

# Cauchy-Riemann Equation

# Functions of a complex variable

Let  $S$  be a set of complex numbers.

A function defined on  $S$  is a rule that assigns to each  $z$  in  $S$  a complex number  $w$ .

value of  $f$  at  $z$ , or  $f(z)$

or

$$w = f(z)$$

$S$  is the domain of definition of  $f$

$w = \frac{1}{z}$  sometimes refer to the function  $f$  itself, for simplicity.

$$w = z^2 + 1$$

Both a domain of definition and a rule are needed in order for a function to be well defined.

Suppose  $w = u + iv$  is the value of a function at  $f$   $z = x + iy$

$$u + iv = f(x + iy)$$

$$\text{or } f(z) = u(x, y) + iv(x, y)$$

  
real-valued functions of real variables  $x, y$

$$\text{or } f(z) = u(r, \theta) + iv(r, \theta)$$

Ex.

$$f(z) = z^2$$

$$f(x + iy) = x^2 - y^2 + i2xy$$

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy$$

$$f(re^{i\theta}) = r^2 \cos 2\theta + ir^2 \sin 2\theta$$

$$u(r, \theta) = r^2 \cos 2\theta \quad v(r, \theta) = r^2 \sin 2\theta$$

when  $v=0$

$f(z)$  is a real-valued function of a complex variable.

$f(z) = P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$  is a polynomial of degree  $n$ .

$\frac{P(z)}{Q(z)}$  : rational function, defined when  $Q(z) \neq 0$

For multiple-valued functions : usually assign one to get single-valued function

Ex.  $z = re^{i\theta}, \quad z \neq 0$

$z^{\frac{1}{2}} = \pm \sqrt{r}e^{i\theta/2}, \quad -\pi < \theta \leq \pi$      $n$ th root

If we choose  $f(z) = \sqrt{r}e^{i\theta/2} \quad (r > 0, -\pi < \theta < \pi)$

$$-\frac{\pi}{2} < \frac{\theta}{2} < \frac{\pi}{2}$$

and  $f(0) = 0$ ,

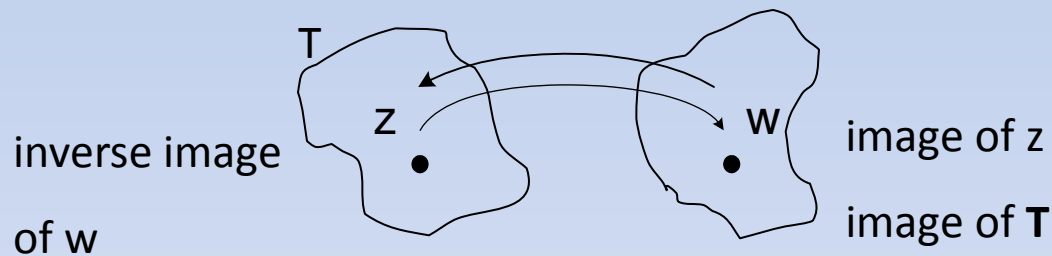
then  $f$  is well defined on the entire complex plane except the ray  $\theta = \pi$ .

## Mappings

$w=f(z)$  is not easy to graph as real functions are.

One can display some information about the function by indicating pairs of corresponding points  $z=(x,y)$  and  $w=(u,v)$ . (draw  $z$  and  $w$  planes separately).

When a function  $f$  is thought of in this way, it is often referred to as a mapping, or transformation.



Mapping can be translation, rotation, reflection. In such cases it is convenient to consider  $z$  and  $w$  planes to be the same.

$w=z+1$  translation  $+1$

$w=iz$  rotation  $\frac{\theta}{2}$

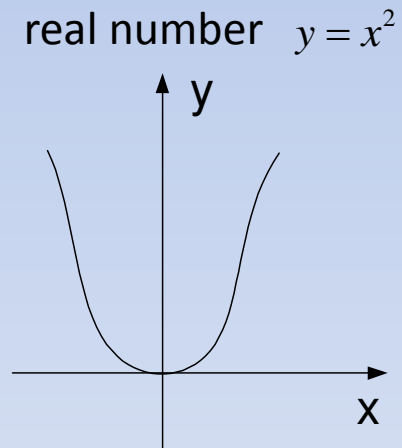
$w = \bar{z}$  reflection in real axis.

Ex. image of curves

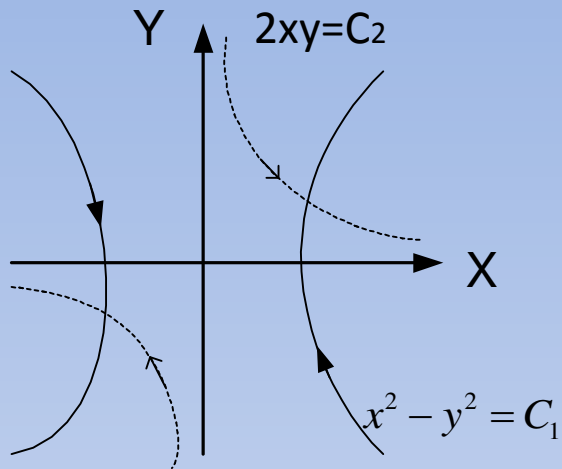
$$w = z^2$$

$$u = x^2 - y^2$$

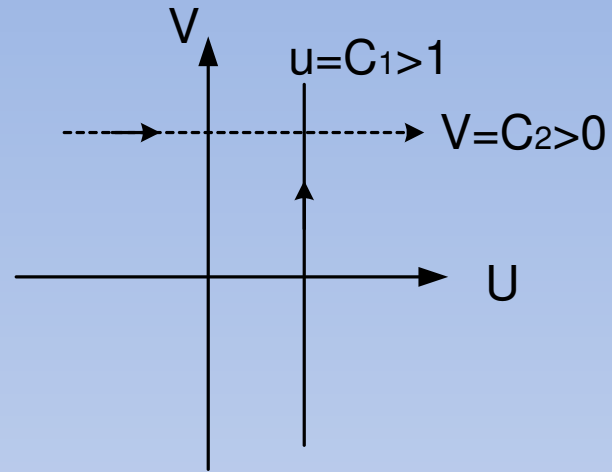
$$v = 2xy$$



a hyperbola  $x^2 - y^2 = c_1$  is mapped in a one to one manner onto the line  $u = c_1$



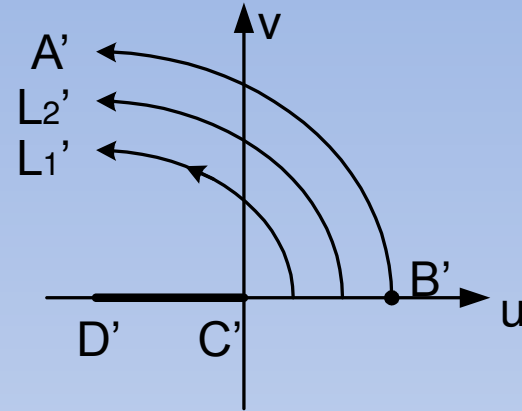
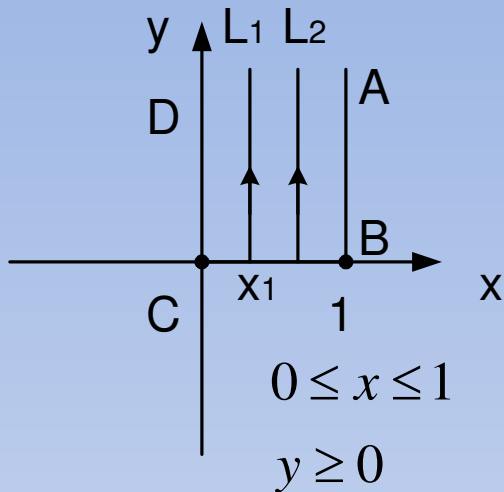
right hand branch  $x > 0$ ,  
 left hand branch  $x < 0$



image

$$\begin{aligned}
 u=C_1, & \quad V = 2y\sqrt{y^2 + c_1} & (-\infty < y < \infty) \\
 u=C_1, & \quad V = -2y\sqrt{y^2 + c_1} & (-\infty < y < \infty)
 \end{aligned}$$

Ex 2.



When  $0 < x_1 < 1$ , point  $(x_1, y)$  moves up a vertical half line,  $L_1$ , as  $y$  increases from  $y = 0$ .

$$u = x^2 - y^2, \quad v = 2x_1 y$$

$$y = \frac{v}{2x_1}$$

$$u = x_1^2 - \left(\frac{v}{2x_1}\right)^2, \quad v^2 = -4x_1^2(u - x_1^2) \quad \leftarrow \text{ a parabola with vertex at } (x_1^2, 0)$$

half line CD is mapped of half line C'D'

$$(0, y)$$

$$(-y^2, 0)$$

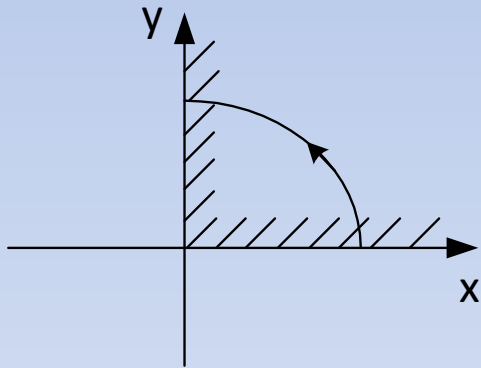


Ex 3.

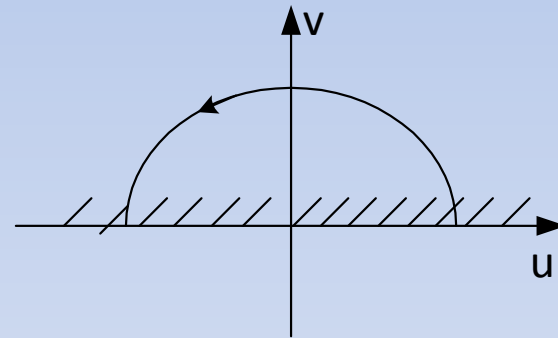
$$w = z^2 = r^2 e^{i2\theta}$$

$$\text{let } w = \rho e^{i\phi}$$

$$\rho = r^2, \quad \phi = 2\theta + 2n\pi \quad (n = 0, \pm 1, \pm 2, \dots)$$



$$r \geq 0, \quad 0 \leq \theta \leq \pi/2$$



one to one  $\longrightarrow$   $\rho \geq 0, \quad 0 \leq \phi \leq \pi$

# Limits

Let a function  $f$  be defined at all points  $z$  in some deleted neighborhood of  $z_0$

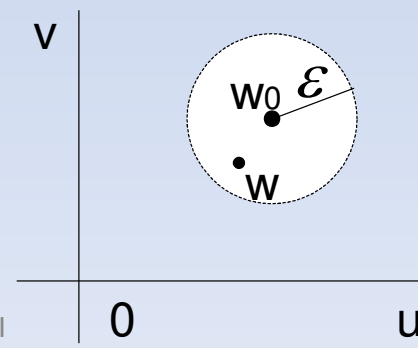
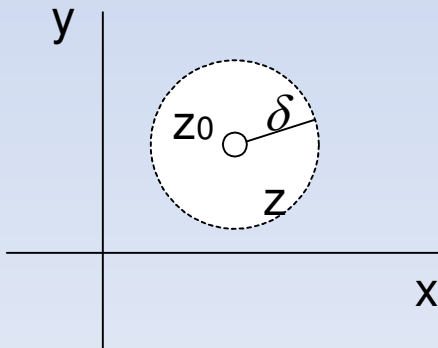
$$\lim_{z \rightarrow z_0} f(z) = w_0 \quad (1)$$

means: the limit of  $f(z)$  as  $z$  approaches  $z_0$  is  $w_0$

$w = f(z)$  can be made arbitrarily close to  $w_0$  if we choose the point  $z$  close enough to  $z_0$  but distinct from it.

(1) means that, for each positive number  $\epsilon$ , there is a positive number  $\delta$  such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta \quad (2)$$



Note:

(2) requires that  $f$  be defined at all points in some deleted neighborhood of  $z_0$

such a deleted neighborhood always exists when  $z_0$  is an interior point of a region on which  $f$  is defined. We can extend the definition of limit to the case in which  $z_0$  is a boundary point of the region by agreeing that left of (2) be satisfied by only those points  $z$  that lie in both the region and the domain

$0 < |z - z_0| < \delta$   
Example 1. show if

$$f(z) = \frac{iz}{2} \quad \text{in } |z| < 1, \quad \text{then}$$

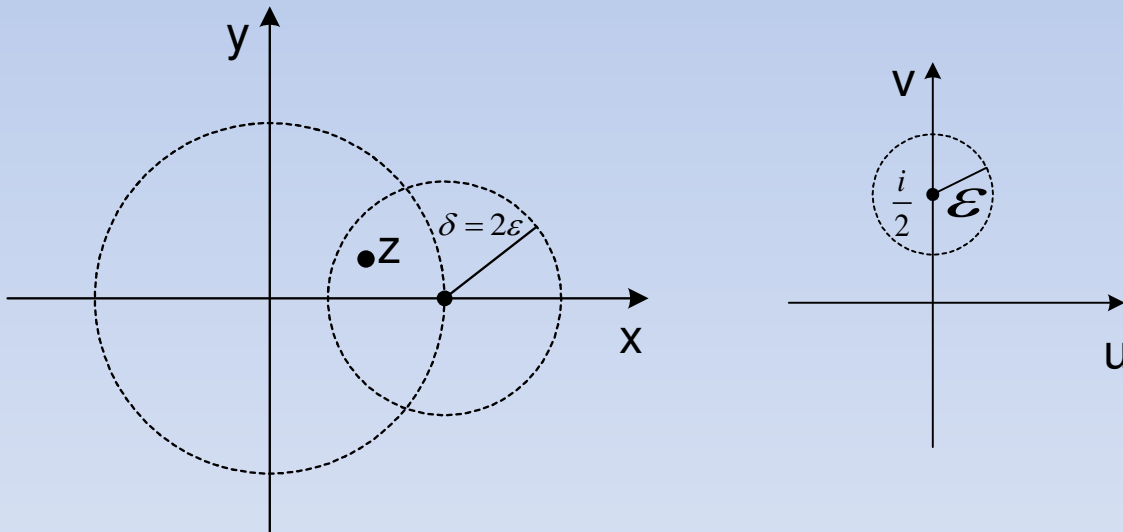
$$\lim_{z \rightarrow 1} f(z) = \frac{i}{2}$$

when  $z$  in  $|z| < 1$

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \frac{|z-1|}{2}$$

For any such  $z$  and any positive number  $\varepsilon$

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \quad \text{whenever} \quad 0 < |z-1| < 2\varepsilon$$



When a limit of a function  $f(z)$  exists at a point  $z_0$ , it is unique.

If not, suppose  $\lim_{z \rightarrow z_0} f(z) = w_0$  and  $\lim_{z \rightarrow z_0} f(z) = w_1$

Then  $|f(z) - w_0| < \varepsilon$  whenever  $0 < |z - z_0| < \delta_0$

$|f(z) - w_1| < \varepsilon$  whenever  $0 < |z - z_0| < \delta_1$

Let  $\delta = \min(\delta_0, \delta_1)$

if  $0 < |z - z_0| < \delta$

$$|[f(z) - w_0] - [f(z) - w_1]| \leq |f(z) - w_0| + |f(z) - w_1| < 2\varepsilon$$

$$|w_1 - w_0| < 2\varepsilon$$

Hence  $|w_1 - w_0|$  is a nonnegative constant, and can be chosen arbitrarily small.

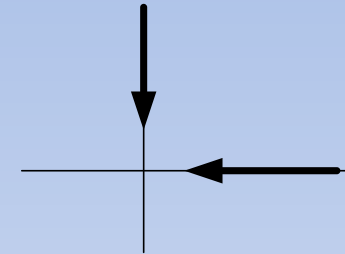
$$w_1 - w_0 = 0, \quad \text{or} \quad w_1 = w_0$$

Ex 2. If  $f(z) = \frac{z}{z}$  (4)  
 then  $\lim_{z \rightarrow 0} f(z)$  does not exist.

$$\lim_{z \rightarrow 0} f(z)$$

show: when  $z = (x, 0)$   $f(z) = \frac{x+i0}{x-i0} = 1$

when  $z = (0, y)$   $f(z) = \frac{0+iy}{0-iy} = -1$



since a limit is unique, limit of (4) does not exist.

(2) provides a means of testing whether a given point  $w_0$  is a limit, it does not directly provide a method for determining that limit.

# Theorems on limits

Thm 1. Suppose that

$$f(z) = u(x, y) + iv(x, y), \quad z_0 = x_0 + iy_0$$

*and*  $w_0 = u_0 + iv_0$

Then  $\lim_{z \rightarrow z_0} f(z) = w_0$  iff

$$\lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = v_0$$

pf: "  $\Leftarrow$   $|u - u_0| < \frac{\varepsilon}{2}$  whenever  $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1$

$|v - v_0| < \frac{\varepsilon}{2}$  whenever  $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2$

let  $\delta = \min(\delta_1, \delta_2)$

since

$$|(u + iv) - (u_0 + iv_0)| = |(u - u_0) + i(v - v_0)| \leq |u - u_0| + |v - v_0|$$

and

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = |(x - x_0) + i(y - y_0)| = |(x + iy) - (x_0 + iy_0)|$$

$$\therefore |(u + iv) - (u_0 + iv_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\text{whenever } 0 < |(x + iy) - (x_0 + iy_0)| < \delta$$

“ ”

$\Rightarrow$

$$\text{But } |(u + iv) - (u_0 + iv_0)| < \varepsilon \quad \text{whenever } 0 < |(x + iy) - (x_0 + iy_0)| < \delta$$

$$|u - u_0| \leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)| < \varepsilon$$

$$\text{and } |v - v_0| \leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)| < \varepsilon$$

$$|(x + iy) - (x_0 + iy_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$$\therefore |u - u_0| < \varepsilon \quad \text{and} \quad |v - v_0| < \varepsilon$$

$$\text{whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$



Thm 2. suppose that

$$\lim_{z \rightarrow z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \rightarrow z_0} F(z) = W_0 \quad (7)$$

Then  $\lim_{z \rightarrow z_0} [f(z) + F(z)] = w_0 + W_0$

$$\lim_{z \rightarrow z_0} [f(z) \cdot F(z)] = w_0 W_0 \quad (9)$$

and if  $W_0 \neq 0$

$$\lim_{z \rightarrow z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}$$

pf: utilize Thm 1.

for (9).  $f(z) = u(x, y) + iv(x, y)$

$$F(z) = U(x, y) + iV(x, y)$$

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0, \quad W_0 = U_0 + iV_0$$

use Thm 1. and (7)

$f(z)F(z) = (uU - vV) + i(vU + uV)$  have the limits

$$\begin{array}{ccc} & \Downarrow & \Downarrow \\ & u_0U_0 - v_0V_0 & v_0U_0 + u_0V_0 \\ & = w_0W_0 & \end{array}$$

An immediate consequence of Thm. 1:

•  $\lim_{z \rightarrow z_0} c = c$

•  $\lim_{z \rightarrow z_0} z = z_0$

•  $\lim_{z \rightarrow z_0} z^n = z_0^n \quad (n = 1, 2, \dots)$

by property (9) and math induction.

•  $P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n \quad (11)$

$\lim_{z \rightarrow z_0} P(z) = P(z_0)$

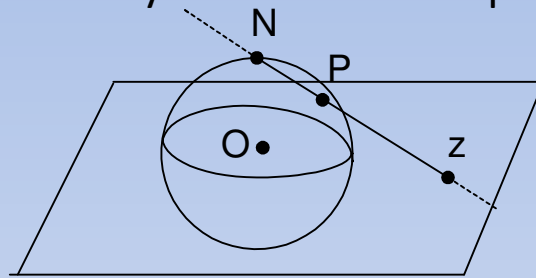
• if  $\lim_{z \rightarrow z_0} f(z) = w_0$ , then  $\lim_{z \rightarrow z_0} |f(z)| = |w_0|$

$\left| |f(z)| - |w_0| \right| \leq |f(z) - w_0| < \varepsilon$  whenever  $0 < |z - z_0| < \delta$

# Limits involving the point at Infinity

It is sometime convenient to include with the complex plane the point at infinity, denoted by  $\infty$ , and to use limits involving it.

Complex plane + infinity = extended complex plane.



complex plane passing thru the equator of a unit sphere.

To each point  $z$  in the plane there corresponds exactly one point  $P$  on the surface of the sphere.

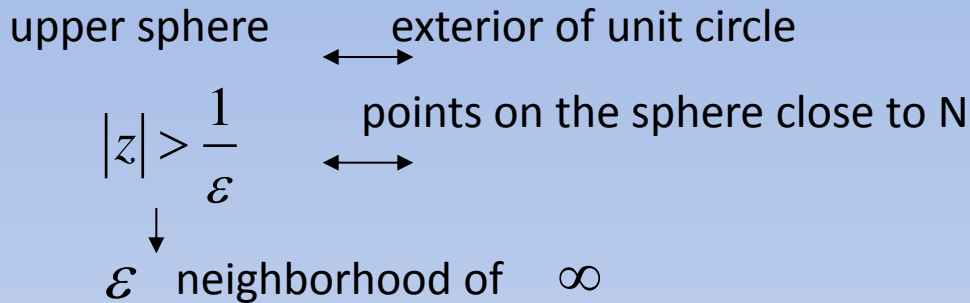


intersection of the line  $z$ - $N$  with the surface.

↑  
north pole

To each point  $P$  on the surface of the sphere, other than the north pole  $N$ , there corresponds exactly one point  $z$  in the plane.

By letting the point N of the sphere correspond to the point at infinity, we obtain a one-to-one correspondence between the points of the sphere and the points of the extended complex plane.



- $\lim_{z \rightarrow z_0} f(z) = \infty$

$$\Leftrightarrow |f(z)| > \frac{1}{\varepsilon} \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

$$\Leftrightarrow \left| \frac{1}{f(z)} - 0 \right| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

$$\therefore \lim_{z \rightarrow z_0} f(z) = \infty \quad \text{iff} \quad \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0$$

Ex1.  $\lim_{z \rightarrow -1} \frac{iz + 3}{z + 1} = \infty$  since  $\lim_{z \rightarrow -1} \frac{z + 1}{iz + 3} = 0$

- $\lim_{z \rightarrow \infty} f(z) = w_0$

$$\Leftrightarrow |f(z) - w_0| < \varepsilon \quad \text{whenever} \quad |z| > \frac{1}{\delta}$$

$$\Leftrightarrow \left| f\left(\frac{1}{z}\right) - w_0 \right| < \varepsilon \quad \text{whenever} \quad 0 < |z - 0| < \delta$$

$$\therefore \lim_{z \rightarrow \infty} f(z) = w_0 \quad \text{iff} \quad \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0$$

EX 2.  $\lim_{z \rightarrow \infty} \frac{2z + i}{z + 1} = 2$  since  $\lim_{z \rightarrow 0} \frac{\frac{2}{z} + i}{\frac{1}{z} + 1} = \lim_{z \rightarrow 0} \frac{2 + iz}{1 + z} = 2$

- $\lim_{z \rightarrow \infty} f(z) = \infty$

$$\Leftrightarrow |f(z)| > \frac{1}{\varepsilon} \quad \text{whenever} \quad |z| > \frac{1}{\delta}$$

$$\Leftrightarrow \left| f\left(\frac{1}{z}\right) \right| > \frac{1}{\varepsilon} \quad \text{whenever} \quad \left| \frac{1}{z} \right| > \frac{1}{\delta}$$

$$\Leftrightarrow \left| \frac{1}{f(1/z)} - 0 \right| < \varepsilon \quad \text{whenever} \quad 0 < |z - 0| < \delta$$

$$\therefore \lim_{z \rightarrow \infty} f(z) = \infty \quad \text{iff} \quad \lim_{z \rightarrow 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$$

Ex 3.  $\lim_{z \rightarrow \infty} \frac{2z^3 - 1}{z^2 + 1} = \infty$

since  $\lim_{z \rightarrow 0} \frac{\frac{1}{z^2} + 1}{\frac{2}{z^3} - 1} = \lim_{z \rightarrow 0} \frac{z + z^3}{2 - z^3} = 0$

# Continuity

A function  $f$  is continuous at a point  $z_0$  if

$$\lim_{z \rightarrow z_0} f(z) \text{ exists,} \quad (1)$$

$$f(z_0) \text{ exists,} \quad (2)$$

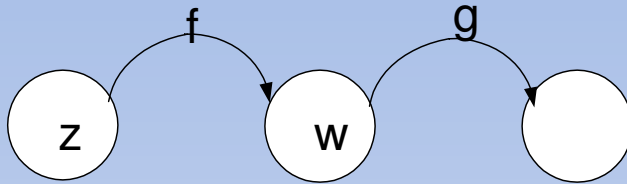
$$\lim_{z \rightarrow z_0} f(z) = f(z_0) \quad (3) \quad ((3) \text{ implies } (1)(2))$$

$$(|f(z) - f(z_0)| < \varepsilon \quad \textit{whenever} \quad |z - z_0| < \delta)$$

- if  $f_1, f_2$  continuous at  $z_0$ , then  $f_1 + f_2, f_1 f_2$   
also continuous at  $z_0$ .

$$\text{So is } \frac{f_1}{f_2} \quad \textit{if} \quad f_2(z_0) \neq 0$$

- A polynomial is continuous in the entire plane because of (11), section 12. p.37
- A composition of continuous function is continuous.



$$|g[f(z)] - g[f(z_0)]| < \varepsilon, \quad \text{whenever } |f(z) - f(z_0)| < r,$$

$$\text{whenever } |z - z_0| < \delta$$

- If a function  $f(z)$  is continuous and non zero at a point  $z_0$ , then  $f(z) \neq 0$  throughout some neighborhood of that point.

when  $f(z_0) \neq 0$  let  $\varepsilon = \frac{|f(z_0)|}{2}$

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2} \quad \text{whenever } |z - z_0| < \delta$$

if there is a point  $z$  in the  $|z - z_0| < \delta$  at which  $f(z) = 0$

$$|f(z_0)| < \frac{|f(z_0)|}{2} \quad \text{a contradiction.}$$



From Thm 1

a function  $f$  of a complex variable is continuous at a point  
iff its component functions  $u$  and  $v$  are continuous there.

$$z_0 = (x_0, y_0)$$

Ex. The function

$f(z) = \cos(x^2 - y^2) \cosh 2xy - i \sin(x^2 - y^2) \sinh 2xy$   
is continuous everywhere in the complex plane since

- (i)  $x^2 - y^2$  are continuous (polynomial)  
 $2xy$
- (ii)  $\cos, \sin, \cosh, \sinh$  are continuous
- (iii) real and imaginary component are continuous

complex function is continuous.



$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

# Derivatives

Let  $f$  be a function whose domain of definition  $f'(z_0)$  contain a neighborhood of a point  $z_0$ . The derivative of  $f$  at  $z_0$ , written  $f'(z_0)$ , is

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided this limit exists.

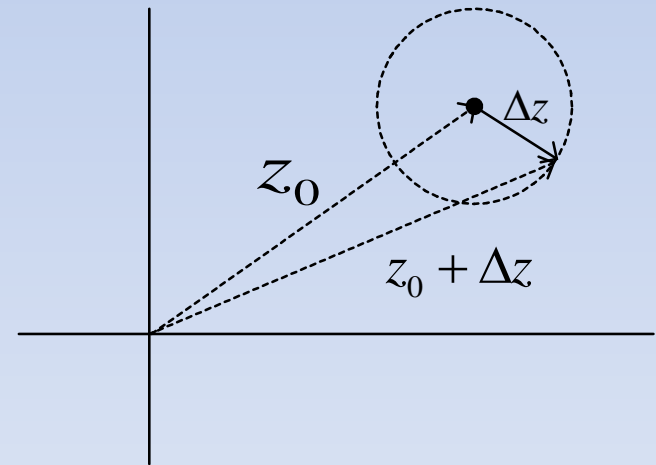
$f$  is said to be differentiable at  $z_0$ .

let  $\Delta z = z - z_0$

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

let  $\Delta w = f(z + \Delta z) - f(z)$ .

$$f'(z) = \frac{dw}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z}$$



Ex1. Suppose  $f(z) = z^2$  at any point  $z$

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \rightarrow 0} (2z + \Delta z) = 2z + 0 = 2z$$

since  $2z + \Delta z$  a polynomial in  $\Delta z$ .

$$\therefore f'(z) = \frac{dw}{dz} = 2z$$

Ex2.  $f(z) = |z|^2$

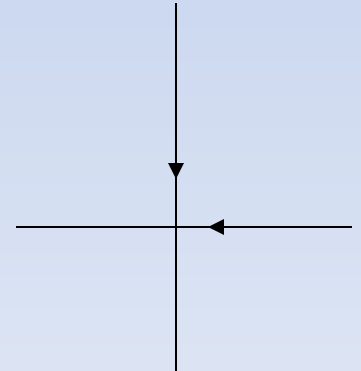
$$\frac{\Delta w}{\Delta z} = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z} = \bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z}$$

when  $\Delta z \rightarrow 0$  thru  $(\Delta x, 0)$  on the real axis  $\overline{\Delta z} = \Delta z$

Hence if the limit of  $\frac{\Delta w}{\Delta z}$  exists, its value =  $\bar{z} + z$

when  $\Delta z \rightarrow 0$  thru  $(0, \Delta y)$  on the imaginary axis.

$\overline{\Delta z} = -\Delta z$ , limit =  $\bar{z} - z$  if it exists.



since limits are unique,

$$\overline{z} + z = \overline{z} - z, \quad \text{or} \quad z = 0 \quad \text{if} \quad \frac{dw}{dz} \text{ is to exist.}$$

observe that  $\frac{\Delta w}{\Delta z} \rightarrow \overline{\Delta z}$  when  $z = 0$

$\therefore \frac{dw}{dz}$  exists only at  $z = 0$ , its value = 0

- Example 2 shows that

a function can be differentiable at a certain point but nowhere else in any neighborhood of that point.


- Re  $|z|^2 = x^2 + y^2$  are continuous, partially

Im  $|z|^2 = 0$  differentiable at a point.


but  $|z|^2$  may not be differentiable there.

- $f(z) = |z|^2$  is continuous at each point in the plane since its components are continuous at each point.

∴ continuity  $\xrightarrow{\text{not necessarily}}$  derivative exists.



existence of derivative  $\Rightarrow$  continuity.



$$\begin{aligned} \lim_{z \rightarrow z_0} [f(z) - f(z_0)] &= \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \rightarrow z_0} (z - z_0) \\ &= f'(z_0) \cdot 0 = 0 \\ \therefore \lim_{z \rightarrow z_0} f(z) &= f(z_0) \end{aligned}$$

## 16. Differentiation Formulas

$$\frac{d}{dz} C = 0 \quad C : \text{complex constant}$$

$$\frac{d}{dz} z = 1$$

$$\frac{d}{dz} [cf(z)] = cf'(z)$$

$$\frac{d}{dz} z^n = nz^{n-1} \quad n \text{ a positive integer.}$$

$$\frac{d}{dz} [f(z) + F(z)] = f'(z) + F'(z)$$

$$\frac{d}{dz} [f(z)F(z)] = f(z)F'(z) + f'(z)F(z) \quad (4)$$

when  $F(z) \neq 0$

$$\frac{d}{dz} \left[ \frac{f(z)}{F(z)} \right] = \frac{F(z)f'(z) - f(z)F'(z)}{[F(z)]^2}$$

*pf* : (4)

$$f(z + \Delta z)F(z + \Delta z) - f(z)F(z)$$

$$= f(z)[F(z + \Delta z) - F(z)] + [f(z + \Delta z) - f(z)]F(z + \Delta z)$$

$$\frac{f(z + \Delta z)F(z + \Delta z) - f(z)F(z)}{\Delta z} = f(z) \frac{F(z + \Delta z) - F(z)}{\Delta z} + \frac{f(z + \Delta z) - f(z)}{\Delta z} F(z + \Delta z)$$

$$\text{as } \Delta z \rightarrow 0 \quad \frac{d}{dz}[fF] = f(z)F'(z) + f'(z)F(z + \Delta z)$$

$$= f(z)F'(z) + f'(z)F(z) \quad (F \text{ continuous at } z)$$

$f$  has a derivative at  $z_0$

$g$  has a derivative at  $f(z_0)$

$F(z)=g[f(z)]$  has a derivative at  $z_0$

and  $F'(z_0) = g'[f(z_0)]f'(z_0)$  chain rule (6)

$$\frac{dW}{dz} = \frac{dW}{dw} \frac{dw}{dz}$$

pf of (6)

choose a  $z_0$  at which  $f'(z_0)$  exists.

let  $w_0 = f(z_0)$  and assume  $g'(w_0)$  exists.

Then, there is  $|w - w_0| < \varepsilon$  of  $w_0$  such that

we can define a function  $\Phi$ , with  $\Phi(w_0) = 0$

$$\Phi(w) = \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \quad \text{when } w \neq w_0 \quad (7)$$

$\lim_{w \rightarrow w_0} \Phi(w) = 0$ , Hence  $\Phi$  is continuous at  $w_0$



$$(7) \Rightarrow g(w) - g(w_0) = [g'(w_0) + \Phi(w)](w - w_0) \quad (|w - w_0| < \varepsilon) \quad (9)$$

valid even when  $w = w_0$

since  $f'(z_0)$  exists and therefore  $f$  is continuous at  $z_0$ , then we can

have  $f(z)$  lies in  $|w - w_0| < \varepsilon$  of  $w_0$  if  $|z - z_0| < \delta$

substitute  $w$  by  $f(z)$  in (9) when  $z$  in  $|z - z_0| < \delta$

(9) becomes

$$\frac{g[f(z)] - g[f(z_0)]}{z - z_0} = \{g'[f(z_0)] + \Phi[f(z)]\} \frac{f(z) - f(z_0)}{z - z_0} \quad (10)$$

$$(0 < |z - z_0| < \delta)$$

since  $f$  is continuous at  $z_0$ ,  $\Phi$  is continuous at  $w_0 = f(z_0)$

$\therefore \Phi[f(z)]$  is continuous at  $z_0$ , and since  $\Phi(w_0) = 0$

$$\lim_{z \rightarrow z_0} \Phi[f(z)] = 0$$

so (10) becomes  $F'(z_0) = g'[f(z_0)] f'(z_0)$  as  $z \rightarrow z_0$

# Cauchy-Riemann Equations

Suppose that  $f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$  exists.

writing  $z_0 = x_0 + iy_0$ ,  $\Delta z = \Delta x + i\Delta y$

Then by Thm. 1

$$\operatorname{Re}[f'(z_0)] = \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \operatorname{Re}\left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}\right] \quad (3)$$

$$\operatorname{Im}[f'(z_0)] = \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \operatorname{Im}\left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}\right] \quad (4)$$

where

$$\begin{aligned} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} \\ &+ \frac{i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]}{\Delta x + i\Delta y} \end{aligned} \quad (5)$$

Let  $(\Delta x, \Delta y)$  tend to  $(0,0)$  horizontally through  $(\Delta x, 0)$   $\Delta y = 0$

$$\therefore \operatorname{Re}[f'(z_0)] = \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$

$$\operatorname{Im}[f'(z_0)] = \lim_{\Delta x \rightarrow 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

$$\therefore f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \quad (6)$$

Let  $(\Delta x, \Delta y)$  tend to  $(0,0)$  vertically through  $(0, \Delta y)$   $\Delta x = 0$

$$f'(z_0) = \left( \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + \frac{i[v(x_0, y_0 + \Delta y) - v(x_0, y_0)]}{i\Delta y} \right)$$

$$= v_y(x_0, y_0) - iu_y(x_0, y_0) \quad (7)$$

$$= -iu_y + v_y$$

$$(6) = (7)$$

$$\therefore u_x(x_0, y_0) = v_y(x_0, y_0) \quad (8)$$

$$u_y(x_0, y_0) = -v_x(x_0, y_0)$$

**Cauchy-Riemann Equations.**

Thm : suppose  $f(z) = u(x, y) + iv(x, y)$

$f'(z)$  exists at a point  $z_0 = x_0 + iy_0$

Then  $u_x, u_y, v_x, v_y$  exist at  $(x_0, y_0)$

and  $u_x = v_y, u_y = -v_x$ ; also  $f'(z) = u_x + iv_x$

Ex 1.  $f(z) = z^2 = x^2 - y^2 + i2xy$

$$u_x = 2x \quad v_x = 2y$$

$$u_y = -2y \quad v_y = 2x$$

$$u_x = v_y, \quad u_y = -v_x$$

$$f'(z) = 2x + i2y = 2(x + iy) = 2z$$

Cauchy-Riemann equations are Necessary conditions for the existence of the derivative of a function  $f$  at  $z_0$ .

➡ Can be used to locate points at which  $f$  does not have a derivative.

Ex 2.  $f(z) = |z|^2$ ,

$$u(x, y) = x^2 + y^2 \quad v(x, y) = 0$$

$$u_x = 2x \quad v_x = 0 \quad u_x \neq v_y, \quad f'(z) \text{ does not exist}$$

$$u_y = 2y \quad v_y = 0 \quad \text{at any nonzero point.}$$

The above Thm does not ensure the existence of  $f'(z_0)$   
(say)

# Sufficient Conditions For Differentiability

$$f'(z_0) \text{ exist} \rightarrow u_x = v_y, \quad u_y = -v_x$$

but not  
"←"

Thm.

Let  $f(z) = u(x, y) + iv(x, y)$  be defined throughout some neighborhood of a point

$$z_0 = x_0 + iy_0$$

suppose  $u_x, u_y, v_x, v_y$  exist everywhere in the neighborhood and are **continuous** at  $(x_0, y_0)$

Then, if  $u_x = v_y, \quad u_y = -v_x$  at  $(x_0, y_0)$   
 $\Rightarrow f'(z_0)$  exists.

*pf* : let  $\Delta z = \Delta x + i\Delta y$ , where  $0 < |\Delta z| < \varepsilon$

$$\Delta w = f(z_0 + \Delta z) - f(z_0) \quad \curvearrowright$$

Thus  $\Delta w = \Delta u + i\Delta v \iff u(z_0 + \Delta z) - u(z_0) + i[v(z_0 + \Delta z) - v(z_0)]$

where  $\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$

$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$$

⇒ Now in view of the continuity of the first-order partial derivatives of  $u$  and  $v$  at the point  $(x_0, y_0)$

$$\begin{aligned} \Delta u = & u(x_0, y_0) + u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + u_{xy}(x_0, y_0)\Delta x\Delta y \\ & + u_{xx}(x_0, y_0)\frac{\Delta x^2}{2!} \\ & + u_{yy}(x_0, y_0)\frac{\Delta y^2}{2!} \\ & - u(x_0, y_0) \qquad \qquad \qquad + \dots \end{aligned}$$

$$= u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_2 \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\varepsilon_1, \varepsilon_2 \rightarrow 0, \text{ as } (\Delta x, \Delta y) \rightarrow (0, 0)$$

$$\Delta w = \Delta u + i\Delta v \quad \leftarrow \text{ where } \varepsilon_1 \text{ and } \varepsilon_2 \text{ tend to 0 as } (\Delta x, \Delta y) \rightarrow (0, 0) \text{ in the } z\text{-plane. } \Delta z$$

$$= \text{above} \quad (3)$$

assuming that the Cauchy-Riemann equations are satisfied at  $(x_0, y_0)$  we can replace

$$u_y \text{ by } -v_x, \text{ and } v_y \text{ by } u_x \text{ in (3), and divide thru by } \Delta z$$

to get

$$\frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_2) \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta z} \quad (4)$$

$$\text{but } \sqrt{(\Delta x)^2 + (\Delta y)^2} = |\Delta z|$$

$$\text{so } \left| \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta z} \right| = 1$$

also  $\varepsilon_1 + i\varepsilon_2$  tends to 0, as  $(\Delta x, \Delta y) \rightarrow (0, 0)$

The last term in (4) tends to 0 as  $\Delta z \rightarrow 0$

$\therefore$  The limit of  $\frac{\Delta w}{\Delta z}$  exists, and  $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$ .



Ex 1.  $f(z) = e^x (\cos y + i \sin y)$

$$u(x, y) = e^x \cos y$$

$$v(x, y) = e^x \sin y$$

$$u_x = v_y, \quad u_y = -v_x \quad \text{everywhere, and continuous.}$$

$\Rightarrow f'(z)$  exists everywhere, and

$$f'(z) = u_x + iv_x = e^x (\cos y + i \sin y)$$

Ex 2.  $f(z) = |z|^2$

$$u(x, y) = x^2 + y^2 \quad u_x = 2x \quad u_y = 2y$$

$$v(x, y) = 0 \quad v_x = 0 \quad v_y = 0$$

has a derivative at  $z=0$ .

$$f'(0) = 0 + i0$$

can not have derivative at any nonzero point.

# Polar Coordinates

$$x = r \cos \theta \quad y = r \sin \theta$$

$$z = x + iy = re^{i\theta} \quad (z \neq 0)$$

Suppose that  $u_x, u_y, v_x, v_y$  exist everywhere in some neighborhood of a given non-zero point  $z_0$  and are continuous at that point.

$u_r, u_\theta, v_r, v_\theta$  also have these properties, and ( by chain rule )

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta}$$

$$u_r = u_x \cos \theta + u_y \sin \theta \quad (2)$$

$$u_\theta = -u_x r \sin \theta + u_y r \cos \theta$$

Similarly,

$$v_r = v_x \cos \theta + v_y \sin \theta \quad (3)$$

$$v_\theta = -v_x r \sin \theta + v_y r \cos \theta$$

$$\text{If } u_x = v_y, u_y = -v_x$$

$$v_r = -u_y \cos \theta + u_x \sin \theta \quad (5)$$

$$v_\theta = u_y r \sin \theta + u_x r \cos \theta$$

$$\text{from (2) (5), } u_r = \frac{1}{r} v_\theta \quad \text{at } z_0 \quad (6)$$

$$\frac{1}{r} u_\theta = -v_r$$

Thm. p53...

$$f'(z_0) = u_x + iv_x$$

$$= ?$$

$$u_r = u_x \cos \theta + u_y \sin \theta$$

$$u_r \cos \theta = u_x \cos^2 \theta + u_y \sin \theta \cos \theta$$

$$\boxed{\therefore u_r \cos \theta + v_r \sin \theta = u_x}$$

$$u_r = v_y \cos \theta - v_x \sin \theta$$

$$u_r \sin \theta = v_y \cos \theta \sin \theta - v_x \sin^2 \theta$$

$$\boxed{v_r \cos \theta - u_r \sin \theta = v_x}$$

$$v_r = v_x \cos \theta + v_y \sin \theta$$

$$= -u_y \cos \theta + u_x \sin \theta$$

$$v_r \sin \theta = -u_y \cos \theta \sin \theta + u_x \sin^2 \theta$$

$$v_r = v_x \cos \theta + v_y \sin \theta$$

$$\cos \theta v_r = v_x \cos^2 \theta + v_y \sin \theta \cos \theta$$

$$\begin{aligned} \therefore f'(z_0) &= u_r \cos \theta + v_r \sin \theta + i(v_r \cos \theta - u_r \sin \theta) \\ &= (\cos \theta - i \sin \theta)(u_r + iv_r) \\ &= e^{-i\theta} (u_r + iv_r) \end{aligned} \quad (7)$$

Ex : Consider

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}}$$

$$u(r, \theta) = \frac{1}{r} \cos \theta \quad v(r, \theta) = -\frac{1}{r} \sin \theta$$

$$u_r = -\frac{1}{r^2} \cos \theta \quad v_r = \frac{1}{r^2} \sin \theta$$

$$u_\theta = -\frac{1}{r} \sin \theta \quad v_\theta = -\frac{1}{r} \cos \theta$$

$$\Rightarrow u_r = \frac{1}{r} v_\theta, \quad \frac{1}{r} u_\theta = -v_r \quad \text{at any non-zero point} \quad z = re^{i\theta}$$

$\therefore f'$  exists

$$f' = e^{-i\theta} \left( -\frac{1}{r^2} \cos \theta + \frac{i}{r^2} \sin \theta \right)$$

$$= \frac{1}{r^2} (-e^{-i\theta}) e^{-i\theta} = -\frac{1}{r^2} e^{-i2\theta} = -\frac{1}{z^2}$$