positive value of $\pm\sqrt{r}$

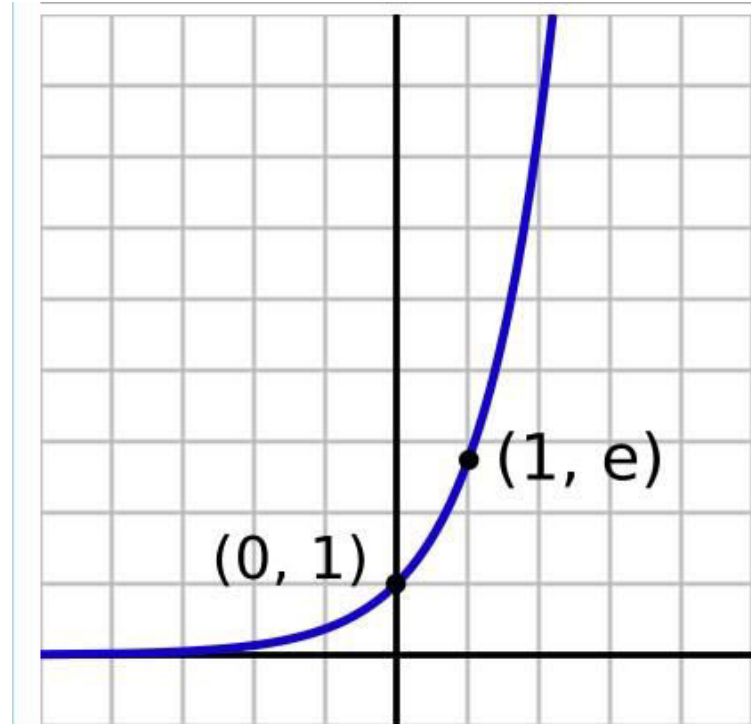
$$z^{1/2} = \sqrt{r} \exp\left(i\frac{\theta}{2}\right), r > 0 \quad \text{Single-valued function}$$

Mappings

- Graphs of Real-value functions



$$f = \tan(x)$$



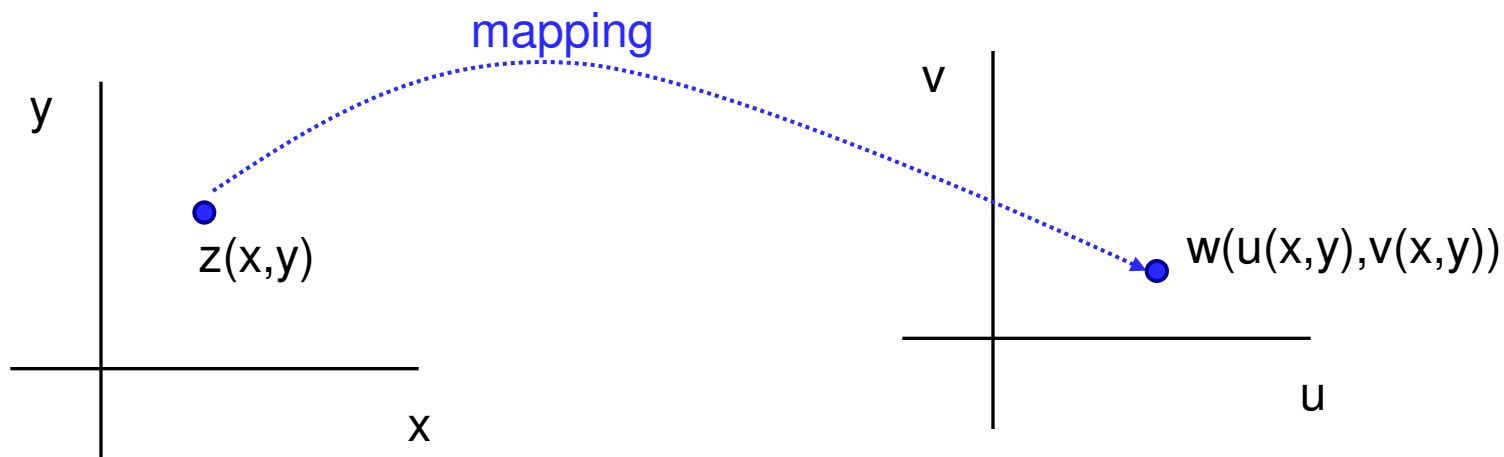
$$f = e^x$$

Note that both x and $f(x)$ are real values.

Mappings

- Complex-value functions

$$f(z) = f(x + yi) = u(x, y) + iv(x, y)$$



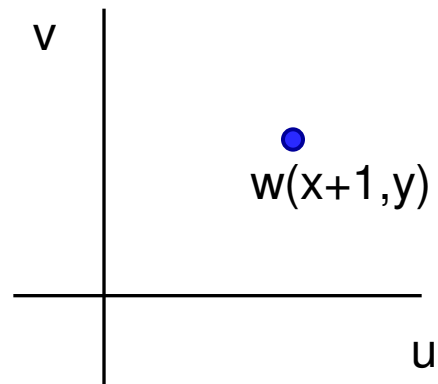
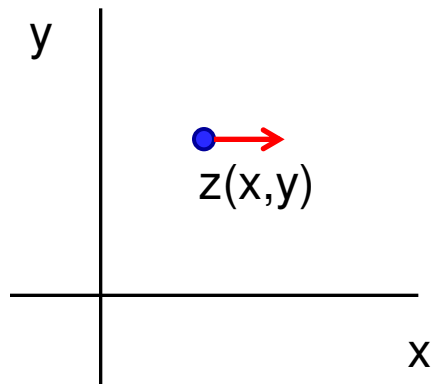
Note that here x , y , $u(x, y)$ and $v(x, y)$ are all real values.

Mappings

■ Examples

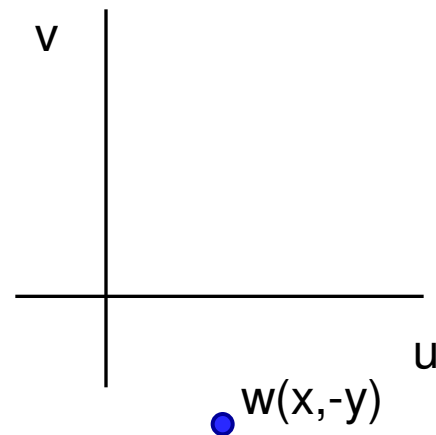
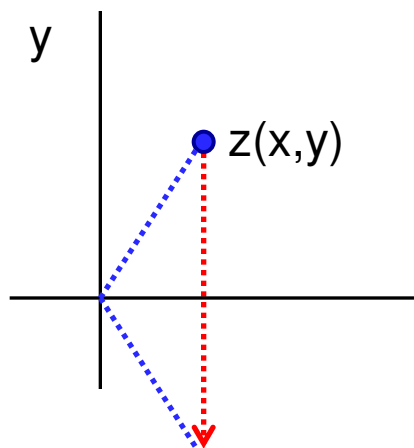
$$w = z + 1 = (x + 1) + iy$$

Translation Mapping



$$w = \bar{z} = x - yi$$

Reflection Mapping

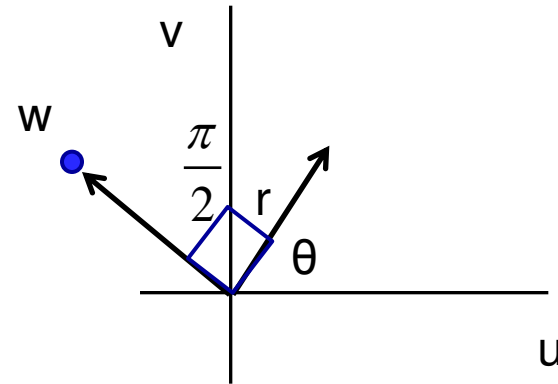
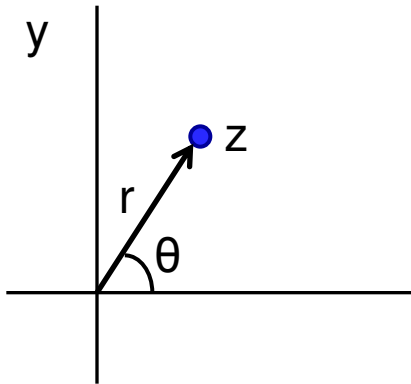


Mappings

- Example

$$w = iz = i(re^{i\theta}) = r \exp(i(\theta + \frac{\pi}{2}))$$

Rotation Mapping



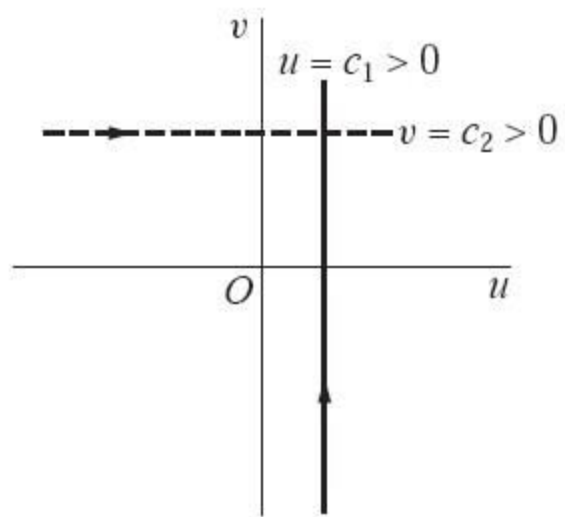
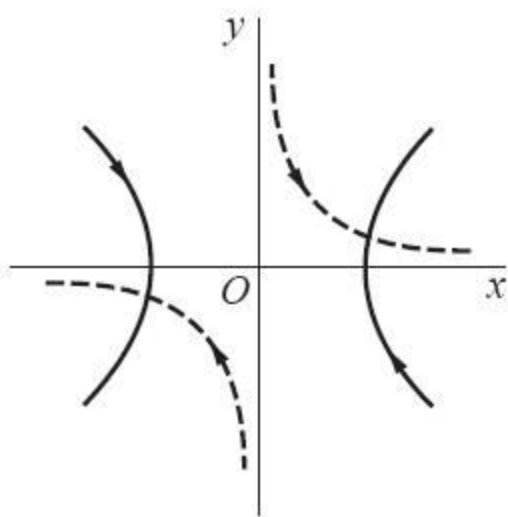
Mappings

■ Example 1

$$w = z^2 \quad u = x^2 - y^2, v = 2xy$$

Let $u=c_1>0$ in the w plane, then $x^2-y^2=c_1$ in the z plane

Let $v=c_2>0$ in the w plane, then $2xy=c_2$ in the z plane



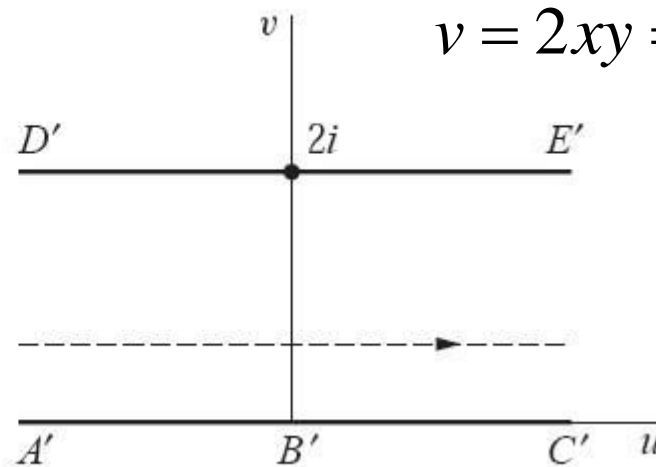
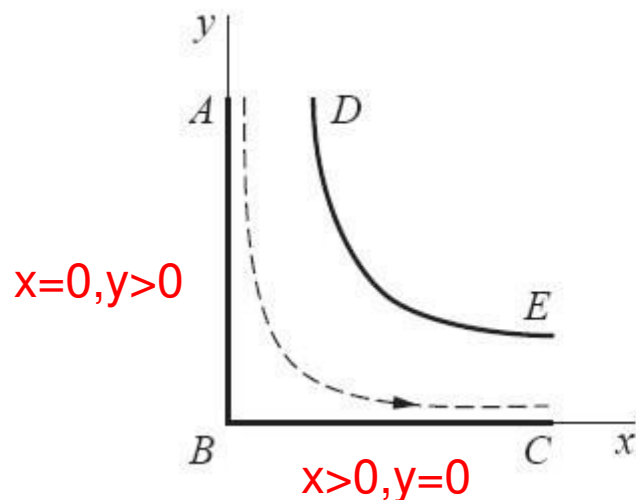
Mappings

■ Example 2

The domain $x > 0, y > 0, xy < 1$ consists of all points lying on the upper branches of hyperbolas

$$u = x^2 - y^2;$$

$$v = 2xy = 2 \Rightarrow xy = 1$$

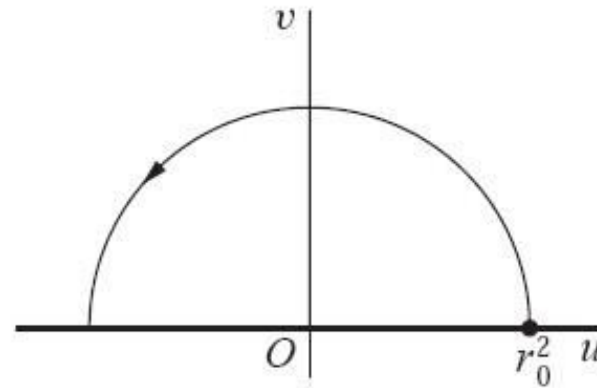
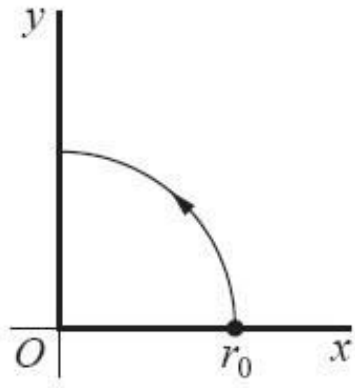


Mappings

- Example 3

$$w = z^2 = r^2 e^{i2\theta}$$

In polar coordinates

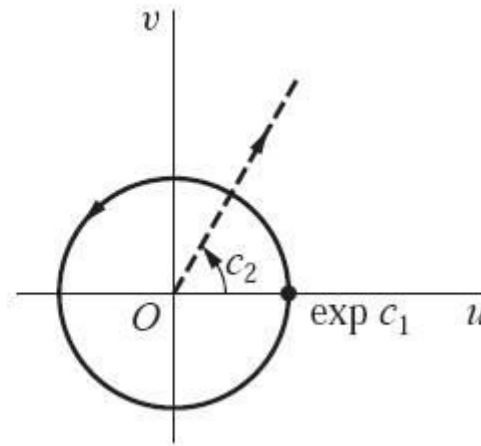
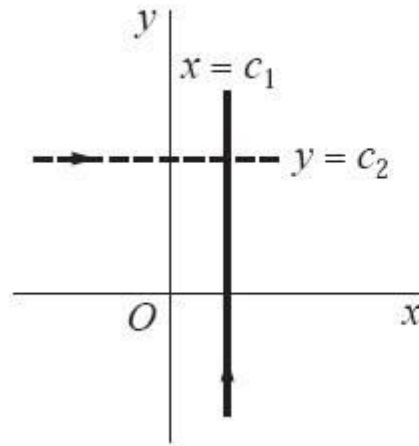


Mappings by the Exponential Function

- The exponential function

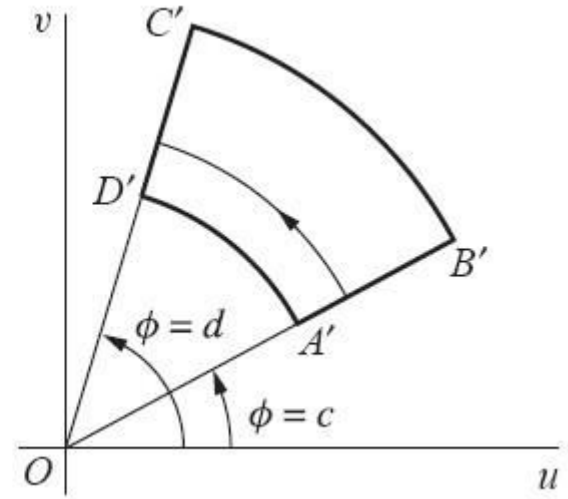
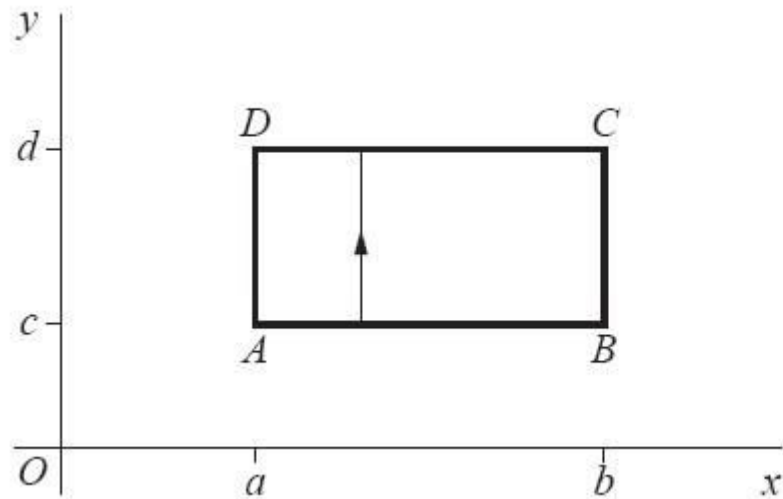
$$w = e^z = e^{x+iy} = e^x e^{iy}, z = x + iy$$

$\rho e^{i\theta} \quad \rho = e^x, \theta = y$



Mappings by the Exponential Function

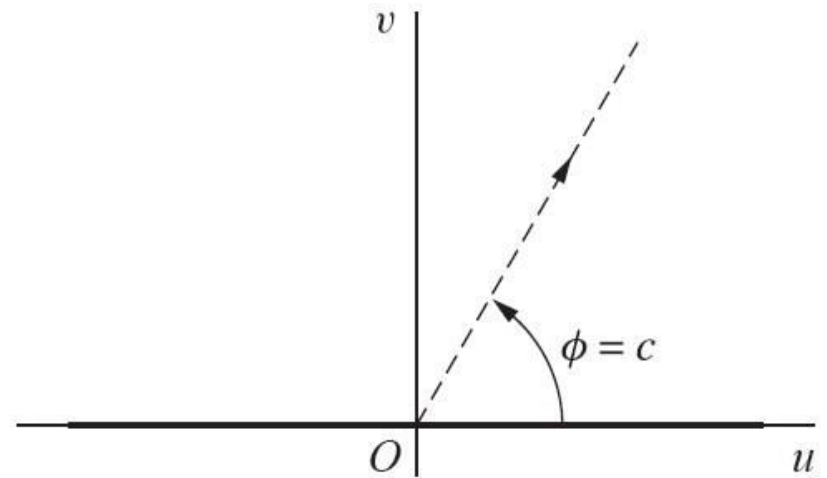
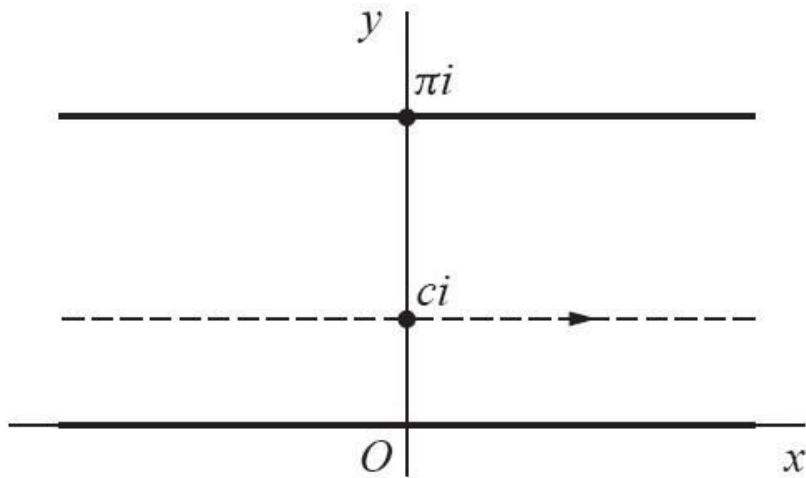
■ Example 2



$$w = \exp(z)$$

Mappings by the Exponential Function

- Example 3



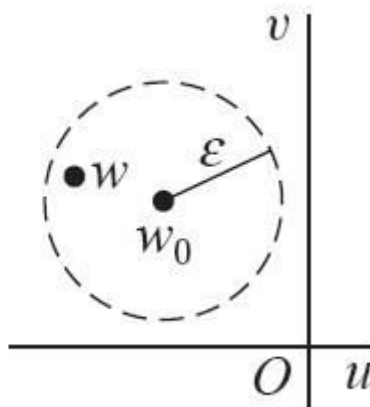
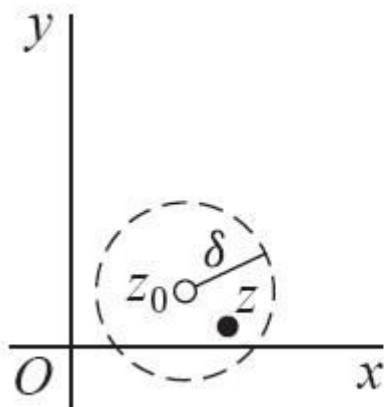
$$w = \exp(z) = e^{x+yi}$$

Limits

- For a given positive value ε , there exists a positive value δ (depends on ε) such that

when $0 < |z - z_0| < \delta$, we have $|f(z) - w_0| < \varepsilon$

meaning the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it.



$$\lim_{z \rightarrow z_0} f(z) = w_0$$

Limits

- The uniqueness of limit

If a limit of a function $f(z)$ exists at a point z_0 , it is unique.

Proof: suppose that $\lim_{z \rightarrow z_0} f(z) = w_0$ & $\lim_{z \rightarrow z_0} f(z) = w_1$

then $\forall \varepsilon / 2 > 0, \exists \delta_0 > 0, \exists \delta_1 > 0$

when $0 < |z - z_0| < \delta_0 \implies |f(z) - w_0| < \varepsilon / 2;$

$0 < |z - z_0| < \delta_1 \implies |f(z) - w_1| < \varepsilon / 2;$

Let $\delta = \min(\delta_0, \delta_1)$, when $0 < |z - z_0| < \delta$, we have

$$\implies |w_1 - w_0| = |(f(z) - w_0) - (f(z) - w_1)|$$

$$\leq |f(z) - w_0| + |f(z) - w_1| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Limits

■ Example 1

Show that $f(z) = i\bar{z}/2$ in the open disk $|z| < 1$, then

$$\lim_{z \rightarrow 1} f(z) = \frac{i}{2}$$

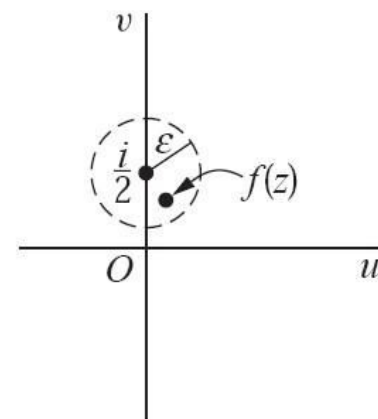
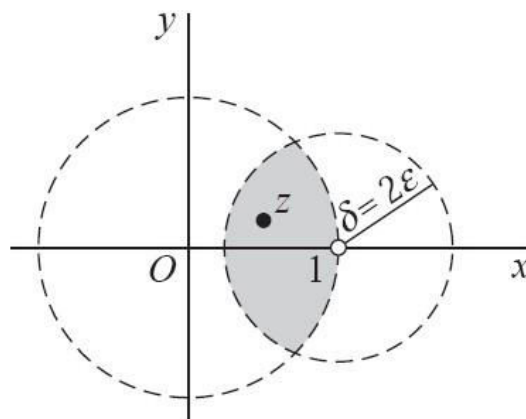
Proof:

$$|f(z) - \frac{i}{2}| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| = \frac{|i||\bar{z} - 1|}{2} = \frac{|z - 1|}{2}$$

$\forall \varepsilon > 0, \exists \delta = 2\varepsilon, s.t.$

when $0 < |z - 1| < \delta (= 2\varepsilon)$

$$\Rightarrow 0 < \frac{|z - 1|}{2} < \varepsilon \Rightarrow |f(z) - \frac{i}{2}| < \varepsilon$$



Limits

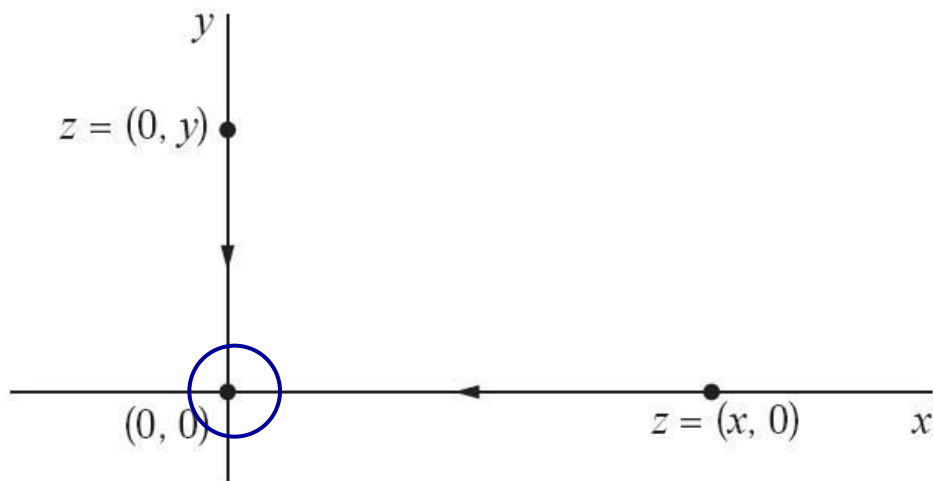
■ Example 2

If $f(z) = \frac{\bar{z}}{z}$ then the limit $\lim_{z \rightarrow 0} f(z)$ does not exist.

$$z = (x, 0) \quad \lim_{x \rightarrow 0} \frac{x + i0}{x - i0} = 1$$

\neq

$$z = (0, y) \quad \lim_{y \rightarrow 0} \frac{0 + iy}{0 - iy} = -1$$



Theorems on Limits

■ Theorem 1

Let $f(z) = u(x, y) + iv(x, y)$ $z = x + iy$

and $z_0 = x_0 + iy_0; w_0 = u_0 + iv_0$

then

$$\lim_{z \rightarrow z_0} f(z) = w_0 \quad (\text{a})$$

if and only if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0 \quad (\text{b})$$

Theorems on Limits

- Proof: (b) \rightarrow (a)

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y) = u_0 \quad \& \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y) = v_0 \quad \Rightarrow \quad \lim_{z \rightarrow z_0} f(z) = w_0$$

$\forall \varepsilon / 2 > 0, \exists \delta_1 > 0, \exists \delta_2 > 0$ s.t.

$$\text{When } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1 \quad \Rightarrow \quad |u(x,y) - u_0| < \frac{\varepsilon}{2}$$

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2 \quad \Rightarrow \quad |v(x,y) - v_0| < \frac{\varepsilon}{2}$$

Let $\delta = \min(\delta_1, \delta_2)$ When $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta, i.e. 0 < |z - z_0| < \delta$

$$|f(z) - w_0| = |(u(x,y) + iv(x,y)) - (u_0 + iv_0)| = |u(x,y) - u_0 + i(v(x,y) - v_0)|$$

$$\leq |u(x,y) - u_0| + |v(x,y) - v_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Theorems on Limits

■ Proof: (a) \rightarrow (b)

$$\lim_{z \rightarrow z_0} f(z) = w_0 \quad \Rightarrow \quad \lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y) = u_0 \quad \& \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y) = v_0$$

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \text{When } 0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z) - w_0| < \varepsilon$$

$$|f(z) - w_0| = |u(x,y) + iv(x,y) - (u_0 + iv_0)|$$

$$= |(u(x,y) - u_0) + i(v(x,y) - v_0)| < \varepsilon$$

$$|u(x,y) - u_0| \leq |(u(x,y) - u_0) + i(v(x,y) - v_0)| < \varepsilon$$

$$|v(x,y) - v_0| \leq |(u(x,y) - u_0) + i(v(x,y) - v_0)| < \varepsilon$$

$$\text{Thus } |u(x,y) - u_0| < \varepsilon; |v(x,y) - v_0| < \varepsilon$$

When $(x,y) \rightarrow (x_0,y_0)$

Theorems on Limits

■ Theorem 2

Let $\lim_{z \rightarrow z_0} f(z) = w_0$ and $\lim_{z \rightarrow z_0} F(z) = W_0$

then $\lim_{z \rightarrow z_0} [f(z) \pm F(z)] = w_0 \pm W_0$

$$\lim_{z \rightarrow z_0} [f(z)F(z)] = w_0W_0$$

$$\lim_{z \rightarrow z_0} \left[\frac{f(z)}{F(z)} \right] = \frac{w_0}{W_0}, W_0 \neq 0$$

Theorems on Limits

$$\lim_{z \rightarrow z_0} f(z) = w_0 \quad \& \quad \lim_{z \rightarrow z_0} F(z) = W_0 \quad \Longrightarrow \quad \lim_{z \rightarrow z_0} [f(z)F(z)] = w_0 W_0$$

Let $f(z) = u(x, y) + iv(x, y), F(z) = U(x, y) + iV(x, y)$

$$z_0 = x_0 + iy_0; w_0 = u_0 + iv_0; W_0 = U_0 + iV_0$$

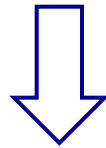
$$f(z)F(z) = (uU - vV) + i(vU + uV)$$

$$\lim_{z \rightarrow z_0} f(z) = w_0$$



When $(x, y) \rightarrow (x_0, y_0);$
 $u(x, y) \rightarrow u_0; v(x, y) \rightarrow v_0; \& U(x, y) \rightarrow U_0; V(x, y) \rightarrow V_0;$

$$\lim_{z \rightarrow z_0} F(z) = W_0$$



$$\text{Re}(f(z)F(z)): (u_0 U_0 - v_0 V_0)$$

$$\text{Im}(f(z)F(z)): (v_0 U_0 + u_0 V_0)$$

$w_0 W_0$

Theorems on Limits

It is easy to verify the limits

$$\lim_{z \rightarrow z_0} c = c$$

$$\lim_{z \rightarrow z_0} z = z_0$$

$$\lim_{z \rightarrow z_0} z^n = z_0^n \quad (n = 1, 2, \dots)$$

For the polynomial

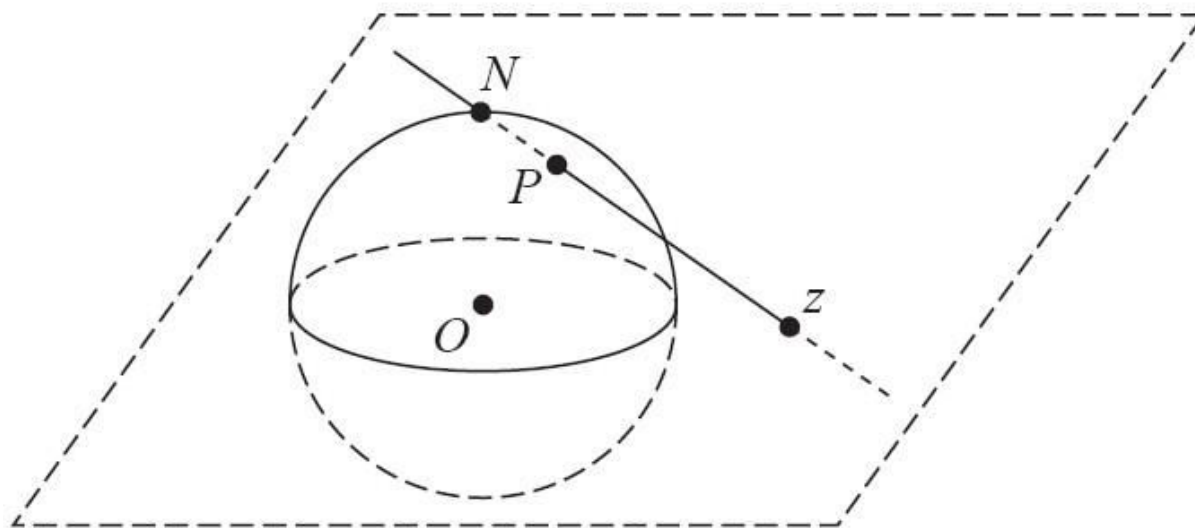
$$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

We have that

$$\lim_{z \rightarrow z_0} P(z) = P(z_0)$$

Limits Involving the Point at Infinity

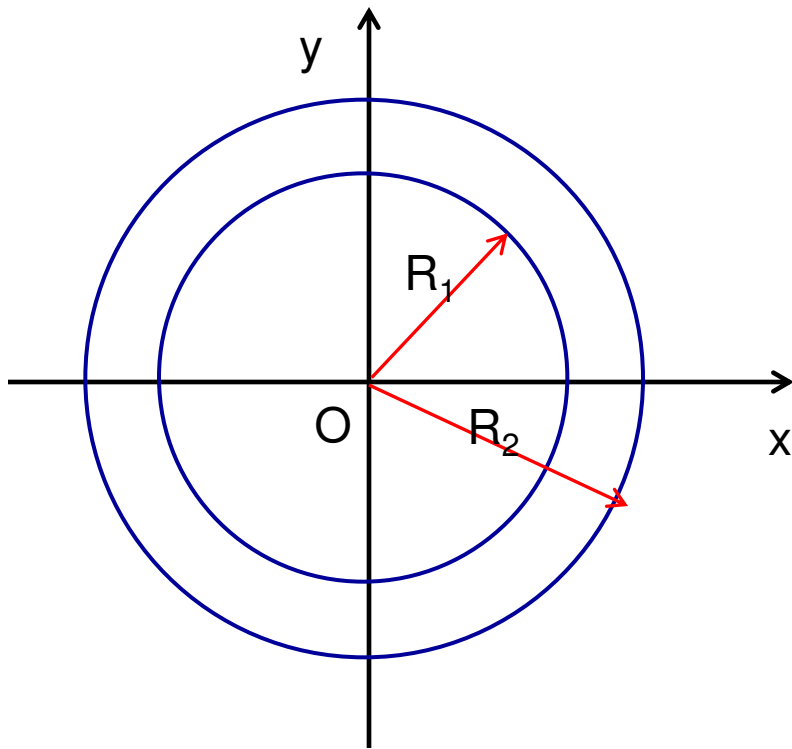
- Riemann sphere & Stereographic Projection



N: the north pole

Limits Involving the Point at Infinity

- The ε Neighborhood of Infinity



When the radius R is large enough

i.e. for each small positive number ε

$$R=1/\varepsilon$$

The region of $|z|>R=1/\varepsilon$ is called the ε Neighborhood of Infinity(∞)

Limits Involving the Point at Infinity

■ Theorem

If z_0 and w_0 are points in the z and w planes, respectively, then

$$\lim_{z \rightarrow z_0} f(z) = \infty \quad \text{iff} \quad \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0$$

$$\lim_{z \rightarrow \infty} f(z) = w_0 \quad \text{iff} \quad \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0$$

$$\lim_{z \rightarrow \infty} f(z) = \infty \quad \text{iff} \quad \lim_{z \rightarrow 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$$

Limits Involving the Point at Infinity

- Examples

$$\lim_{z \rightarrow -1} \frac{iz + 3}{z + 1} = \infty \quad \text{since} \quad \lim_{z \rightarrow -1} \frac{z + 1}{iz + 3} = 0$$

$$\lim_{z \rightarrow \infty} \frac{2z + i}{z + 1} = 2 \quad \text{since} \quad \lim_{z \rightarrow 0} \frac{(2/z) + i}{(1/z) + 1} = \lim_{z \rightarrow 0} \frac{2 + iz}{1 + z} = 2.$$

$$\lim_{z \rightarrow \infty} \frac{2z^3 - 1}{z^2 + 1} = \infty \quad \text{since} \quad \lim_{z \rightarrow 0} \frac{(1/z^2) + 1}{(2/z^3) - 1} = \lim_{z \rightarrow 0} \frac{z + z^3}{2 - z^3} = 0.$$

Continuity

■ Continuity

A function is continuous at a point z_0 if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

meaning that

1. the function f has a limit at point z_0 and
2. the limit is equal to the value of $f(z_0)$

For a given positive number ε , there exists a positive number δ , s.t.

$$\text{When } |z - z_0| < \delta \quad |f(z) - f(z_0)| < \varepsilon$$

$$0 < |z - z_0| < \delta ?$$

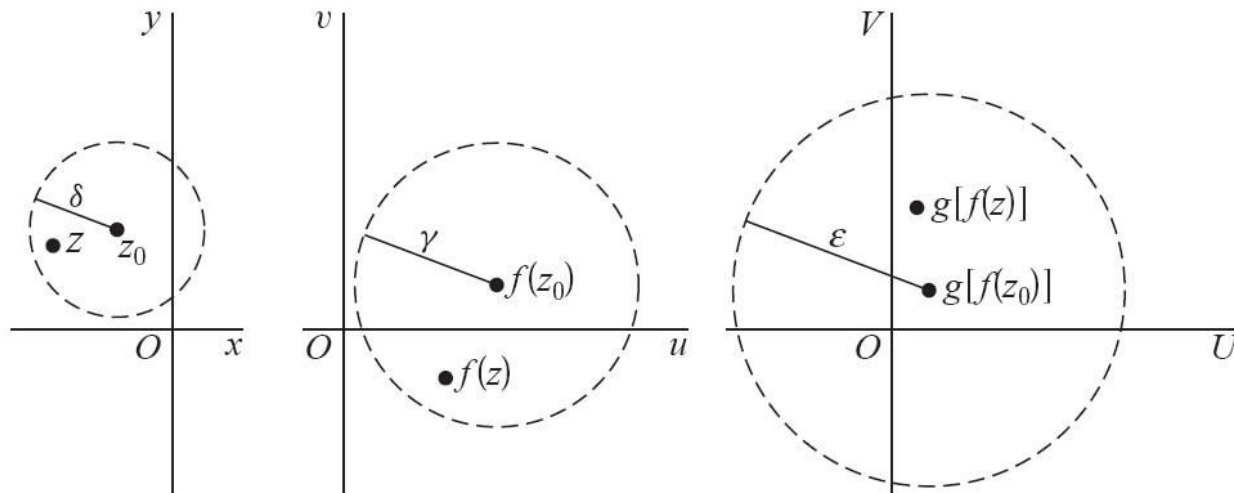
Continuity

■ Theorem 1

A composition of continuous functions is itself continuous.

Suppose $w=f(z)$ is a continuous at the point z_0 ;
 $g=g(f(z))$ is continuous at the point $f(z_0)$

Then the composition $g(f(z))$ is continuous at the point z_0



Continuity

■ Theorem 2

If a function $f(z)$ is continuous and nonzero at a point z_0 , then $f(z) \neq 0$ throughout some neighborhood of that point.

Proof $\lim_{z \rightarrow z_0} f(z) = f(z_0) \neq 0$

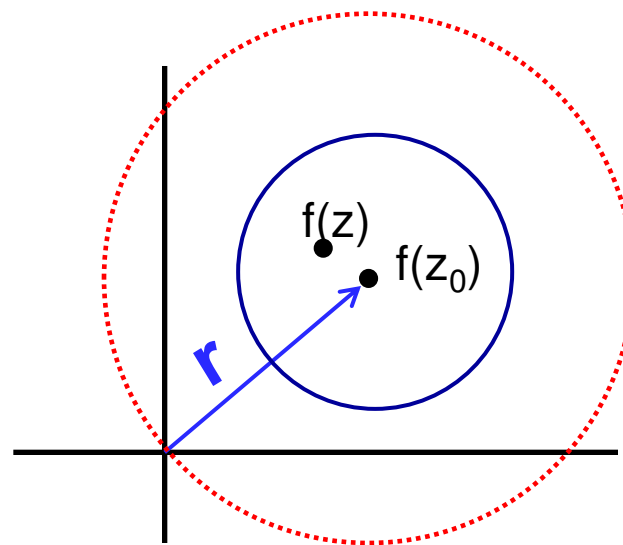
$$\forall \varepsilon = \frac{|f(z_0)|}{2} > 0, \exists \delta > 0, \text{ s.t.}$$

When $|z - z_0| < \delta$

$$|f(z) - f(z_0)| < \varepsilon = \frac{|f(z_0)|}{2}$$

If $f(z) = 0$, then $|f(z_0)| < \frac{|f(z_0)|}{2}$

Contradiction!



$$\forall \varepsilon \leq |f(z_0)|$$

Continuity

■ Theorem 3

If a function f is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M such that

$$|f(z)| \leq M \quad \text{for all points } z \text{ in } R$$

where equality holds for at least one such z .

Note: $|f(z)| = \sqrt{u^2(x, y) + v^2(x, y)}$

where $u(x, y)$ and $v(x, y)$ are continuous real functions

Derivatives

- Derivative

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \varepsilon$ of a point z_0 . The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

And the function f is said to be differentiable at z_0 when $f'(z_0)$ exists.

Derivatives

■ Illustration of Derivative

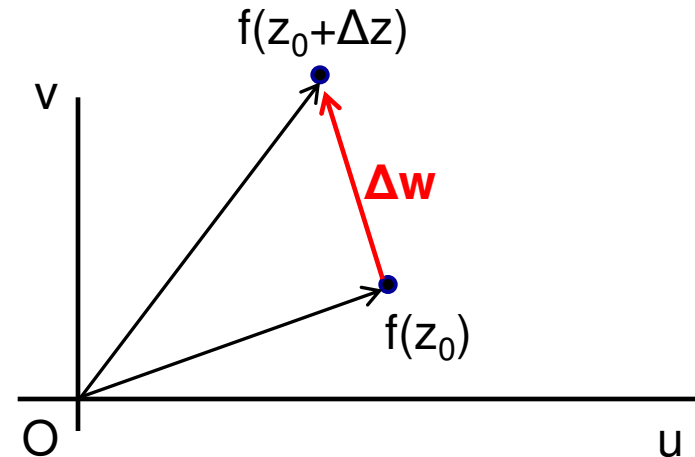
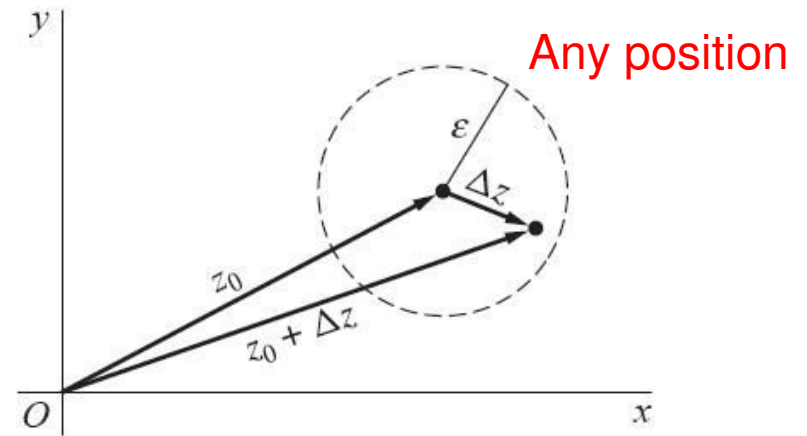
$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$z = z_0 + \Delta z$$

$$\Delta w = f(z_0 + \Delta z) - f(z_0)$$

$$\frac{dw}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z}$$



Derivatives

- Example 1

Suppose that $f(z)=z^2$. At any point z

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \rightarrow 0} (2z + \Delta z) = 2z$$

since $2z + \Delta z$ is a polynomial in Δz . Hence $dw/dz=2z$ or $f'(z)=2z$.

Derivatives

■ Example 2

$$\text{If } f(z) = \bar{z}, \text{ then } \frac{\Delta w}{\Delta z} = \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} = \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

$\Delta z = (\Delta x, \Delta y) \rightarrow (0, 0)$ In any direction

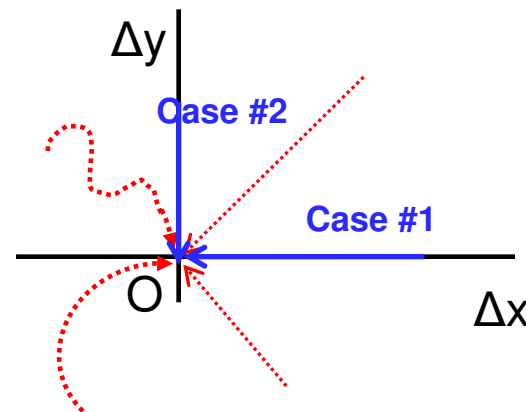
Case #1: $\Delta x \rightarrow 0, \Delta y = 0$

$$\lim_{\Delta x \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i0}{\Delta x + i0} = 1$$

Case #2: $\Delta x = 0, \Delta y \rightarrow 0$

$$\lim_{\Delta y \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} = \frac{0 - i\Delta y}{0 + i\Delta y} = -1$$

Since the limit is not unique, this function does not exist anywhere



Derivatives

■ Example 3

Consider the real-valued function $f(z)=|z|^2$. Here

$$\frac{\Delta w}{\Delta z} = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z} = \bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z}$$

Case #1: $\Delta x \rightarrow 0, \Delta y = 0$

$$\lim_{\Delta x \rightarrow 0} (\bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z}) = \lim_{\Delta x \rightarrow 0} (\bar{z} + \Delta x + z \frac{\Delta x - i0}{\Delta x + i0}) = \bar{z} + z$$

Case #2: $\Delta x = 0, \Delta y \rightarrow 0$

$$\lim_{\Delta y \rightarrow 0} (\bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z}) = \lim_{\Delta y \rightarrow 0} (\bar{z} - i\Delta y + z \frac{0 - i\Delta y}{0 + i\Delta y}) = \bar{z} - z$$

$$\bar{z} + z = \bar{z} - z \implies z = 0$$

dw/dz can not exist when z is not 0

Derivatives

■ Continuity & Derivative

Continuity $\not\Rightarrow$ Derivative

For instance,

$f(z)=|z|^2$ is continuous at each point, however, dw/dz does not exist when z is not 0

Derivative \Rightarrow Continuity

$$\lim_{z \rightarrow z_0} [f(z) - f(z_0)] = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \rightarrow z_0} (z - z_0) = f'(z_0)0 = 0$$

Note: The existence of the derivative of a function at a point implies the continuity of the function at that point.

Differentiation Formulas

■ Differentiation Formulas

$$\frac{d}{dz} c = 0; \frac{d}{dz} z = 1; \frac{d}{dz} [cf(z)] = cf'(z)$$

$$\frac{d}{dz} [z^n] = nz^{n-1} \quad \text{Refer to pp.7 (13)}$$

$$\frac{d}{dz} [f(z) \pm g(z)] = f'(z) \pm g'(z)$$

$$\frac{d}{dz} [f(z) \cdot g(z)] = f(z) \cdot g'(z) + f'(z) \cdot g(z)$$

$$\frac{d}{dz} \left[\frac{f(z)}{g(z)} \right] = \frac{f'(z) \cdot g(z) - f(z) \cdot g'(z)}{[g(z)]^2}$$

$$F(z) = g(f(z))$$

$$F'(z_0) = g'(f(z_0))f'(z_0)$$

$$\frac{dW}{dz} = \frac{dW}{dw} \frac{dw}{dz}$$

Differentiation Formulas

- Example

To find the derivative of $(2z^2+i)^5$, write $w=2z^2+i$ and $W=w^5$. Then

$$\frac{d}{dz}(2z^2+i)^5 = (5w^4)w' = 5(2z^2+i)^4(4z) = 20z(2z^2+i)^4$$

Analytic Function

- Analytic at a point z_0

A function f of the complex variable z is analytic at a point z_0 if it has a derivative at each point in some neighborhood of z_0 .

Note that if f is analytic at a point z_0 , it must be analytic at each point in some neighborhood of z_0

- Analytic function

A function f is analytic in an open set if it has a derivative everywhere in that set.

Note that if f is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S .

Analytic Function

- Analytic vs. Derivative

- For a point

Analytic \rightarrow Derivative ✓

Derivative \rightarrow Analytic ✗

- For all points in an open set

Analytic \rightarrow Derivative ✓

Derivative \rightarrow Analytic ✓

f is analytic in an open set D iff f is derivative in D

Analytic Function

- Singular point (singularity)

If function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0 , then z_0 is called a singular point.

For instance, the function $f(z)=1/z$ is analytic at every point in the finite plane except for the point of $(0,0)$. Thus $(0,0)$ is the singular point of function $1/z$.

- Entire Function

An entire function is a function that is analytic at each point in the entire finite plane.

For instance, the polynomial is entire function.

Analytic Function

■ Property 1

If two functions are analytic in a domain D , then

- their sum and product are both analytic in D
- their quotient is analytic in D provided the function in the denominator does not vanish at any point in D

■ Property 2

From the chain rule for the derivative of a composite function, a composition of two analytic functions is analytic.

$$\frac{d}{dz} g(f(z)) = g'[f(z)]f'(z)$$

Analytic Function

■ Theorem

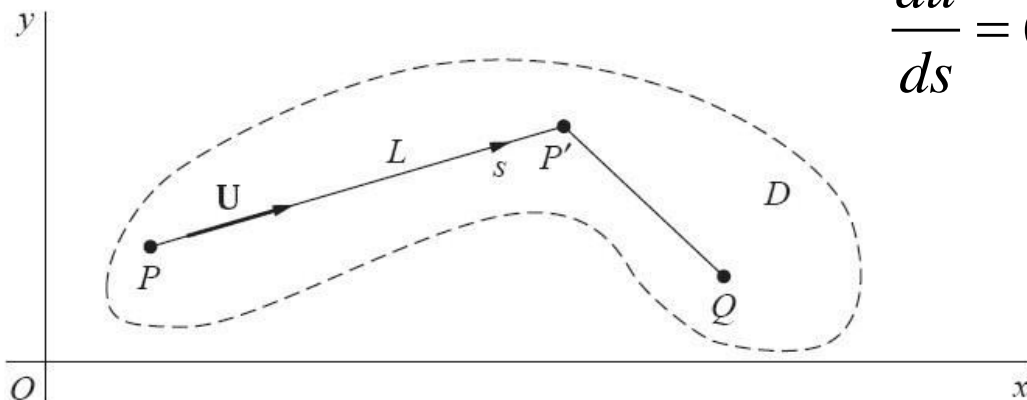
If $f'(z) = 0$ everywhere in a domain D , then $f(z)$ must be constant throughout D .

$$f'(z) = u_x + iv_x = v_y - iu_y = 0$$

$$u_x = u_y = 0 \text{ \& } v_x = v_y = 0$$

$$\frac{du}{ds} = (\text{gradu}) \cdot U \quad \text{gradu} = u_x i + u_y j$$

U is the unit vector along L



Example z^2 is Analytic

$$z = x + iy$$

$$f(z) = z^2 = x^2 - y^2 + 2i xy = u + i v$$

$$\begin{aligned} \rightarrow \quad u = x^2 - y^2 & \quad \rightarrow \quad \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \\ v = 2xy & \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} \end{aligned}$$

$\therefore f'$ exists & single-valued \forall finite z .

i.e., z^2 is an entire function.

Example: z^* is Not Analytic

$$z = x + iy$$

$$f(z) = z^* = x - iy = u + iv$$

$$\rightarrow \begin{array}{l} u = x \\ v = -y \end{array} \rightarrow \begin{array}{l} \frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = 0 \neq -\frac{\partial v}{\partial x} \end{array}$$

$\therefore f'$ doesn't exist $\forall z$, even though it is continuous every where.

i.e., z^* is nowhere analytic.

Examples

■ Example

Suppose that a function $f(z) = u(x, y) + iv(x, y)$ and its conjugate $\bar{f}(z) = u(x, y) - iv(x, y)$ are both analytic in a given domain D . Show that $f(z)$ must be constant throughout D .

Proof: $f(z) = u(x, y) + iv(x, y)$ is analytic, then $u_x = v_y, u_y = -v_x$

$\bar{f}(z) = u(x, y) - iv(x, y)$ is analytic, then $u_x = -v_y, u_y = v_x$

$$\begin{array}{ccc} \Rightarrow & u_x = 0, v_x = 0 & \Rightarrow & f'(z) = u_x + iv_x = 0 \end{array}$$

Based on the Theorem in pp. 74, we have that f is constant throughout D

Examples

■ Example

Suppose that f is analytic throughout a given region D , and the modulus $|f(z)|$ is constant throughout D , then the function $f(z)$ must be constant there too.

Proof:

$$|f(z)| = c, \quad \text{for all } z \text{ in } D$$

where c is real constant.

If $c=0$, then $f(z)=0$ everywhere in D .

If $c \neq 0$, we have

$$f(z)\overline{f(z)} = c^2 \quad \Longrightarrow \quad \overline{f(z)} = \frac{c^2}{f(z)}, \quad f(z) \neq 0 \text{ in } D$$

Both f and its conjugate are analytic, thus f must be constant in D . (Refer to Ex. 3)

Uniquely Determined Analytic Function

■ Lemma

Suppose that

- a) A function f is analytic throughout a domain D ;
- b) $f(z)=0$ at each point z of a domain or line segment contained in D .

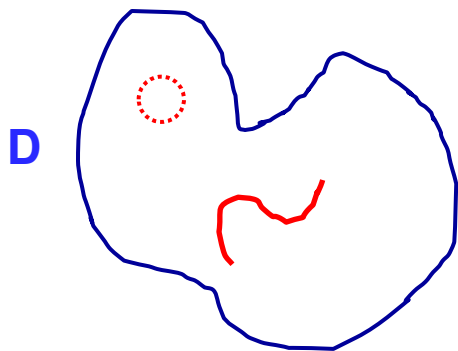
Then $f(z) \equiv 0$ in D ; that is, $f(z)$ is identically equal to zero throughout D .

Refer to Chap. 6 for the proof.

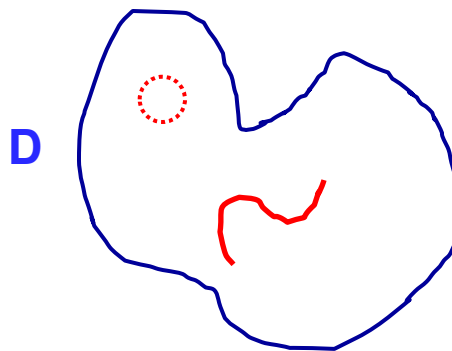
Uniquely Determined Analytic Function

- Theorem

A function that is analytic in a domain D is uniquely determined over D by its values in a domain, or along a line segment, contained in D .



$f(z)$



$g(z)$

$$f(z) \equiv g(z)$$

Reflection Principle

■ Theorem

Suppose that a function f is analytic in some domain D which contains a segment of the x axis and whose lower half is the reflection of the upper half with respect to that axis. Then

$$\overline{f(z)} = f(\bar{z})$$

for each point z in the domain if and only if $f(x)$ is real for each point x on the segment.

