DERIVATIVE OF ANALYTIC FUNCTION

Derivatives of Analytic Functions

$z = x + i\,y$

Let $f\,(z)$ be analytic around z, then

$$\frac{d\,f\,(x)}{d\,x} = g\,(x) \qquad \rightarrow \qquad \frac{d\,f\,(z)}{d\,z} = g\,(z)$$

Proof :

$f\,(z)$ analytic $\rightarrow$ $$f'(z) = \frac{\partial f\,(x + i\,y)}{\partial x} = \left.\frac{d\,f\,(x)}{d\,x}\right|_{x=z} = g\,(z)$$

E.g. $$\frac{d\,x^n}{d\,x} = n\,x^{n-1} \qquad \rightarrow \qquad \frac{d\,z^n}{d\,z} = n\,z^{n-1}$$

$\therefore$ Analytic functions can be defined by Taylor series of the same coefficients as their real counterparts.

Derivative of Logarithm

CRCs $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{d \ln z}{dz} = \frac{1}{z}$$ for z within each branch.

Proof : $\ln z = \ln r + i\left(\theta + 2\pi n\right) = u + iv \rightarrow$ $u = \ln r$, $v = \theta + 2\pi n$

$$\frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{x}{r^2} = \frac{\partial v}{\partial y}$$

$$\frac{\partial \theta}{\partial y} = \left(1 + \frac{y^2}{x^2}\right)^{-1}\frac{1}{x} = \frac{x}{r^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{r}\frac{\partial r}{\partial y} = \frac{y}{r^2} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial \theta}{\partial x} = \left(1 + \frac{y^2}{x^2}\right)^{-1}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2}$$

$\rightarrow$ $\ln z$ is analytic within each branch.

$$\therefore \quad \frac{d \ln z}{dz} = \frac{\partial \ln z}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{r^2} - i\frac{y}{r^2} = \frac{1}{x + iy} = \frac{1}{z}$$

$$r^2 = z z^*$$

QED

Derivatives of Functions w(t)

- Consider derivatives of complex-valued functions w of real variable t

$$w(t) = u(t) + iv(t)$$

where the function u and v are real-valued functions of t. The derivative

$$w'(t), or \frac{d}{dt} w(t)$$

of the function w(t) at a point t is defined as

$$w'(t) = u'(t) + iv'(t)$$

Derivatives of Functions w(t)

- Properties

For any complex constant $z_0=x_0+iy_0$,

u(t) v(t)

$$\frac{d}{dt}[z_0 w(t)] = [(x_0+iy_0)(u+iv)]' = [(x_0u \quad y_0v)+i(y_0u+x_0v)]'$$

$$=(x_0u \quad y_0v)'+i(y_0u+x_0v)'$$

$$=(x_0u' \quad y_0v')+i(y_0u'+x_0v')$$

$$=(x_0+iy_0)(u'+iv')=z_0w'(t)$$

Derivatives of Functions w(t)

- Properties

$$\frac{d}{dt} e^{z_0 t} = z_0 e^{z_0 t}$$

where $z_0 = x_0 + iy_0$. We write

u(t) v(t)

$$e^{z_0 t} = e^{(x_0 + iy_0)t} = e^{x_0 t} \cos y_0 t + i e^{x_0 t} \sin y_0 t$$

$$\frac{d}{dt} e^{z_0 t} = (e^{x_0 t} \cos y_0 t)' + i(e^{x_0 t} \sin y_0 t)'$$

Similar rules from calculus and some simple algebra then lead us to the expression

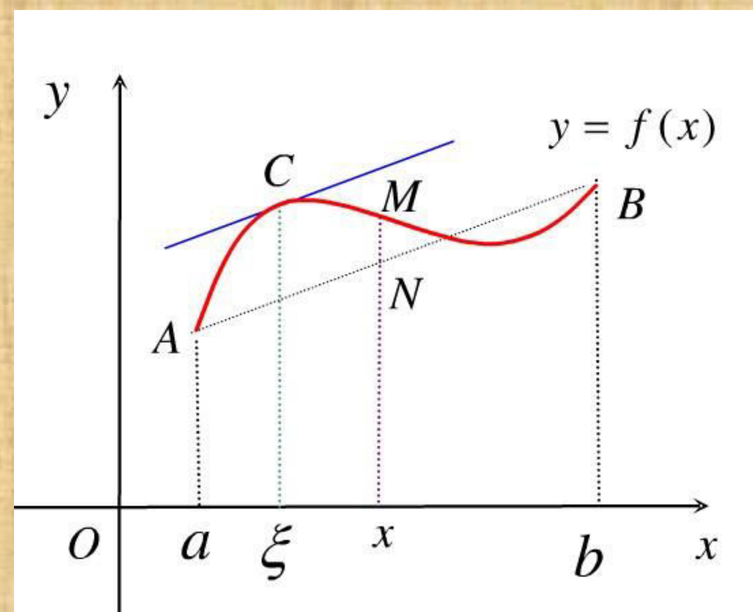
$$\frac{d}{dt} e^{z_0 t} = (x_0 + iy_0) e^{(x_0 + iy_0)t} = z_0 e^{z_0 t}$$

Derivatives of Functions w(t)

- Example

Suppose that the ***real function*** f(t) is continuous on an interval a≤ t ≤b, if f'(t) exists when a<t<b, the mean value theorem for derivatives tells us that there is a number ζ in the interval a<ζ<b such that

$$f'(\ \ )=\frac{f(b)\quad f(a)}{b\quad a}$$



Derivatives of Functions w(t)

- Example (Cont')

The mean value theorem no longer applies for some ***complex functions***. For instance, the function

$$w(t) = e^{it}$$

on the interval $0 \le t \le 2\pi$.

Please note that

$$|w'(t)| = |ie^{it}| = 1$$

And this means that the derivative w'(t) is never zero, while

$$w(2\pi) - w(0) = 0 \Longrightarrow w'(\varsigma) \neq \frac{w(2\pi) - w(0)}{2\pi - 0} = 0, \forall \varsigma \in (0, 2\pi)$$

Note: not every rules from calculus holds for complex functions