

# Complex Fourier Transforms

# Complex Fourier transform

- Consider the expression

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} F_n e^{in\omega_0 t} = \sum_{n=-\infty}^{\infty} F_n \cos(n\omega_0 t) + iF_n \sin(n\omega_0 t) \\ &= \sum_{n=0}^{\infty} (F_n + F_{-n}) \cos(n\omega_0 t) + i(F_n - F_{-n}) \sin(n\omega_0 t) \end{aligned}$$

- So  $a_n = F_n + F_{-n}$  and  $b_n = i(F_n - F_{-n})$

- Since  $a_n$  and  $b_n$  are real, we can let  $F_{-n} = \overline{F_n}$

and get  $a_n = 2\operatorname{Re}(F_n)$  and  $b_n = -2\operatorname{Im}(F_n)$

$$\operatorname{Re}(F_n) = \frac{a_n}{2} \quad \text{and} \quad \operatorname{Im}(F_n) = -\frac{b_n}{2}$$

■ Thus

$$\begin{aligned} F_n &= \frac{1}{T} \left( \int_{t_0}^{t_0+T} f(t) \cos(n\omega_0 t) dt - i \int_{t_0}^{t_0+T} f(t) \sin(n\omega_0 t) dt \right) \\ &= \frac{1}{T} \int_{t_0}^{t_0+T} f(t) (\cos(n\omega_0 t) dt - i \sin(n\omega_0 t)) dt \\ &= \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-in\omega_0 t} dt \\ &= |F_n| e^{i\varphi_n} \end{aligned}$$

■ So you could also write  $f(t) = \sum_{n=-\infty}^{\infty} |F_n| e^{i(n\omega_0 t + \varphi_n)}$

The Fourier transform  $G(k)$  and the original function  $g(x)$  are both in general complex.

$$\mathfrak{F}\{g(x)\} = G_r(k) + iG_i(k)$$

The Fourier transform can be written as,

$$\mathfrak{F}\{g(x)\} = G(k) = A(k)e^{i\Theta(k)}$$

$$A = |G| = \sqrt{G_r^2 + G_i^2}$$

$A \equiv$  amplitude spectrum, or magnitude spectrum

$\Theta \equiv$  phase spectrum

$$A^2 = |G|^2 = G_r^2 + G_i^2 \equiv \text{power spectrum}$$