

Harmonic Function

Harmonic Functions

- A Harmonic Function

A *real-valued* function H of *two real variables* x and y is said to be harmonic in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$H_{xx}(x, y) + H_{yy}(x, y) = 0$$

Known as Laplace's equation.

Harmonic Functions

- Theorem 1

If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D , then its component functions u and v are harmonic in D .

Proof: $f(z) = u(x, y) + iv(x, y)$ is analytic in D

$$\Rightarrow u_x = v_y \ \& \ u_y = -v_x$$

Differentiating both sides of these equations with respect to x and y respectively, we have

$$\begin{array}{ccc} u_{xx} = v_{yx} \ \& \ u_{yx} = -v_{xx} & \xrightarrow{\text{continuity}} \ u_{xx} = v_{xy} \ \& \ u_{xy} = -v_{xx} \\ u_{xy} = v_{yy} \ \& \ u_{yy} = -v_{xy} & \xrightarrow{\hspace{1cm}} \ u_{xy} = v_{yy} \ \& \ u_{yy} = -v_{xy} \end{array}$$

$$u_{xx} + u_{yy} = 0 \ \& \ v_{xx} + v_{yy} = 0$$

Theorem in Sec.52:

a function is analytic at a point, then its real and imaginary components have continuous partial derivatives of all order at that point.

Harmonic Functions

- Example 3

The function $f(z)=i/z^2$ is analytic whenever $z \neq 0$ and since

$$\frac{i}{z^2} = \frac{i \times (\bar{z})^2}{z^2 \times (\bar{z})^2} = \frac{2xy + i(x^2 - y^2)}{(x^2 + y^2)^2}$$

The two functions

$$u(x, y) = \frac{2xy}{(x^2 + y^2)^2} \quad v(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

are harmonic throughout any domain in the xy plane that does not contain the origin.

Harmonic Functions

- Harmonic conjugate

If *two* given function u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy-Riemann equation throughout D , then v is said to be a harmonic conjugate of u .

Is the definition symmetry for u and v ?

Cauchy-Riemann equation $u_x = v_y \ \& \ u_y = -v_x$

If u is a harmonic conjugate of v , then

$$u_x = -v_y \ \& \ u_y = v_x$$

Harmonic Functions

- Theorem 2

A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u .

- Example 4

The function $f(z) = z^2$ is entire function, and its real and imaginary components are $u(x, y) = x^2 - y^2$ & $v(x, y) = 2xy$

Based on the Theorem 2, v is a harmonic conjugate of u throughout the plane. However, u is not the harmonic conjugate of v , since $g(z) = 2xy + i(x^2 - y^2)$ is not an analytic function.

Harmonic Functions

- Example 5

Obtain a harmonic conjugate of a given function.

$$u(x, y) = y^3 - 3x^2y$$

Suppose that v is the harmonic conjugate of the given function

Then $u_x = v_y$ & $u_y = -v_x$

$$u_x = -6xy = v_y \quad \Longrightarrow \quad v = -3xy^2 + \phi(x)$$

$$u_y = 3y^2 - 3x^2 = -v_x \quad \Longrightarrow \quad 3y^2 - 3x^2 = -(-3y^2 + \phi'(x))$$

$$\phi'(x) = 3x^2 \Rightarrow \phi(x) = x^3 + C$$

$$v = -3xy^2 + x^3 + C$$