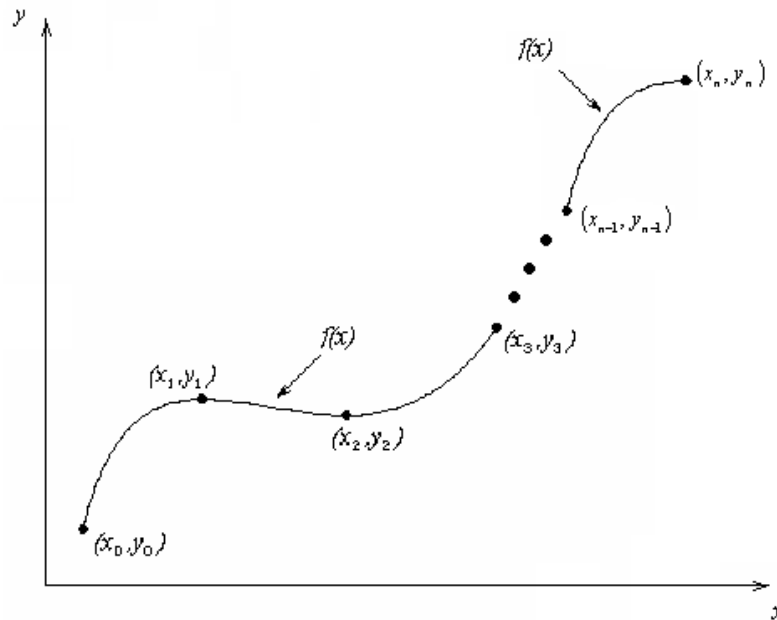


Newton's Forward and Backward Interpolation

WHAT IS INTERPOLATION?

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, finding the value of 'y' at a value of 'x' in (x_0, x_n) is called **interpolation**.



INTERPOLANTS

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate,
- Differentiate, and
- Integrate.

NEWTONS DIVIDED DIFFERENCE

What is divided difference?

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_1}$$

$$f[x_0, x_1, \dots, x_{k-1}, x_k] = \frac{f[x_1, x_2, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0}$$

for $k = 3, 4, \dots, n$.

These Ist, IInd... and kth order differences are denoted by $\Delta f, \Delta^2 f, \dots, \Delta^k f$.

INTERPOLATION USING DIVIDED DIFFERENCE

The *divided difference interpolation polynomial* is:

$$P(x) = f(x_0) + (x - x_0) f[x_0, x_1] + \Lambda + (x - x_0) \Lambda \\ (x - x_{n-1}) f[x_0, x_1, \dots, x_n]$$

Example

For the data

x:	-1	0	2	5
f(x) :	7	10	22	235

Find the divided difference polynomial and estimate $f(1)$.

Solution

X	f	Δf	$\Delta^2 f$	$\Delta^3 f$
-1	7			
0	10	3		
2	22	6	1	
5	235	71	13	2

$$\begin{aligned} P(x) &= f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \\ &\quad (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3] \\ &= 7 + (x + 1) \times 3 + (x + 1)(x - 0) \times 1 + (x + 1)(x - 0)(x - 2) \times 2 \\ &= 2x^3 - x^2 + 10 \end{aligned}$$

$$P(1) = 11$$

NEWTON FORWARD INTERPOLATION

For convenience we put $p = \frac{x - x_0}{h}$ and $f_0 = y_0$. Then we have

$$P(x_0 + ph) = y_0 + pDy_0 + \frac{p(p-1)}{2!} D^2y_0 + \frac{p(p-1)(p-2)}{3!} D^3y_0 + \dots +$$

$$\frac{p(p-1)(p-2)\dots(p-n+1)}{n!} D^ny_0$$

Example

Estimate $f(3.17)$ from the data using Newton Forward Interpolation.

x:	3.1	3.2	3.3	3.4	3.5
f(x):	0.6	1.0	1.2	1.3	

Solution

First let us form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3.1	0				
		0.6			
3.2	0.6		- 0.2		
		0.4		0	
3.3	1.0		- 0.2		0.1
		0.2		0.1	
3.4	1.2		-0.1		
		0.1			
3.5	1.3				

Here $x_0 = 3.1$, $x = 3.17$, $h = 0.1$.

Solution

$$P = \frac{X - X_0}{h} = \frac{0.07}{0.1} = 0.7$$

Newton forward formula is:

$$P(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$\begin{aligned} P(3.17) &= 0 + 0.7 \times 0.6 + \frac{0.7(0.7-1)}{2} \times (-0.2) + \frac{0.7(0.7-1)(0.7-2)}{6} \times 0 + \frac{0.7(0.7-1)(0.7-2)(0.7-3)}{24} \times 0.1 \\ &= 0.4384 \end{aligned}$$

Thus $f(3.17) = 0.4384$.

NEWTON BACKWARD INTERPOLATION FORMULA

Taking $p = \frac{x - x_n}{h}$, we get the interpolation formula as:

$$P(x_n + ph) = y_0 + p \nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots + \frac{p(p+1)(p+2)\dots(p+n-1)}{n!} \nabla^n y_n$$

Example

Estimate $f(42)$ from the following data using **newton backward interpolation**.

x:	20	25	30	35	40	45	
f(x):	354	332	332	291	260	231	204

Solution

The difference table is:

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$
20	354	- 22				
25	332	- 41	- 19			
30	291	- 31	10	29	-37	
35	260	- 29	2	- 8	8	45
40	231	- 27	2	0		
45	204					

Here $x_n = 45$, $h = 5$, $x = 42$

and $p = - 0.6$

Solution

Newton backward formula is:

$$P(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!} \nabla^5 y_n$$

$$P(42) = 204 + (-0.6)(-27) + \frac{(-0.6)(0.4)}{2} \times 2 + \frac{(-0.6)(0.4)(1.4)}{6} \times 0 + \frac{(-0.6)(0.4)(1.4)(2.4)}{24} \times 8 + \frac{(-0.6)(0.4)(1.4)(2.4)(3.4)}{120} \times 45 = 219.1430$$

Thus, $f(42) = 219.143$

INTERPOLATION USING CENTRAL DIFFERENCES

Suppose the values of the function $f(x)$ are known at the points $a - 3h, a - 2h, a - h, a, a + h, a + 2h, a + 3h, \dots$ etc. Let these values be $y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3 \dots$, and so on. Then we can form the central difference table as:

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
a-3h	y₋₃						
		Δy_{-3}					
a-2h	y₋₂		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
a-h	y₋₁		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
a	y₀		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
a+h	y₁		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
a+2h	y₂		$\Delta^2 y_1$				
		Δy_2					
a+3h	y₃						

We can relate the central difference operator δ with Δ and E using the operator relation $\delta = \Delta E^{1/2}$.