Kurtosis

Introduction

The usual kurtosis measure is

$$\beta_2(F) = \frac{\mu_4(F)}{\mu_2^2(F)}$$

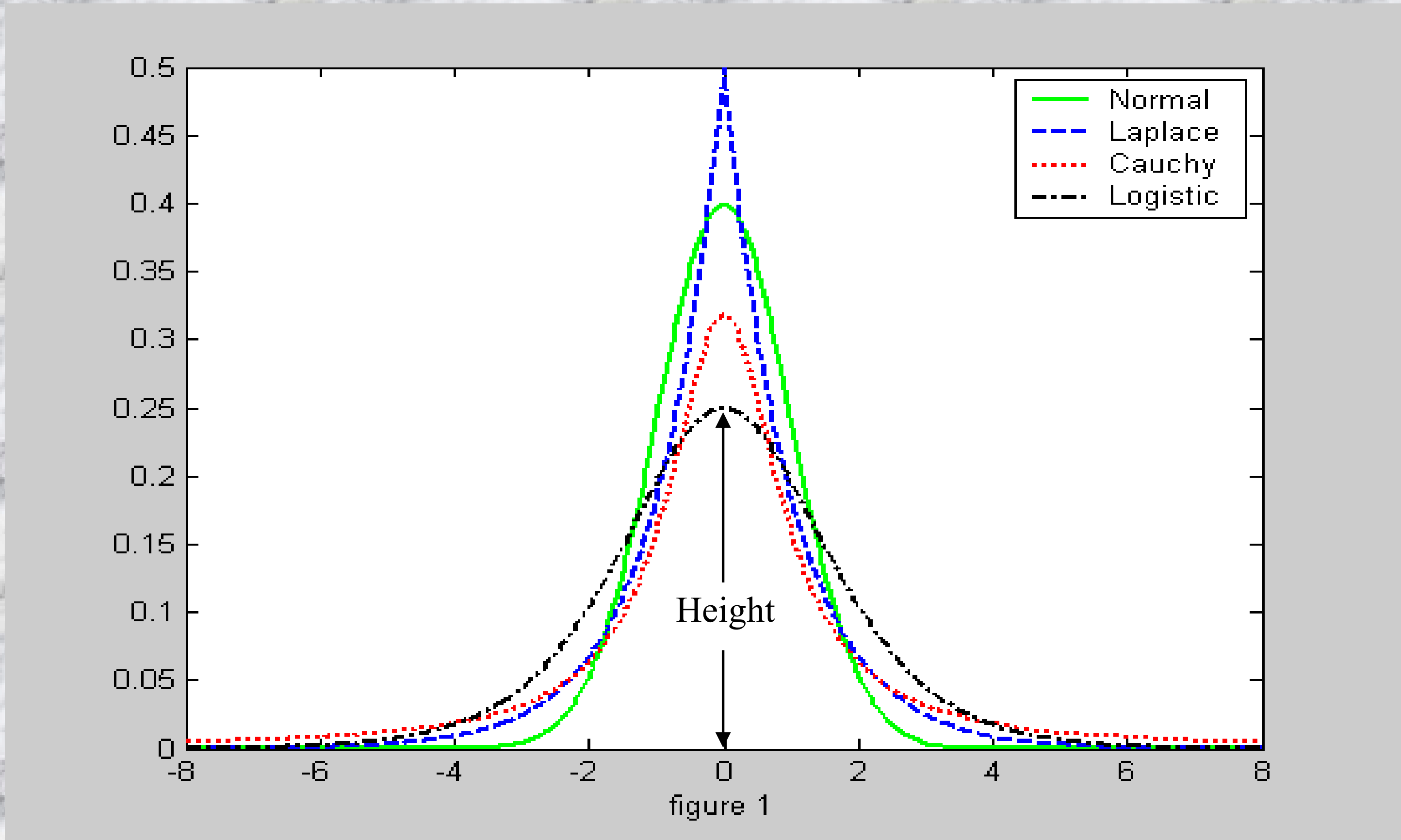
where :

$$\mu_k(F) = E_F\left(X - E_F(x)\right)^k$$

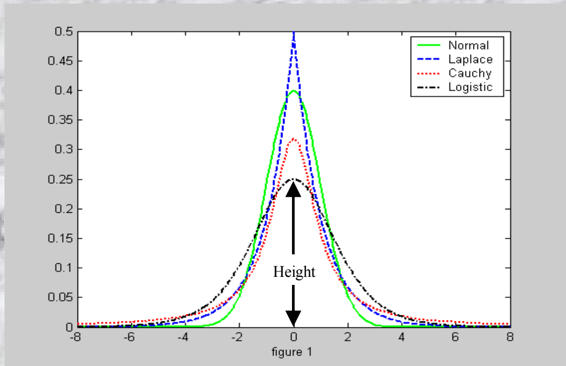
with X$\sim F(.)$.

What is **kurtosis** ?

β_2 does not sort the distributions based on the height



What is **kurtosis** ?



We compute β_2 for some distributions, Uniform, Normal, Logistic, Laplace, and Cauchy for the above distributions the kurtosis Measures, β_2 ,are 1.8, 3, 4.2, 6, and ∞ respectively.

What is **kurtosis** ?

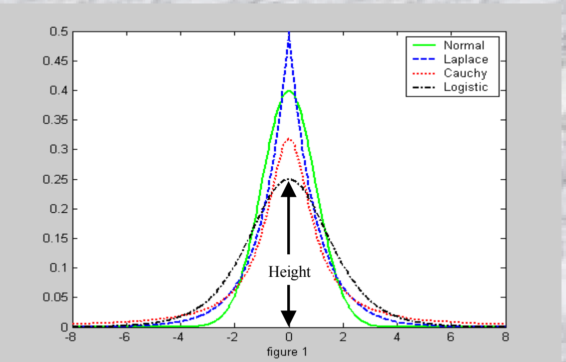


figure 1

As the height of distributions increases, the order of distributions will be Logistic, Cauchy, Normal, Laplace, and Uniform respectively.

Disadvantages of β_2

1- It's infinite for heavy tail distributions.

2- It doesn't work well for some distributions such as Ali's scale contaminated normal distributions.

$$F_k\left(x\right) = \left(1 - \frac{1}{k^2 - 1}\right)\Phi\left(x\right) + \frac{1}{k^2 - 1}\Phi\left(\frac{x}{k}\right), \quad k = 2, 3, \cdots$$

where $\Phi\left(x\right)$ is the standard normal distribution function.

$$\beta_2 = 3(k^2+1)/4 \to \infty$$

when $k \to \infty$

But this sequence converges in distribution to the standard normal distribution as $k \to \infty$.

3- It can be misleading as a departure from normality.

$\beta_2 = 3$ is not a sufficient condition for normality.

A Modified Measure of Kurtosis

$$\beta_p^q(F) = \frac{E_F[(X - E_F(x))I_{(p,q)}(X)]^4}{E_F^2[(X - E_F(X))^2 I_{(p,q)}(X)]}$$

Where p and q are quantile of order p and q, respectively

with X$\sim F(.)$.

Properties of β_p^q

P.1. $\lim_{p\to 0} \beta_{F^{-1}(p)}^{F^{-1}(1-p)} = \beta_2$ If f(x) > 0 for all x.

P.2. $\lim_{p\to \frac{1}{2}} \beta_{F^{-1}(p)}^{F^{-1}(1-p)} = 0,$

P.3. $\beta_a^b(F_k) = \beta_a^b(\Phi(x))$

$k \to \infty$

Proof:

since $$\beta_a^b(F_k(x)) = \frac{\left(1-\frac{1}{k^2-1}\right)\int_a^b x^4\phi(x)\,dx + A(k)}{\left(\left(1-\frac{1}{k^2-1}\right)\int_a^b x^2\phi(x)\,dx + B(k)\right)^2}$$

where A(k) = $\frac{k^5}{k(k^2-1)}\int_{\frac{b}{k}}^{\frac{a}{k}} x^4\phi(x)\,dx$ And B(k) = $\frac{k^3}{k(k^2-1)}\int_{\frac{b}{k}}^{\frac{a}{k}} x^2\phi(x)\,dx$

Lim A(k) = lim B(k) = 0 as $k \to \infty$

We show that the treatment of β_p^q is as the same as β_2

Properties of kurtosis measure

Oja (1981) says a location and scale invariant functionl T can be named a kurtosis measure if $T(G) \geq T(F)$ whenever G has at least as much Kurtosis as F according to the definition of relative kurtosis.

T is a kurtosis measure if :

1- It must be location and scale invariant i.e.

$$T(ax+b) = T(x) \quad \text{for } a > 0$$

2- It must preserve one of the orderings.

Ordering << were defined in such a way that F<< G means, in some location and scale free, that G has at least as much mass in the center and tails as F i.e.

$$\text{if } F <_s G \text{ then } T(F) <= T(G)$$