

Evaluation of Real Integrals

Complex variables

Integrals of sinusoidal functions

$$\int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta \quad \text{set } z = \exp i\theta \text{ in unit circle}$$

$$\Rightarrow \cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right), \quad \sin \theta = \frac{1}{2i} \left(z - \frac{1}{z} \right), \quad d\theta = -iz^{-1} dz$$

$$\text{Ex : Evaluate } I = \int_0^{2\pi} \frac{\cos 2\theta}{a^2 + b^2 - 2ab \cos \theta} d\theta \quad \text{for } b > a > 0$$

$$\cos n\theta = \frac{1}{2} (z^n + z^{-n}) \Rightarrow \cos 2\theta = \frac{1}{2} (z^2 + z^{-2})$$

$$\begin{aligned} \frac{\cos 2\theta}{a^2 + b^2 - 2ab \cos \theta} d\theta &= \frac{\frac{1}{2} (z^2 + z^{-2}) (-iz^{-1}) dz}{a^2 + b^2 - 2ab \cdot \frac{1}{2} (z + z^{-1})} = \frac{-\frac{1}{2} (z^4 + 1) idz}{z^2 (za^2 + zb^2 - abz^2 - ab)} \\ &= \frac{i}{2ab} \frac{(z^4 + 1) dz}{z^2 \left(z^2 - z \left(\frac{a}{b} - + \frac{b}{a} \right) + 1 \right)} = \frac{i}{2ab} \frac{(z^4 + 1)}{z^2 \left(z - \frac{a}{b} \right) \left(z - \frac{b}{a} \right)} dz \end{aligned}$$

Complex variables

$$I = \frac{i}{2ab} \oint_C \frac{z^4 + 1}{z^2(z - \frac{a}{b})(z - \frac{b}{a})} dz \quad \text{double poles at } z = 0 \text{ and } z = a/b \text{ within the unit circle}$$

$$\text{Residue : } R(z_0) = \lim_{z \rightarrow z_0} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \right\}$$

(1) pole at $z = 0, m = 2$

$$\begin{aligned} R(0) &= \lim_{z \rightarrow 0} \left\{ \frac{1}{1!} \frac{d}{dz} \left[z^2 \frac{z^4 + 1}{z^2(z - a/b)(z - b/a)} \right] \right\} \\ &= \lim_{z \rightarrow 0} \left\{ \frac{4z^3}{(z - a/b)(z - b/a)} + \frac{(z^4 + 1)(-1)[2z - (a/b + b/a)]}{(z - a/b)^2(z - b/a)^2} \right\} = a/b + b/a \end{aligned}$$

(2) pole at $z = a/b, m = 1$

$$R(a/b) = \lim_{z \rightarrow a/b} \left[(z - a/b) \frac{z^4 + 1}{z^2(z - a/b)(z - b/a)} \right] = \frac{(a/b)^4 + 1}{(a/b)^2(a/b - b/a)} = \frac{-(a^4 + b^4)}{ab(b^2 - a^2)}$$

$$I = 2\pi i \times \frac{i}{2ab} \left[\frac{a^2 + b^2}{ab} - \frac{a^4 + b^4}{ab(b^2 - a^2)} \right] = \frac{2\pi a^2}{b^2(b^2 - a^2)}$$

Complex variables

Some infinite integrals

$$\int_{-\infty}^{\infty} f(x) dx$$

$f(z)$ has the following properties

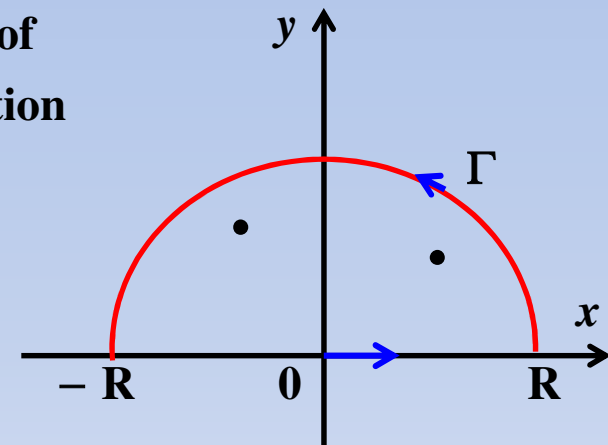
(1) $f(z)$ is analytic in the upper half-plane, $\text{Im}z \geq 0$, except for a finite number of poles, none of which is on the real axis.

(2) on a semicircle Γ of radius R , R times the maximum of $|f|$ on Γ tends to zero as $R \rightarrow \infty$ (a sufficient condition is that $zf(z) \rightarrow 0$ as $|z| \rightarrow \infty$).

(3) $\int_{-\infty}^0 f(x) dx$ and $\int_0^{\infty} f(x) dx$ both exist

$$\Rightarrow \int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum_j R_j$$

for $|\int_{\Gamma} f(z) dz| \leq 2\pi R \times (\text{maximum of } |f| \text{ on } \Gamma)$, the integral along Γ tends to zero as $R \rightarrow \infty$.



Complex variables

Ex : Evaluate $I = \int_0^{\infty} \frac{dx}{(x^2 + a^2)^4}$ a is real

$$\oint_C \frac{dz}{(z^2 + a^2)^4} = \int_{-R}^R \frac{dx}{(x^2 + a^2)^4} + \int_{\Gamma} \frac{dz}{(z^2 + a^2)^4} \text{ as } R \rightarrow \infty$$

$$\Rightarrow \int_{\Gamma} \frac{dz}{(z^2 + a^2)^4} \rightarrow 0 \Rightarrow \oint_C \frac{dz}{(z^2 + a^2)^4} = \int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^4}$$

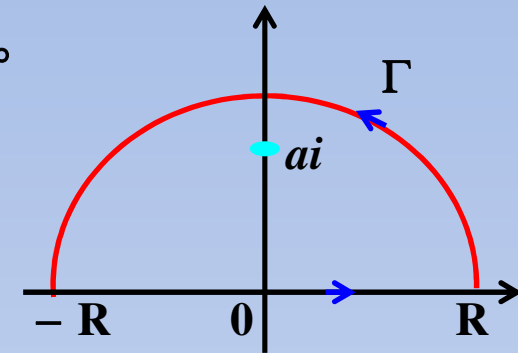
$(z^2 + a^2)^4 = 0 \Rightarrow$ poles of order 4 at $z = \pm ai$,

only $z = ai$ at the upper half - plane

$$\text{set } z = ai + \xi, \xi \rightarrow 0 \Rightarrow \frac{1}{(z^2 + a^2)^4} = \frac{1}{(2ai\xi + \xi^2)^4} = \frac{1}{(2ai\xi)^4} \left(1 - \frac{i\xi}{2a}\right)^{-4}$$

$$\text{the coefficient of } \xi^{-1} \text{ is } \frac{1}{(2a)^4} \frac{(-4)(-5)(-6)}{3!} \left(\frac{-i}{2a}\right)^3 = \frac{-5i}{32a^7}$$

$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)^4} = 2\pi i \left(\frac{-5i}{32a^7}\right) = \frac{10\pi}{32a^7} \Rightarrow I = \frac{1}{2} \times \frac{10\pi}{32a^7} = \frac{5\pi}{32a^7}$$



Complex variables

For poles on the real axis:

Principal value of the integral, defined as $\rho \rightarrow 0$

$$P \int_{-R}^R f(x) dx = \int_{-R}^{z_0 - \rho} f(x) dx + \int_{z_0 + \rho}^R f(x) dx$$

for a closed contour C

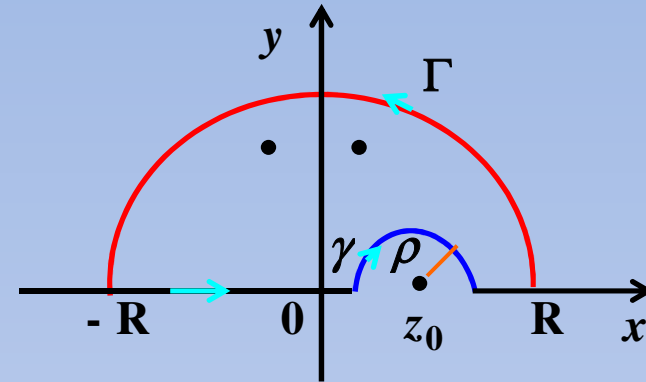
$$\begin{aligned} \oint_C f(z) dz &= \int_{-R}^{z_0 - \rho} f(x) dx + \int_{\gamma} f(z) dz + \int_{z_0 + \rho}^R f(x) dx + \int_{\Gamma} f(z) dz \\ &= P \int_{-R}^R f(x) dx + \int_{\gamma} f(z) dz + \int_{\Gamma} f(z) dz \end{aligned}$$

(1) for $\int_{\gamma} f(z) dz$ has a pole at $z = z_0 \Rightarrow \int_{\gamma} f(z) dz = -\pi i a_1$

(2) for $\int_{\Gamma} f(z) dz$ set $z = R e^{i\theta}$ $dz = i R e^{i\theta} d\theta$

$$\Rightarrow \int_{\Gamma} f(z) dz = \int_{\Gamma} f(R e^{i\theta}) i R e^{i\theta} d\theta$$

If $f(z)$ vanishes faster than $1/R^2$ as $R \rightarrow \infty$, the integral is zero



Complex variables

Jordan's lemma

- (1) $f(z)$ is analytic in the upper half - plane except for a finite number of poles in $\text{Im } z > 0$
- (2) the maximum of $|f(z)| \rightarrow 0$ as $|z| \rightarrow \infty$ in the upper half - plane
- (3) $m > 0$, then

$$I_{\Gamma} = \int_{\Gamma} e^{imz} f(z) dz \rightarrow 0 \text{ as } R \rightarrow \infty, \Gamma \text{ is the semicircular contour}$$

for $0 \leq \theta \leq \pi/2$, $1 \geq \sin \theta / \theta \geq \pi/2$

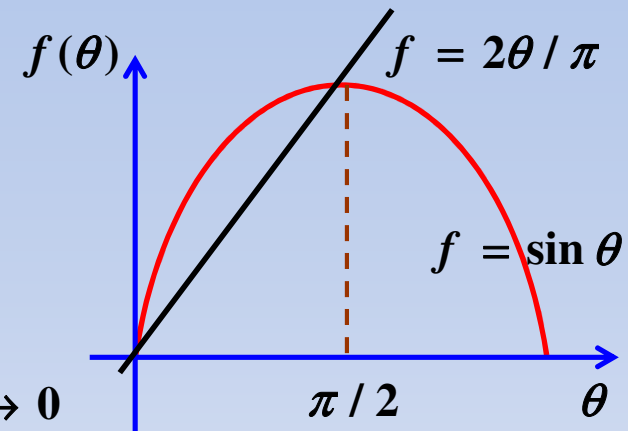
$$|\exp(imz)| = |\exp(-mR \sin \theta)|$$

$$\begin{aligned} I_{\Gamma} &\leq \int_{\Gamma} |e^{imz} f(z)| |dz| \leq MR \int_0^{\pi} e^{-mR \sin \theta} d\theta \\ &= 2MR \int_0^{\pi/2} e^{-mR \sin \theta} d\theta \end{aligned}$$

M is the maximum of $|f(z)|$ on $|z| = R$, $R \rightarrow \infty$ $M \rightarrow 0$

$$I_{\Gamma} < 2MR \int_0^{\pi/2} e^{-mR(2\theta/\pi)} d\theta = \frac{\pi M}{m} (1 - e^{-mR}) < \frac{\pi M}{m}$$

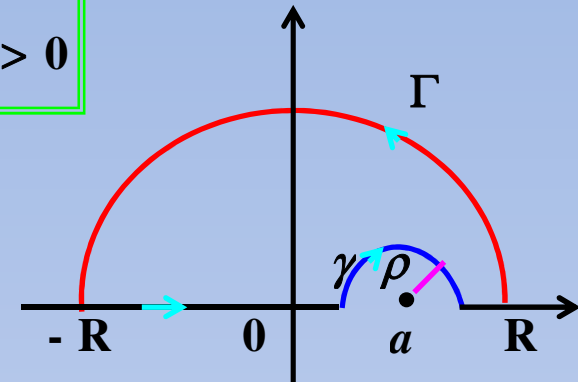
as $R \rightarrow \infty \Rightarrow M \rightarrow 0 \Rightarrow I_{\Gamma} \rightarrow 0$



Complex variables

Ex : Find the principal value of $\int_{-\infty}^{\infty} \frac{\cos mx}{x-a} dx$ a real, $m > 0$

Consider the integral $I = \oint_C \frac{e^{imz}}{z-a} dz = 0$ no pole in the upper half - plane, and $|(z-a)^{-1}| \rightarrow 0$ as $|z| \rightarrow \infty$



$$I = \oint_C \frac{e^{imz}}{z-a} dz$$

$$= \int_{-R}^{a-\rho} \frac{e^{imx}}{x-a} dx + \int_{\gamma} \frac{e^{imz}}{z-a} dz + \int_{a+\rho}^R \frac{e^{imx}}{x-a} dx + \int_{\Gamma} \frac{e^{imz}}{z-a} dz = 0$$

$$\text{As } R \rightarrow \infty \text{ and } \rho \rightarrow 0 \Rightarrow \int_{\Gamma} \frac{e^{imz}}{z-a} dz \rightarrow 0$$

$$\Rightarrow P \int_{-\infty}^{\infty} \frac{e^{imx}}{x-a} dx - i\pi a_{-1} = 0 \text{ and } a_{-1} = e^{ima}$$

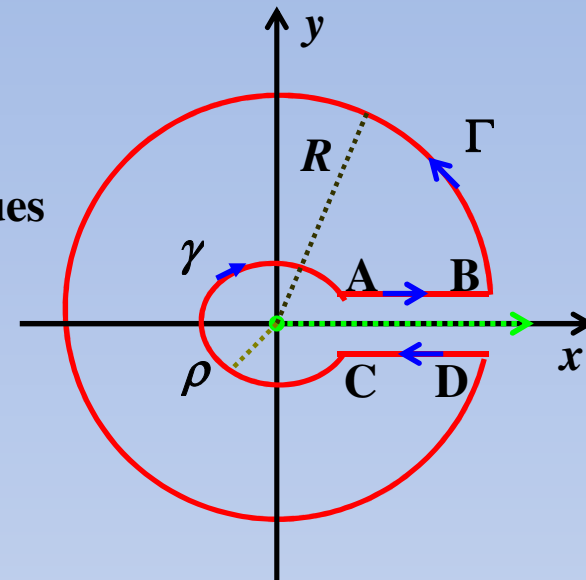
$$\Rightarrow P \int_{-\infty}^{\infty} \frac{\cos mx}{x-a} dx = -\pi \sin ma \text{ and } P \int_{-\infty}^{\infty} \frac{\sin mx}{x-a} dx = \pi \cos ma$$

Complex variables

Integral of multivalued functions

Multivalued functions such as $z^{1/2}, \text{Ln}z$

Single branch point is at the origin. We let $R \rightarrow \infty$ and $\rho \rightarrow 0$. The integrand is multivalued, its values along two lines AB and CD joining $z = \rho$ to $z = R$ are not equal and opposite.



$$\text{Ex : } I = \int_0^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} \text{ for } a > 0$$

(1) the integrand $f(z) = (z+a)^{-3} z^{-1/2}$, $|zf(z)| \rightarrow 0$ as $\rho \rightarrow 0$ and $R \rightarrow \infty$
the two circles make no contribution to the contour integral

(2) pole at $z = -a$, and $(-a)^{1/2} = a^{1/2} e^{i\pi/2} = ia^{1/2}$

$$\begin{aligned} R(-a) &= \lim_{z \rightarrow -a} \frac{1}{(3-1)!} \frac{d^{3-1}}{dz^{3-1}} \left[(z+a)^3 \frac{1}{(z+a)^3 z^{1/2}} \right] \\ &= \lim_{z \rightarrow -a} \frac{1}{2!} \frac{d^2}{dz^2} z^{-1/2} = \frac{-3i}{8a^{5/2}} \end{aligned}$$

Complex variables

$$\int_{AB} dz + \int_{\Gamma} dz + \int_{DC} dz + \int_{\gamma} dz = 2\pi i \left(\frac{-3i}{8a^{5/2}} \right)$$

$$\text{and } \int_{\Gamma} dz = 0 \text{ and } \int_{\gamma} dz = 0$$

along line AB $\Rightarrow z = xe^{i0}$, along line CD $\Rightarrow z = xe^{i2\pi}$

$$\int_{0,A \rightarrow B}^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} + \int_{\infty,C \rightarrow D}^0 \frac{dx}{(xe^{i2\pi} + a)^3 x^{1/2} e^{(1/2 \times 2\pi i)}} = \frac{3\pi}{4a^{5/2}}$$

$$\Rightarrow \left(1 - \frac{1}{e^{i\pi}} \right) \int_0^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} = \frac{3\pi}{4a^{5/2}}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{(x+a)^3 x^{1/2}} = \frac{3\pi}{8a^{5/2}}$$

Complex variables

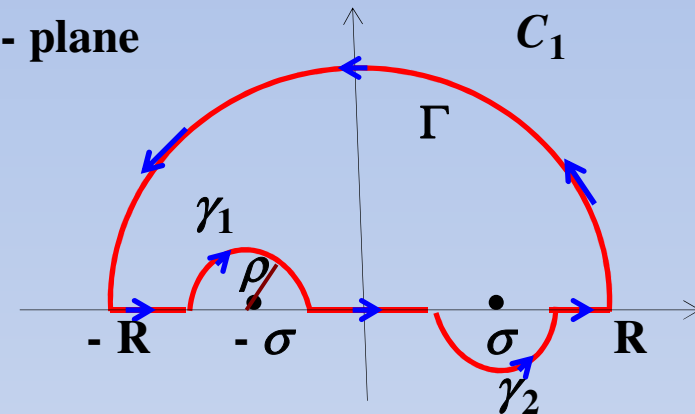
Ex : Evaluate $I(\sigma) = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2} dx$

$$\oint_C \frac{z \sin z}{z^2 - \sigma^2} dz = \frac{1}{2i} \oint_{C_1} \frac{ze^{iz}}{z^2 - \sigma^2} dz - \frac{1}{2i} \oint_{C_2} \frac{ze^{-iz}}{z^2 - \sigma^2} dz = I_1 + I_2$$

(1) for I_1 , the contour is chosen on the upper half - plane

due to the term e^{iz} , and only one pole at $z = \sigma$.

$$\begin{aligned} I_1 &= \frac{1}{2i} \oint_{C_1} \frac{ze^{iz}}{z^2 - \sigma^2} dz = \frac{1}{2i} \int_{-R}^{-\sigma-\rho} \frac{xe^{ix}}{x^2 - \sigma^2} dx \\ &+ \frac{1}{2i} \int_{-\sigma+\rho}^{\sigma-\rho} \frac{xe^{ix}}{x^2 - \sigma^2} dx + \frac{1}{2i} \int_{\sigma+\rho}^{\infty} \frac{xe^{ix}}{x^2 - \sigma^2} dx \\ &+ \frac{1}{2i} \int_{\gamma_1} \frac{ze^{iz}}{z^2 - \sigma^2} dz + \frac{1}{2i} \int_{\gamma_2} \frac{ze^{iz}}{z^2 - \sigma^2} dz + \frac{1}{2i} \int_{\Gamma} \frac{ze^{iz}}{z^2 - \sigma^2} dz \\ &= \frac{1}{2i} 2\pi i \times \text{Res}(z = \sigma) = \pi \frac{\sigma e^{i\sigma}}{2\sigma} = \frac{\pi}{2} e^{i\sigma} \end{aligned}$$



Complex variables

As $\rho \rightarrow 0$ and $R \rightarrow \infty \Rightarrow \int_{\Gamma} dz \rightarrow 0$

$$\frac{1}{2i} \int_{\gamma_1} \frac{ze^{iz}}{(z+\sigma)(z-\sigma)} dz = \frac{1}{2i} \times (-\pi i) \text{Res}(z = -\sigma) = \frac{-\pi}{4} e^{-i\sigma}$$

$$\frac{1}{2i} \int_{\gamma_2} \frac{ze^{iz}}{(z+\sigma)(z-\sigma)} dz = \frac{1}{2i} \times \pi i \text{Res}(z = \sigma) = \frac{\pi}{4} e^{i\sigma}$$

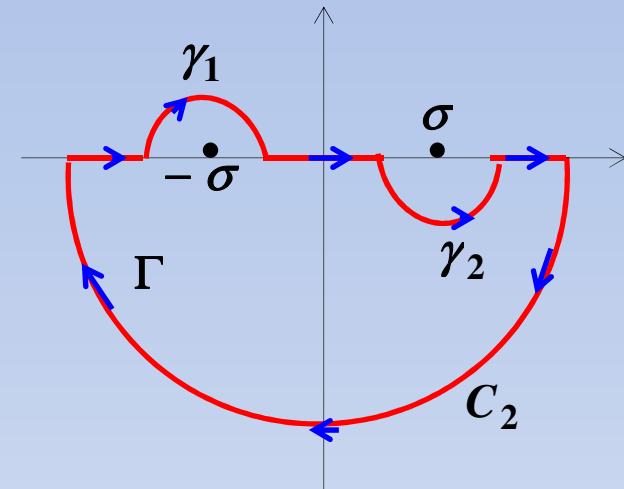
$$I_1 = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 - \sigma^2} dx + \frac{\pi}{4} (e^{i\sigma} - e^{-i\sigma}) = \frac{\pi}{2} e^{i\sigma}$$

(2) for I_2 , we choose the lower half - plane by the term e^{-iz} , only one pole at $z = -\sigma$

$$I_2 = \frac{-1}{2i} \oint_{C_2} \frac{ze^{-iz}}{z^2 - \sigma^2} dz = \frac{-1}{2i} \int_{-R}^{-\sigma-\rho} \frac{xe^{-ix}}{x^2 - \sigma^2} dx$$

$$- \frac{1}{2i} \int_{-\sigma+\rho}^{\sigma-\rho} \frac{xe^{-ix}}{x^2 - \sigma^2} dx - \frac{1}{2i} \int_{\sigma+\rho}^{\infty} \frac{xe^{-ix}}{x^2 - \sigma^2} dx - \frac{1}{2i} \int_{\gamma_1} \frac{ze^{-iz}}{z^2 - \sigma^2} dz$$

$$- \frac{1}{2i} \int_{\gamma_2} \frac{ze^{-iz}}{z^2 - \sigma^2} dz - \frac{1}{2i} \int_{\Gamma} \frac{ze^{-iz}}{z^2 - \sigma^2} dz = \left(\frac{-1}{2i}\right) \times (-2\pi i) \frac{(-\sigma)e^{i\sigma}}{-2\sigma} = \frac{\pi}{2} e^{i\sigma}$$



Complex variables

As $\rho \rightarrow 0, R \rightarrow \infty \Rightarrow \int_{\Gamma} dz \rightarrow 0$

$$\frac{-1}{2i} \int_{\gamma_1} \frac{ze^{-iz}}{(z+\sigma)(z-\sigma)} dz = \left(\frac{-1}{2i}\right)(-\pi i) \frac{(-\sigma)e^{i\sigma}}{-2\sigma} = \frac{\pi}{4} e^{i\sigma}$$

$$\frac{-1}{2i} \int_{\gamma_2} \frac{ze^{-iz}}{(z+\sigma)(z-\sigma)} dz = \left(\frac{-1}{2i}\right)(\pi i) \frac{\sigma e^{-i\sigma}}{2\sigma} = \frac{-\pi}{4} e^{-i\sigma}$$

$$I_2 = \frac{-1}{2i} \int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2 - \sigma^2} dx + \frac{\pi}{4} (e^{i\sigma} - e^{-i\sigma}) = \frac{\pi}{2} e^{i\sigma}$$

$$\Rightarrow \frac{-1}{2i} \int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2 - \sigma^2} dx = \frac{\pi}{2} e^{i\sigma} - \frac{1}{4} (e^{i\sigma} - e^{-i\sigma})$$

$$\begin{aligned} I(\sigma) &= \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2} dx = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 - \sigma^2} dx - \frac{1}{2i} \int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2 - \sigma^2} dx \\ &= \frac{\pi}{2} e^{i\sigma} - \frac{\pi}{4} (e^{i\sigma} - e^{-i\sigma}) + \frac{\pi}{2} e^{i\sigma} - \frac{\pi}{4} (e^{i\sigma} - e^{-i\sigma}) \\ &= \pi e^{i\sigma} - \frac{\pi}{2} e^{i\sigma} + \frac{\pi}{2} e^{-i\sigma} = \frac{\pi}{2} (e^{i\sigma} + e^{-i\sigma}) = \pi \cos \sigma \end{aligned}$$