Introduction

Jean Baptiste Joseph Fourier
(Mar21st 1768 –May16th 1830)
French mathematician, physicist

Main Work:
Théorie analytique de la chaleur
(The Analytic Theory of Heat)

- Any function of a variable, whether continuous or discontinuous, can be expanded in a series of sines of multiples of the variable (Incorrect)
- The concept of dimensional homogeneity in equations
- Proposal of his partial differential equation for conductive diffusion of heat

Discovery of the "greenhouse effect"

Fourier Transform

Transition from Fourier integral to Fourier transform

$$f(x) = \int_0^{\infty}[a(\omega)\cos\omega x + b(\omega)\sin\omega x]d\omega. \quad (1a)$$

Where $a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\, dx, \qquad (1b)$

$$b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\, dx.$$

Put (1b) into (1a):

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\{\int_{-\infty}^{\infty} f(\xi)[cos\omega\xi cos\,\omega x + sin\omega\xi sin\omega x]d\xi\}d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(\xi)cos\omega(\xi - x)d\xi d$$

Since cos(A-B)=cosAcosB+sinAsinB and introduce complex exponentials:

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(\xi)\frac{e^{i\omega(\xi-x)}+e^{-i\omega(\xi-x)}}{2}\mathrm{d}\xi d\omega$$
$$=\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(\xi)e^{i\omega(\xi-x)}\mathrm{d}\xi d\omega+\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(\xi)e^{-i\omega(\xi-x)}\mathrm{d}\xi d\omega$$

To combine the two terms on the right-hand side, let us change the dummy integration variable from $\omega$to $-\omega$:

$$f(x)=\frac{1}{2\pi}\int_{0}^{-\infty}\int_{-\infty}^{\infty}f(\xi)e^{-i\omega(\xi-x)}\mathrm{d}\xi(-d\omega)$$
$$+\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(\xi)e^{-i\omega(\xi-x)}\mathrm{d}\xi d\omega$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}f(\xi)e^{-i\omega\xi}\mathrm{d}\xi]e^{i\omega x}d\omega$$

Thus,

$f(x) = a\int_{-\infty}^{\infty} c(\omega)e^{i\omega \mathrm{x}}d\omega \; c(\omega) = b\int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi}\mathrm{d}\xi$ ab=1/2π

There is no longer a need to distinguish x and ξ, so:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(\omega)e^{i\omega \mathrm{x}}d\omega$$
$$c(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\mathrm{d}x.$$

Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(\omega)e^{i\omega\mathrm{x}}d\omega, \quad (1)$$

$$c(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\mathrm{d}x. \quad (2)$$

They can be a transform pair: (2) defines the **Fourier transform**, c(w), of the given function *f*(x), and (1) is called the **inversion formula**.

$$F\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} c(\omega)e^{-i\omega\mathrm{x}}d\mathrm{x},$$

$$F^{-1}\{\hat{f}(\omega)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega\mathrm{x}}d\omega.$$

Example:

Derive the result $F\left\{e^{-a|x|}\right\} = \dfrac{2a}{\omega^2 + a^2}$ (a>0)

Solution:

According to the definition $F\left\{f(x)\right\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

Then

$$F\left\{e^{-a|x|}\right\} = \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\omega x} dx = \int_{-\infty}^{0} e^{ax} e^{-i\omega x} dx + \int_{0}^{\infty} e^{-ax} e^{-i\omega x} dx = \frac{1}{a - i\omega} + \frac{1}{a + i\omega} = \frac{2a}{a + \omega^2}$$

Properties and applications

1.Linearity of the transform and its inverse.

$F\{af+bg\}=aF\{f\}+bF\{g\}$

$F^{-1}\{a\hat{f}+b\hat{g}\}=aF^{-1}\{\hat{f}\}+bF^{-1}\{\hat{g}\}$

2.Transform of *n*th derivative.

$F\left\{f^{(n)}(x)\right\} = (i\omega)^n \hat{f}(\omega).$

3.Fourier convolution.

$(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi.$

Then the Fourier convolution theorem:

$F\{f * g\} = \hat{f}(\omega)\hat{g}(\omega)$ and $F^{-1}\{\hat{f}\hat{g}\} = f * g$

4. Translation formulas, x-shift and ω-shift

$$F\{f(x-a)\} = e^{-ia\omega}\hat{f}(\omega)$$
$$F^{-1}\{\hat{f}(\omega - a)\} = e^{ia\omega}f(x)$$

Example:

Solve the wave equation $c^2 u_{xx} = u_{tt}$; $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$

Take the Fourier Transform of both equations. The initial condition gives

$$\hat{u}(\omega,0) = \hat{f}(\omega)$$

$$\hat{u}_t(\omega,0) = \frac{\partial}{\partial t}\hat{u}(x,t)\bigg|_{t=0} = \hat{g}(\omega)$$

And the PDE gives

$$c^2(-\omega^2 \hat{u}(\omega,t)) = \frac{\partial^2}{\partial t^2}\hat{u}(\omega,t)$$

Which is basically an ODE in t, we can write it as

$$\frac{\partial^2}{\partial t^2}\hat{u}(\omega,t) + c^2\omega^2 \hat{u}(\omega,t) = 0$$

Which has the solution, and derivative

$$\hat{u}(\omega,t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t$$

$$\frac{\partial}{\partial t}\hat{u}(\omega,t) = -c\omega A(\omega)\sin c\omega t + c\omega B(\omega)\cos c\omega t$$

So the first initial condition gives $A(\omega)=\hat{f}(\omega)$ and the second gives

$c\omega B(\omega)=\hat{g}(\omega)$ and make the solution

$$\hat{u}(\omega,t)=\hat{f}(\omega)\cos c\omega t+\frac{\hat{g}(\omega)}{\omega}\frac{\sin c\omega t}{c}$$

Let's first look at

$$\hat{f}(\omega)\cos c\omega t=\hat{f}(\omega)\left(\frac{e^{i\omega ct}+e^{-i\omega t}}{2}\right)=\frac{1}{2}\left(\hat{f}(\omega)e^{ic\omega t}+\hat{f}(\omega)e^{-i\omega ct}\right)$$

Then

$$F^{-1}[\hat{f}(\omega)\cos c\omega t]=\frac{1}{2}(f(x+ct)+f(x-ct))$$

The second piece

$$\frac{\hat{g}(\omega)}{\omega}\frac{\sin c\omega t}{c}=\frac{\hat{g}(\omega)}{i\omega}\frac{\sin c\omega t}{-ic}$$

And now the first factor looks like an integral, as a derivative with respect to x would cancel the iw in bottom. Define:

$$h(x) = \int_{s=0}^{x} g(s)ds$$

By fundamental theorem of calculus

$$h'(x) = g(x) \qquad \hat{g}(\omega) = i\omega \hat{h}(\omega)$$

So $$\frac{\hat{g}(\omega)}{\omega}\frac{\sin c\omega t}{c} = \hat{h}(\omega)\left(\frac{e^{ic\omega t} - e^{-ic\omega t}}{2i}\right)\frac{1}{-ic} = \frac{1}{2c}\left(\hat{h}(\omega)e^{ic\omega t} - \hat{h}(\omega)e^{-ic\omega t}\right)$$

$$F^{-1}\left[\frac{1}{\omega c}\hat{g}(\omega)\sin c\omega t\right] = \frac{1}{2c}(h(x+ct) - h(x-ct)) = \frac{1}{2c}\left(\int_0^{x+ct} g(s)ds - \int_0^{x-ct} g(s)ds\right) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

Putting both piece together we get the solution

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

Cosine and sine transforms

1.Fourier cosine transform

$$F_C\{f(x)\} = \hat{f}_C(\omega) = \int_0^\infty f(x) cos \omega x dx,$$

And its inverse:

$$F_C^{-1}\{\hat{f}_C(\omega)\} = f(x) = \frac{2}{\pi}\int_0^\infty \hat{f}_C(\omega)\, cos \omega x\, d\omega.$$

2.Fourier sine transform

$F_S\{f(x)\} = \hat{f}_S(\omega) = \int_0^\infty f(x) sin\omega x dx,$

And its inverse:

$$F_S^{-1}\{\hat{f}_S(\omega)\} = f(x) = \frac{2}{\pi}\int_0^\infty \hat{f}_S(\omega)\, sin\, \omega x\, d\omega$$

Properties:

$F_C\{f'(x)\} = \omega\hat{f}_S(\omega) - f(0)$

$F_S\{f'(x)\} = -\omega\hat{f}_C(\omega).$

$F_C\{f''(x)\} = -\omega^2\hat{f}_C(\omega) - f'(0).$

$F_S\{f''(x)\} = -\omega^2\hat{f}_S(\omega) + \omega f(0)$

Example:

Solve heat transfer equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ B.C: (1) u(0,t)=0

(2)u(x,0)=P(x) $1 \le x \le 2$ or P(x)=1,

Solution with Fourier Sine Transform:

$$F_s\{u''-u'\} = F_s\{0\}$$

$$F_s\{u''\} = F_s\{u'\}$$

$$-\omega^2 \hat{f}_s + \omega f_0 = -\omega \hat{f}_c(\omega) = \frac{\partial \hat{f}_s}{\partial t}$$

According to the B.C, we can get

$$f_0 = 0 \qquad -\omega^2 \hat{f}_s(\omega,t) = \frac{\partial \hat{f}_s(\omega,t)}{\partial t} \qquad \hat{f}_s(\omega,t) = \hat{f}_s(\omega,0)\exp(-\omega^2 t)$$

Then $\hat{f}_s(\omega,0) = (2/\pi)\int_0^\infty u(x,0)\sin(\omega x)dx = (2/\pi)\int_0^\infty P(x)\sin(\omega x)dx = (2/\omega\pi)[\cos\omega - \cos 2\omega]$

$$\hat{f}_s(\omega,t) = \hat{f}_s(\omega,0)\exp(-\omega^2 t) = (2/\omega\pi)[\cos\omega - \cos 2\omega]\exp(-\omega^2 t)$$

Inverse $\hat{f}_s(\omega,t)$ Gives the complete solution

$$u(x,t) = \int_0^\infty \hat{f}_s(\omega,t)\sin(\omega t)d\omega = \int_0^\infty (2/\omega\pi)\left(\cos(\omega) - \cos(2\omega)\right)\exp(-\omega^2 t)\sin(\omega x)d\omega$$

Fourier Transform of the Unit-Step Function

- Since

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

using the integration property, it is

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

Properties of the Fourier Transform - Summary

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Right or left shift in time	$x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by a power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Multiplication by a complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by sin $\omega_0 t$	$x(t) \sin \omega_0 t \leftrightarrow \frac{j}{2}[X(\omega + \omega_0) - X(\omega - \omega_0)]$
Multiplication by cos $\omega_0 t$	$x(t) \cos \omega_0 t \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in the time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)\, d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
Convolution in the time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in the time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)\, d\omega$
Special case of Parseval's theorem	$\int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\, d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Example :

The property of Fourier transform of derivatives can be used for solution of differential equations:

$$y' - 4y = H(t)e^{-4t}$$

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$F\{y'\} - 4F\{y\} = F\{H(t)e^{-4t}\} = \frac{1}{4 + iw}$$

Setting *F{y(t)}*=*Y(w)*, we have

$$iwY(w) - 4Y(w) = \frac{1}{4 + iw}$$

Example :

Then

$$Y(w) = \frac{1}{(4+iw)(-4+iw)} = -\frac{1}{16+w^2}$$

Therefore

$$y(w) = F^{-1}\{Y(w)\} = -\frac{1}{8}e^{-4|t|}$$