

Taylor's and Laurent's Series

Taylor Series

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C dz \frac{f(z)}{(z-z_0)^{n+1}}$$

$$f(z) = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{z'-z}$$

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \frac{f(z)}{z-z_0}$$

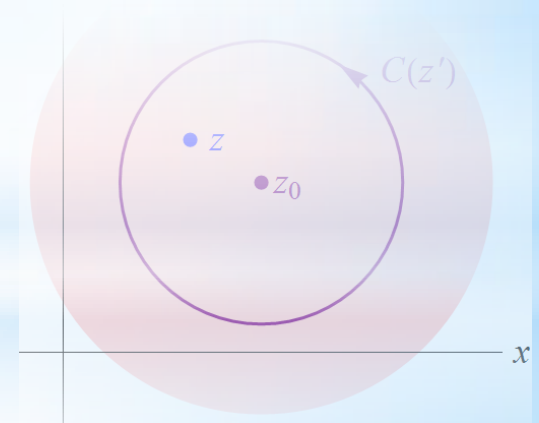
$$\frac{1}{z'-z} = \frac{1}{z'-z_0 - (z-z_0)} = \frac{1}{z'-z_0} \left(1 - \frac{z-z_0}{z'-z_0} \right)^{-1} = \frac{1}{z'-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{z'-z_0} \right)^n$$

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \oint_C dz' \frac{f(z')}{(z'-z)^{n+1}}$$

$$\rightarrow f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0)$$

Taylor series

(f analytic in $R \supset C$)



Let z_1 be the closest singularity from z_0 , then the **radius of convergence** is $|z_1 - z_0|$.

Engineering Mathematics III
i.e., series converges for $|z - z_0| < |z_1 - z_0|$

Laurent Series

Let f be analytic within an annular region

$$r \leq |z - z_0| \leq R$$

$$\rightarrow f(z) = \frac{1}{2\pi i} \left[\oint_{C_1} - \oint_{C_2} \right] dz' \frac{f(z')}{z' - z}$$

$$C_1: \frac{1}{z' - z} = \frac{1}{z' - z_0 - (z - z_0)} = \frac{1}{z' - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0} \right)^n$$

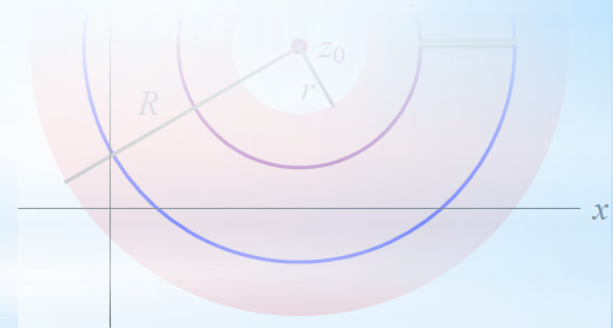
$$C_2: \frac{1}{z' - z} = -\frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{z' - z_0}{z - z_0} \right)^n$$


\rightarrow

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} dz' \frac{f(z')}{(z' - z_0)^{n+1}} + \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{(z - z_0)^{n+1}} \oint_{C_2} dz' (z' - z_0)^n f(z')$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C dz \frac{f(z)}{(z - z_0)^{n+1}}$$

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \frac{f(z)}{z - z_0}$$




$$a_n = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{(z' - z_0)^{n+1}}$$

C within f 's region of analyticity

Laurent Expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{(z' - z_0)^{n+1}}$$

$$f(z) = \frac{1}{z(z-1)}$$

$$z_0 = 0$$

$$0 < |z| < 1$$

Expansion via binomial theorem :

$$f(z) = -\left(\frac{1}{z} + \frac{1}{1-z}\right) = -\left(\frac{1}{z} + \sum_{n=0}^{\infty} z^n\right)$$

Laurent series :

$$a_n = \frac{1}{2\pi i} \oint_C dz \frac{1}{z^{n+1} z(z-1)} = -\frac{1}{2\pi i} \sum_{k=0}^{\infty} \oint_C dz \frac{z^k}{z^{n+2}} = \begin{cases} -1 & n \geq -1 \\ 0 & \text{otherwise} \end{cases}$$

$$\rightarrow f(z) = -\sum_{n=-1}^{\infty} z^n$$