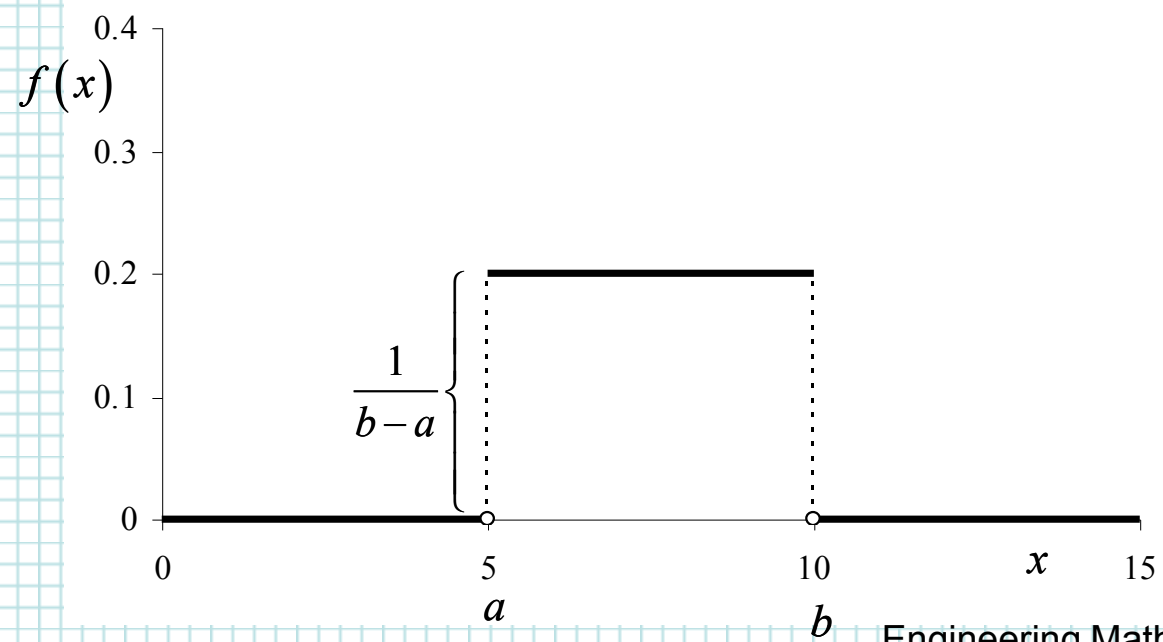


MOMENT GENERATING FUNCTIONS

Continuous Distributions

The Uniform distribution from a to b

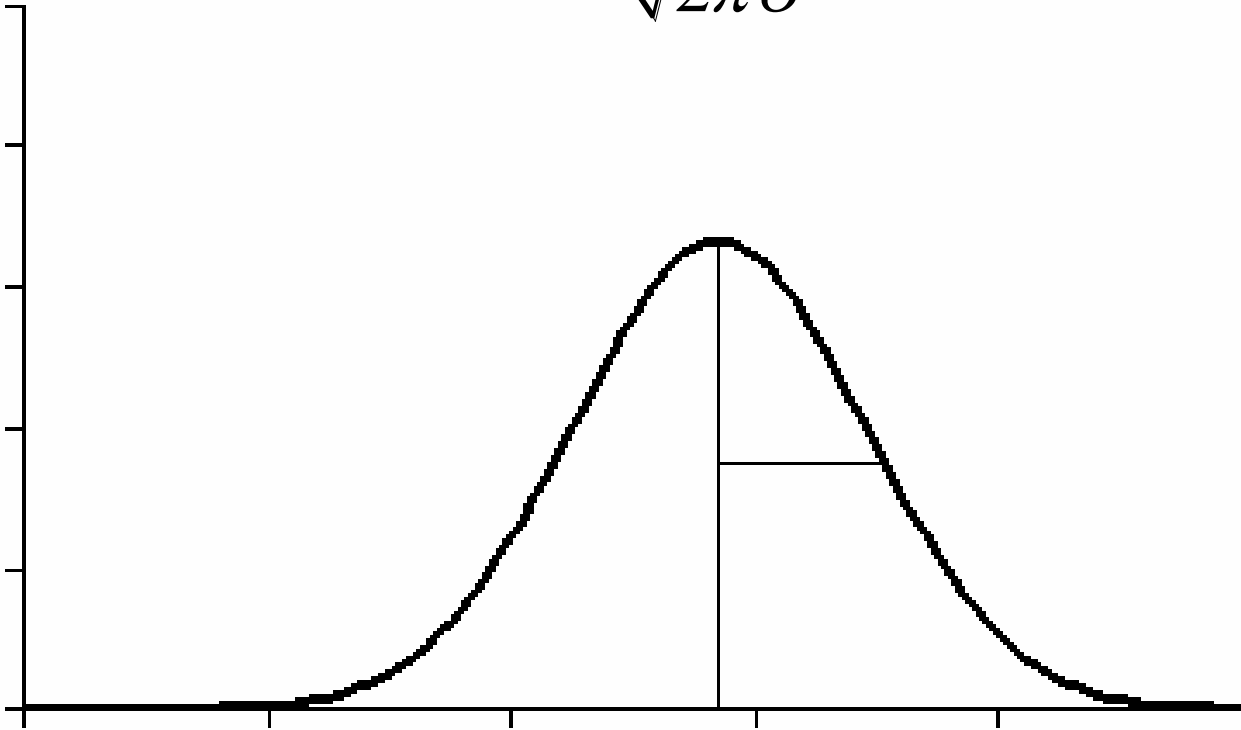
$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$



The Normal distribution

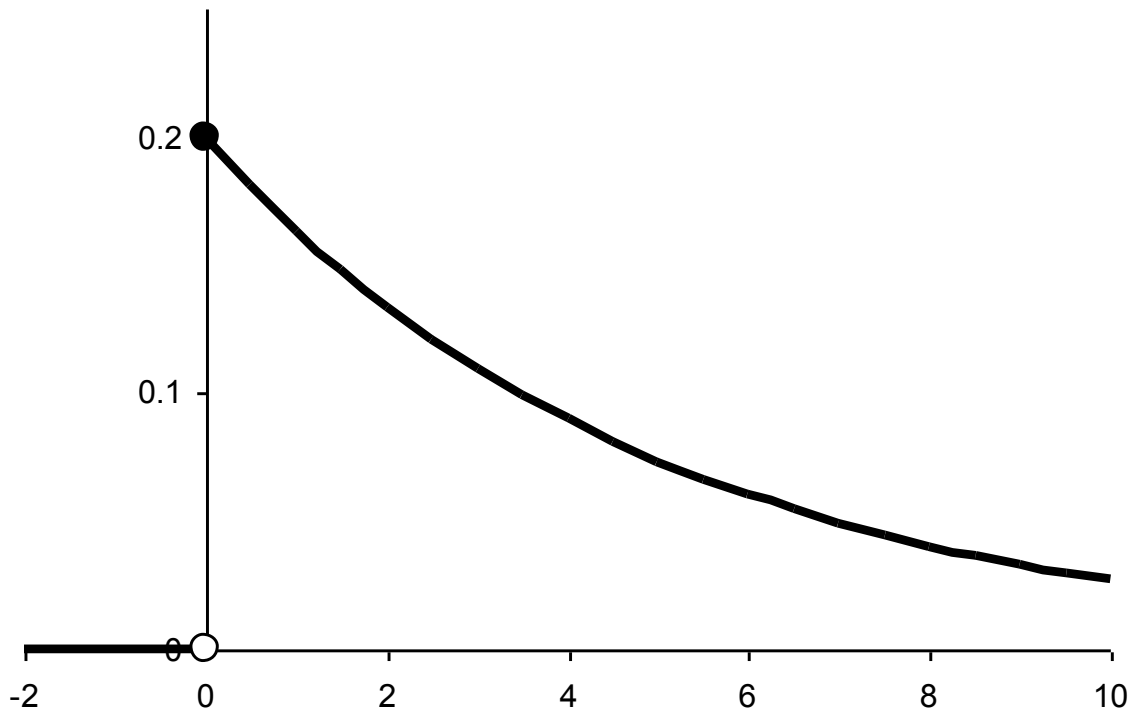
(mean μ , standard deviation σ)

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



The Exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



Weibull distribution with parameters α and β .

$$F(x) = 1 - e^{-\frac{\alpha}{\beta}x^\beta}$$

$$f(x) = F'(x) = \alpha x^{\beta-1} e^{-\frac{\alpha}{\beta}x^\beta} \quad x \geq 0$$

The Gamma distribution

Let the continuous random variable X have density function:

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Then X is said to have a **Gamma distribution** with parameters α and λ .

Expectation of functions of Random Variables

X is discrete

$$E[g(X)] = \sum_x g(x) p(x) = \sum_i g(x_i) p(x_i)$$

X is continuous

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

Moment generating functions

Moment Generating function of a R.V. X

$$m_X(t) = E[e^{tX}] = \begin{cases} \sum_x e^{tx} p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

Examples

1. The Binomial distribution (parameters p, n)

$$m_X(t) = E[e^{tX}] = \sum_x e^{tx} p(x)$$

$$= \sum_{x=0}^n e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^n \binom{n}{x} (e^t p)^x (1-p)^{n-x} = \sum_{x=0}^n \binom{n}{x} a^x b^{n-x}$$

$$= (a+b)^n = (e^t p + 1 - p)^n$$

2. The Poisson distribution (parameter λ)

$$p(x) = \frac{\lambda^x}{x!} e^{-\lambda} \quad x = 0, 1, 2, \dots$$

The moment generating function of X , $m_X(t)$ is:

$$\begin{aligned} m_X(t) &= E[e^{tX}] = \sum_x e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x}{x!} e^{-\lambda} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} \quad \text{using } e^u = \sum_{x=0}^{\infty} \frac{u^x}{x!} \\ &= e^{\lambda(e^t - 1)} \end{aligned}$$

3. The Exponential distribution (parameter λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The moment generating function of X , $m_X(t)$ is:

$$\begin{aligned} m_X(t) &= E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx \\ &= \int_0^{\infty} \lambda e^{(t-\lambda)x} dx = \left[\lambda \frac{e^{(t-\lambda)x}}{t-\lambda} \right]_0^{\infty} \\ &= \begin{cases} \frac{\lambda}{\lambda-t} & t < \lambda \\ \text{undefined} & t \geq \lambda \end{cases} \end{aligned}$$

4. The Standard Normal distribution ($\mu = 0, \sigma = 1$)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The moment generating function of X , $m_X(t)$ is:

$$\begin{aligned} m_X(t) &= E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx \\ &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-2tx}{2}} dx \end{aligned}$$

We will now use the fact that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi a}} e^{-\frac{(x-b)^2}{2a^2}} dx = 1 \quad \text{for all } a > 0, b$$

We have completed the square

$$m_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-2tx}{2}} dx = e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-2tx+t^2}{2}} dx$$

$$= e^{\frac{t^2}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} dx}_{\text{This is 1}} = e^{\frac{t^2}{2}}$$

This is 1

4. The Gamma distribution (parameters α, λ)

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The moment generating function of X , $m_X(t)$ is:

$$\begin{aligned} m_X(t) &= E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx \\ &= \int_0^{\infty} e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx \\ &= \int_0^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx \end{aligned}$$

We use the fact

$$\int_0^{\infty} \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} dx = 1 \quad \text{for all } a > 0, b > 0$$

$$m_X(t) = \int_0^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx$$

$$= \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \underbrace{\int_0^{\infty} \frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx}_{\text{Equal to 1}} = \left(\frac{\lambda}{\lambda-t} \right)^\alpha$$

Equal to 1

Properties of Moment Generating Functions

$$1. \quad m_X(0) = 1$$

$$m_X(t) = E(e^{tX}), \text{ hence } m_X(0) = E(e^{0 \cdot X}) = E(1) = 1$$

Note: the moment generating functions of the following distributions satisfy the property $m_X(0) = 1$

i) Binomial Dist'n $m_X(t) = (e^t p + 1 - p)^n$

ii) Poisson Dist'n $m_X(t) = e^{\lambda(e^t - 1)}$

iii) Exponential Dist'n $m_X(t) = \left(\frac{\lambda}{\lambda - t} \right)$

iv) Std Normal Dist'n $m_X(t) = e^{\frac{t^2}{2}}$

v) Gamma Dist'n $m_X(t) = \left(\frac{\lambda}{\lambda - t} \right)^\alpha$

$$2. \quad m_X(t) = 1 + \mu_1 t + \frac{\mu_2}{2!} t^2 + \frac{\mu_3}{3!} t^3 + \dots + \frac{\mu_k}{k!} t^k + \dots$$

We use the expansion of the exponential function:

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots + \frac{u^k}{k!} + \dots$$

$$\begin{aligned} m_X(t) &= E\left(e^{tX}\right) \\ &= E\left(1 + tX + \frac{t^2}{2!} X^2 + \frac{t^3}{3!} X^3 + \dots + \frac{t^k}{k!} X^k + \dots\right) \\ &= 1 + tE(X) + \frac{t^2}{2!} E(X^2) + \frac{t^3}{3!} E(X^3) + \dots + \frac{t^k}{k!} E(X^k) + \dots \\ &= 1 + t\mu_1 + \frac{t^2}{2!} \mu_2 + \frac{t^3}{3!} \mu_3 + \dots + \frac{t^k}{k!} \mu_k + \dots \end{aligned}$$

$$3. \quad m_X^{(k)}(0) = \left. \frac{d^k}{dt^k} m_X(t) \right|_{t=0} = \mu_k$$

Now

$$m_X(t) = 1 + \mu_1 t + \frac{\mu_2}{2!} t^2 + \frac{\mu_3}{3!} t^3 + \dots \quad \dots$$

$$m'_X(t) = \mu_1 + \frac{\mu_2}{2!} 2t + \frac{\mu_3}{3!} 3t^2 + \dots \quad \dots$$

$$= \mu_1 + \mu_2 t + \frac{\mu_3}{2!} t^2 + \dots \quad \dots$$

and $m'_X(0) = \mu_1$

$$m''_X(t) = \mu_2 + \mu_3 t + \frac{\mu_4}{2!} t^2 + \dots \quad \dots$$

and $m''_X(0) = \mu_2$

continuing we find $m_X^{(k)}(0) = \mu_k$

Property 3 is very useful in determining the moments of a random variable X .

Examples

i) Binomial Dist'n $m_X(t) = (e^t p + 1 - p)^n$

$$m'_X(t) = n(e^t p + 1 - p)^{n-1} (pe^t)$$

$$m'_X(0) = n(e^0 p + 1 - p)^{n-1} (pe^0) = np = \mu_1 = \mu$$

$$m''_X(t) = np \left[(n-1)(e^t p + 1 - p)^{n-2} (e^t p) e^t + (e^t p + 1 - p)^{n-1} e^t \right]$$

$$= npe^t (e^t p + 1 - p)^{n-2} \left[(n-1)(e^t p) + (e^t p + 1 - p) \right]$$

$$= npe^t (e^t p + 1 - p)^{n-2} \left[ne^t p + 1 - p \right]$$

$$= np [np + 1 - p] = np [np + q] = n^2 p^2 + npq = \mu_2$$

ii) Poisson Dist'n $m_X(t) = e^{\lambda(e^t-1)}$

$$m'_X(t) = e^{\lambda(e^t-1)} [\lambda e^t] = \lambda e^{\lambda(e^t-1)+t}$$

$$m''_X(t) = \lambda e^{\lambda(e^t-1)+t} [\lambda e^t + 1] = \lambda^2 e^{\lambda(e^t-1)+2t} + \lambda e^{\lambda(e^t-1)+t}$$

$$m'''_X(t) = \lambda^2 e^{\lambda(e^t-1)+2t} [\lambda e^t + 2] + \lambda e^{\lambda(e^t-1)+t} [\lambda e^t + 1]$$

$$= \lambda^2 e^{\lambda(e^t-1)+2t} [\lambda e^t + 3] + \lambda e^{\lambda(e^t-1)+t}$$

$$= \lambda^3 e^{\lambda(e^t-1)+3t} + 3\lambda^2 e^{\lambda(e^t-1)+2t} + \lambda e^{\lambda(e^t-1)+t}$$

To find the moments we set $t = 0$.

$$\mu_1 = m'_X(0) = \lambda e^{\lambda(e^0-1)+0} = \lambda$$

$$\mu_2 = m''_X(0) = \lambda^2 e^{\lambda(e^0-1)+0} + \lambda e^{\lambda(e^0-1)+0} = \lambda^2 + \lambda$$

$$\mu_3 = m'''_X(0) = \lambda^3 e^0 + 3\lambda^2 e^{0t} + \lambda e^0 = \lambda^3 + 3\lambda^2 + \lambda$$

iii) Exponential Dist'n $m_X(t) = \left(\frac{\lambda}{\lambda - t} \right)$

$$m'_X(t) = \frac{d}{dt} \left(\frac{\lambda}{\lambda - t} \right) = \lambda \frac{d(\lambda - t)^{-1}}{dt}$$

$$= \lambda(-1)(\lambda - t)^{-2}(-1) = \lambda(\lambda - t)^{-2}$$

$$m''_X(t) = \lambda(-2)(\lambda - t)^{-3}(-1) = 2\lambda(\lambda - t)^{-3}$$

$$m'''_X(t) = 2\lambda(-3)(\lambda - t)^{-4}(-1) = 2(3)\lambda(\lambda - t)^{-4}$$

$$m_X^{(4)}(t) = 2(3)\lambda(-4)(\lambda - t)^{-5}(-1) = (4!)\lambda(\lambda - t)^{-5}$$

⋮

$$m_X^{(k)}(t) = (k!)\lambda(\lambda - t)^{-k-1}$$

Thus

$$\mu_1 = \mu = m'_X(0) = \lambda(\lambda)^{-2} = \frac{1}{\lambda}$$

$$\mu_2 = m''_X(0) = 2\lambda(\lambda)^{-3} = \frac{2}{\lambda^2}$$

⋮

$$\mu_k = m_X^{(k)}(0) = (k!) \lambda(\lambda)^{-k-1} = \frac{k!}{\lambda^k}$$

The moments for the exponential distribution can be calculated in an alternative way. This is done by expanding $m_X(t)$ in powers of t and equating the coefficients of t^k to the coefficients in:

$$m_X(t) = 1 + \mu_1 t + \frac{\mu_2}{2!} t^2 + \frac{\mu_3}{3!} t^3 + \dots + \frac{\mu_k}{k!} t^k + \dots$$

$$\begin{aligned} m_X(t) &= \frac{\lambda}{\lambda - t} = \frac{1}{1 - \frac{t}{\lambda}} = \frac{1}{1 - u} = 1 + u + u^2 + u^3 + \dots \\ &= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \frac{t^3}{\lambda^3} + \dots \end{aligned}$$

Equating the coefficients of t^k we get:

$$\frac{\mu_k}{k!} = \frac{1}{\lambda^k} \quad \text{or} \quad \mu_k = \frac{k!}{\lambda^k}$$

The moments for the standard normal distribution

$$m_X(t) = e^{\frac{t^2}{2}}$$

We use the expansion of e^u .

$$e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!} = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots + \frac{u^k}{k!} + \dots$$

$$\begin{aligned} m_X(t) = e^{\frac{t^2}{2}} &= 1 + \left(\frac{t^2}{2}\right) + \frac{\left(\frac{t^2}{2}\right)^2}{2!} + \frac{\left(\frac{t^2}{2}\right)^3}{3!} + \dots + \frac{\left(\frac{t^2}{2}\right)^k}{k!} + \dots \\ &= 1 + \frac{1}{2}t^2 + \frac{1}{2^2 2!}t^4 + \frac{1}{2^3 3!}t^6 + \dots + \frac{1}{2^k k!}t^{2k} + \dots \end{aligned}$$

We now equate the coefficients t^k in:

$$m_X(t) = 1 + \mu_1 t + \frac{\mu_2}{2!} t^2 + \dots + \frac{\mu_k}{k!} t^k + \dots + \frac{\mu_{2k}}{(2k)!} t^{2k} + \dots$$

If k is odd: $\mu_k = 0$.

For even $2k$:
$$\frac{\mu_{2k}}{(2k)!} = \frac{1}{2^k k!}$$

or
$$\mu_{2k} = \frac{(2k)!}{2^k k!}$$

Thus
$$\mu_1 = 0, \mu_2 = \frac{2!}{2} = 1, \mu_3 = 0, \mu_4 = \frac{4!}{2^2 (2!)} = 3$$

Summary

Moments

Moment generating functions

Moments of Random Variables

$$\mu_k = E(X^k) = \begin{cases} \sum_x x^k p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^k f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

The moment generating function

$$m_X(t) = E[e^{tX}] = \begin{cases} \sum_x e^{tx} p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

Examples

1. The Binomial distribution (parameters p, n)

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, 2, \dots$$

$$m_X(t) = (e^t p + 1 - p)^n = (e^t p + q)^n$$

2. The Poisson distribution (parameter λ)

$$p(x) = \frac{\lambda^x}{x!} e^{-\lambda} \quad x = 0, 1, 2, \dots$$

$$m_X(t) = e^{\lambda(e^t - 1)}$$

3. The Exponential distribution (parameter λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$m_X(t) = \begin{cases} \frac{\lambda}{\lambda - t} & t < \lambda \\ \text{undefined} & t \geq \lambda \end{cases}$$

4. The Standard Normal distribution ($\mu = 0, \sigma = 1$)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$m_X(t) = e^{\frac{t^2}{2}}$$

5. The Gamma distribution (parameters α, λ)

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$
$$m_X(t) = \left(\frac{\lambda}{\lambda - t} \right)^\alpha$$

6. The Chi-square distribution (degrees of freedom ν)
($\alpha = \nu/2, \lambda = 1/2$)

$$f(x) = \begin{cases} \frac{\left(\frac{1}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\frac{1}{2}x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$
$$m_X(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t} \right)^{\frac{\nu}{2}} = (1 - 2t)^{-\frac{\nu}{2}}$$

Properties of Moment Generating Functions

$$1. \quad m_X(0) = 1$$

$$2. \quad m_X(t) = 1 + \mu_1 t + \frac{\mu_2}{2!} t^2 + \frac{\mu_3}{3!} t^3 + \dots + \frac{\mu_k}{k!} t^k + \dots$$

$$3. \quad m_X^{(k)}(0) = \left. \frac{d^k}{dt^k} m_X(t) \right|_{t=0} = \mu_k$$

$$\text{i.e.} \quad m'_X(0) = \mu_1$$

$$m''_X(0) = \mu_2$$

$$m'''_X(0) = \mu_3, \quad \text{etc}$$

The log of Moment Generating Functions

Let $l_X(t) = \ln m_X(t)$ = the log of the moment generating function

Then $l_X(0) = \ln m_X(0) = \ln 1 = 0$

$$l'_X(t) = \frac{1}{m_X(t)} m'_X(t) = \frac{m'_X(t)}{m_X(t)} \quad l'_X(0) = \frac{m'_X(0)}{m_X(0)} = \mu_1 = \mu$$

$$l''_X(t) = \frac{m''_X(t) m_X(t) - [m'_X(t)]^2}{[m_X(t)]^2}$$

$$l''_X(0) = \frac{m''_X(0) m_X(0) - [m'_X(0)]^2}{[m_X(0)]^2} = \mu_2 - [\mu_1]^2 = \sigma^2$$

Thus $l_X(t) = \ln m_X(t)$ is very useful for calculating the mean and variance of a random variable

1. $l'_X(0) = \mu$

2. $l''_X(0) = \sigma^2$

Examples

1. The Binomial distribution (parameters p, n)

$$m_X(t) = (e^t p + 1 - p)^n = (e^t p + q)^n$$

$$l_X(t) = \ln m_X(t) = n \ln(e^t p + q)$$

$$l'_X(t) = n \frac{1}{e^t p + q} e^t p$$

$$\mu = l'_X(0) = n \frac{1}{p + q} p = np$$

$$l''_X(t) = n \frac{e^t p (e^t p + q) - e^t p (e^t p)}{(e^t p + q)^2}$$

$$\sigma^2 = l''_X(0) = n \frac{p(p + q) - p(p)}{(p + q)^2} = npq$$

2. The Poisson distribution (parameter λ)

$$m_X(t) = e^{\lambda(e^t - 1)}$$

$$l_X(t) = \ln m_X(t) = \lambda(e^t - 1)$$

$$l'_X(t) = \lambda e^t$$

$$\mu = l'_X(0) = \lambda$$

$$l''_X(t) = \lambda e^t$$

$$\sigma^2 = l''_X(0) = \lambda$$

3. The Exponential distribution (parameter λ)

$$m_X(t) = \begin{cases} \frac{\lambda}{\lambda - t} & t < \lambda \\ \text{undefined} & t \geq \lambda \end{cases}$$

$$l_X(t) = \ln m_X(t) = \ln \lambda - \ln(\lambda - t) \quad \text{if } t < \lambda$$

$$l'_X(t) = \frac{1}{\lambda - t} = (\lambda - t)^{-1}$$

$$l''_X(t) = -1(\lambda - t)^{-2}(-1) = \frac{1}{(\lambda - t)^2}$$

Thus $\mu = l'_X(0) = \frac{1}{\lambda}$ and $\sigma^2 = l''_X(0) = \frac{1}{\lambda^2}$

4. The Standard Normal distribution ($\mu = 0, \sigma = 1$)

$$m_X(t) = e^{\frac{t^2}{2}}$$

$$l_X(t) = \ln m_X(t) = \frac{t^2}{2}$$

$$l'_X(t) = t, \quad l''_X(t) = 1$$

Thus $\mu = l'_X(0) = 0$ and $\sigma^2 = l''_X(0) = 1$

5. The Gamma distribution (parameters α, λ)

$$m_X(t) = \left(\frac{\lambda}{\lambda - t} \right)^\alpha$$

$$l_X(t) = \ln m_X(t) = \alpha [\ln \lambda - \ln(\lambda - t)]$$

$$l'_X(t) = \alpha \left[\frac{1}{\lambda - t} \right] = \frac{\alpha}{\lambda - t}$$

$$l''_X(t) = \alpha (-1)(\lambda - t)^{-2} (-1) = \frac{\alpha}{(\lambda - t)^2}$$

Hence $\mu = l'_X(0) = \frac{\alpha}{\lambda}$ and $\sigma^2 = l''_X(0) = \frac{\alpha}{\lambda^2}$

6. The Chi-square distribution (degrees of freedom ν)

$$m_X(t) = (1 - 2t)^{-\frac{\nu}{2}}$$

$$l_X(t) = \ln m_X(t) = -\frac{\nu}{2} \ln(1 - 2t)$$

$$l'_X(t) = -\frac{\nu}{2} \frac{1}{1 - 2t} (-2) = \frac{\nu}{1 - 2t}$$

$$l''_X(t) = \nu(-1)(1 - 2t)^{-2} (-2) = \frac{2\nu}{(1 - 2t)^2}$$

Hence $\mu = l'_X(0) = \nu$ and $\sigma^2 = l''_X(0) = 2\nu$

Summary of Discrete Distributions

Name	probability function p(x)	Mean	Variance	Moment generating function M _X (t)
Discrete Uniform	$p(x) = \frac{1}{N} \quad x=1,2,\dots,N$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\frac{e^t - e^{tN-1}}{N(e^t-1)}$
Bernoulli	$p(x) = \begin{cases} p & x=1 \\ q & x=0 \end{cases}$	p	pq	q + pe ^t
Binomial	$p(x) = \binom{N}{x} p^x q^{N-x}$	Np	Npq	(q + pe ^t) ^N
Geometric	$p(x) = pq^{x-1} \quad x=1,2,\dots$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^t}{1-qe^t}$
Negative Binomial	$p(x) = \binom{x-1}{k-1} p^k q^{x-k}$ x=k,k+1,...	$\frac{k}{p}$	$\frac{kq}{p^2}$	$\left[\frac{pe^t}{1-qe^t} \right]^k$
Poisson	$p(x) = \frac{\lambda^x}{x!} e^{-\lambda} \quad x=1,2,\dots$	λ	λ	e ^{λ(e^t-1)}
Hypergeometric	$p(x) = \frac{\binom{A}{x} \binom{N-A}{n-x}}{\binom{N}{n}}$	n $\left(\frac{A}{N} \right)$	n $\left(\frac{A}{N} \right) \left(1 - \frac{A}{N} \right) \left(\frac{N-n}{N-1} \right)$	not useful

Summary of Continuous Distributions

Name	probability density function $f(x)$	Mean	Variance	Moment generating function $M_X(t)$
Continuous Uniform	$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{[b-a]t}$
Exponential	$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\left[\frac{\lambda}{\lambda-t} \right]$ for $t < \lambda$
Gamma	$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left[\frac{\lambda}{\lambda-t} \right]^\alpha$ for $t < \lambda$
χ^2 v d.f.	$f(x) = \begin{cases} \frac{(1/2)^{v/2}}{\Gamma(v/2)} x^{v/2-1} e^{-(1/2)x} & x \geq 0 \\ 0 & x < 0 \end{cases}$	v	2v	$\left[\frac{1}{1-2t} \right]^{v/2}$ for $t < 1/2$
Normal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$	μ	σ^2	$e^{t\mu+(1/2)t^2\sigma^2}$
Weibull	$f(x) = \begin{cases} \frac{\gamma}{\theta} x^{\gamma-1} e^{-x^\gamma/\theta} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\theta^{2/\gamma} \Gamma\left(\frac{\gamma+1}{\gamma}\right)$	$\theta^{2/\gamma} \left\{ \Gamma\left(\frac{\gamma+2}{\gamma}\right) - \left[\Gamma\left(\frac{\gamma+1}{\gamma}\right) \right]^2 \right\}$	not avail.