Quadric Surfaces

Quadric Surfaces

The fourth basic type of surface in space is a **quadric surface.** Quadric surfaces are the three-dimensional analogs of conic sections.

Quadric Surface

The equation of a **quadric surface** in space is a second-degree equation in three variables. The **general form** of the equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

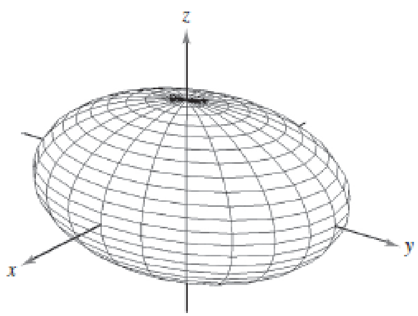
There are six basic types of quadric surfaces: **ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid, and hyperbolic paraboloid.**

Quadric Surfaces

The intersection of a surface with a plane is called the **trace of the surface** in the plane. To visualize a surface in space, it is helpful to determine its traces in some well-chosen planes.

The traces of quadric surfaces are conics. These traces, together with the **standard form** of the equation of each quadric surface, are shown in the following table.

Quadric Surfaces

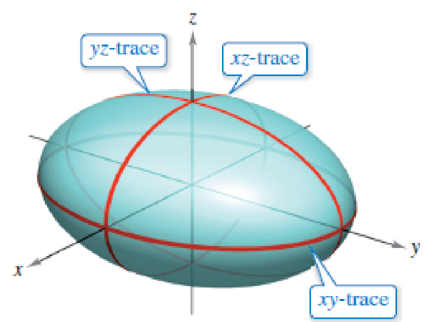


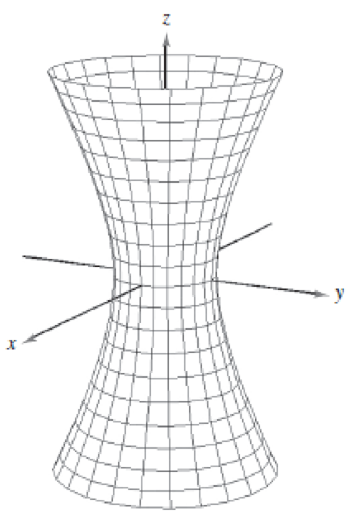
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Trace	**Plane**
Ellipse	Parallel to *xy*-plane
Ellipse	Parallel to *xz*-plane
Ellipse	Parallel to *yz*-plane

The surface is a sphere when $a = b = c \neq 0$.



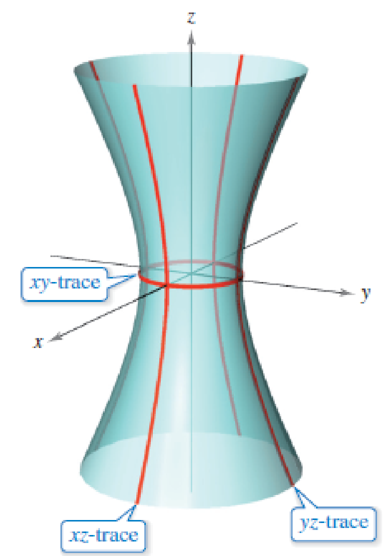


Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Trace	**Plane**
Ellipse	Parallel to *xy*-plane
Hyperbola	Parallel to *xz*-plane
Hyperbola	Parallel to *yz*-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.



Quadric Surfaces

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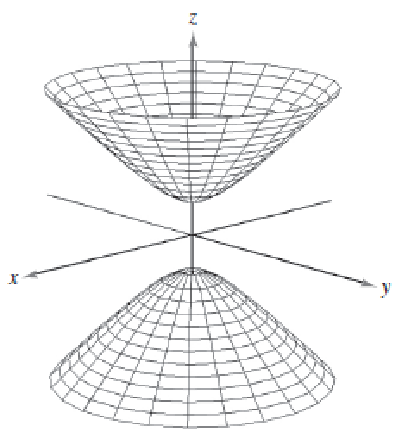
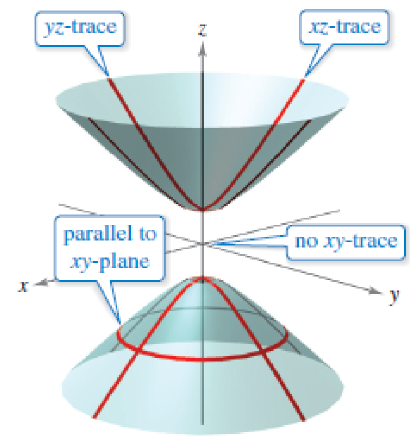


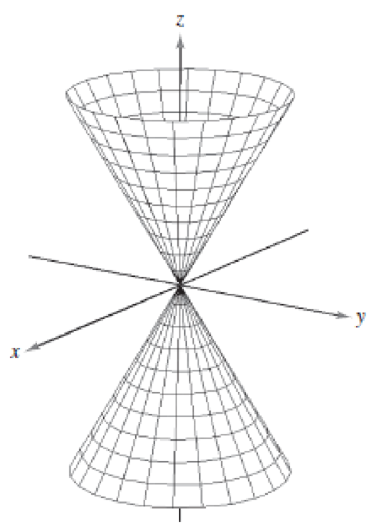
Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Trace	**Plane**
Ellipse	Parallel to *xy*-plane
Hyperbola	Parallel to *xz*-plane
Hyperbola	Parallel to *yz*-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.



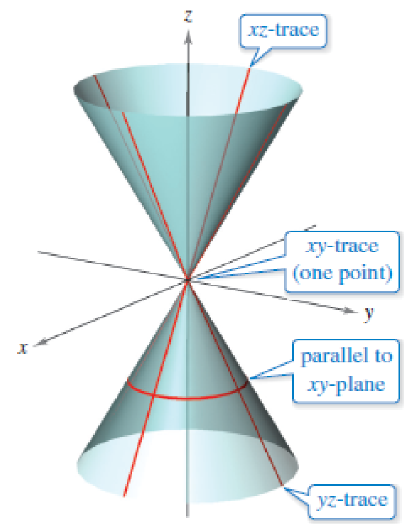


Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Trace	**Plane**
Ellipse	Parallel to *xy*-plane
Hyperbola	Parallel to *xz*-plane
Hyperbola	Parallel to *yz*-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.



Quadric Surfaces

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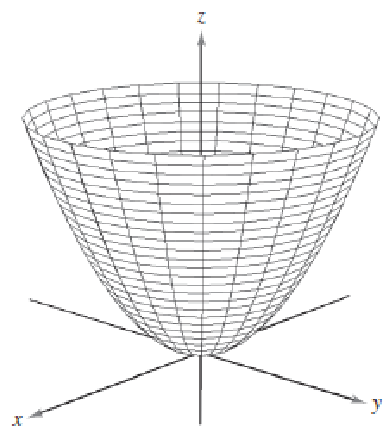
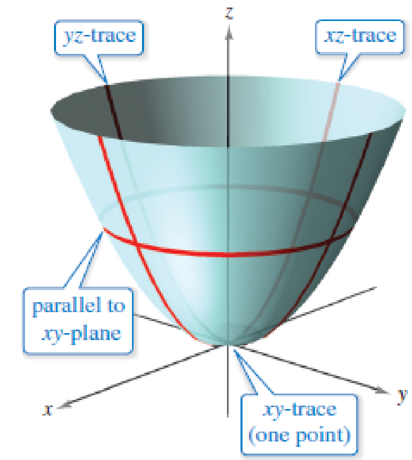


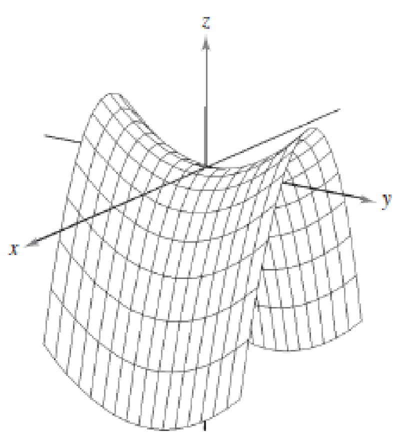
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Trace	**Plane**
Ellipse	Parallel to *xy*-plane
Parabola	Parallel to *xz*-plane
Parabola	Parallel to *yz*-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



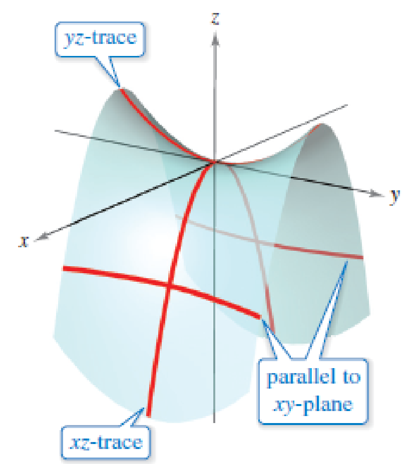


Hyperbolic Paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

Trace	**Plane**
Hyperbola	Parallel to *xy*-plane
Parabola	Parallel to *xz*-plane
Parabola	Parallel to *yz*-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



Example 2 – *Sketching a Quadric Surface*

Classify and sketch the surface given by

$4x^2 - 3y^2 + 12z^2 + 12 = 0$.

Solution:

Begin by writing the equation in standard form.

$$4x^2 - 3y^2 + 12z^2 + 12 = 0$$ Write original equation.

$$\frac{x^2}{-3} + \frac{y^2}{4} - z^2 - 1 = 0$$ Divide by -12.

$$\frac{y^2}{4} - \frac{x^2}{3} - \frac{z^2}{1} = 1$$ Standard form

You can conclude that the surface is a hyperboloid of two sheets with the *y*-axis as its axis.

Example 2 – *Solution*

To sketch the graph of this surface, it helps to find the traces in the coordinate planes.

$$xy\text{-trace } (z = 0): \quad \frac{y^2}{4} - \frac{x^2}{3} = 1 \qquad \text{Hyperbola}$$

$$xz\text{-trace } (y = 0): \quad \frac{x^2}{3} + \frac{z^2}{1} = -1 \qquad \text{No trace}$$

$$yz\text{-trace } (x = 0): \quad \frac{y^2}{4} - \frac{z^2}{1} = 1 \qquad \text{Hyperbola}$$

Example 2 – *Solution*

cont'd

The graph is shown in Figure 11.59.

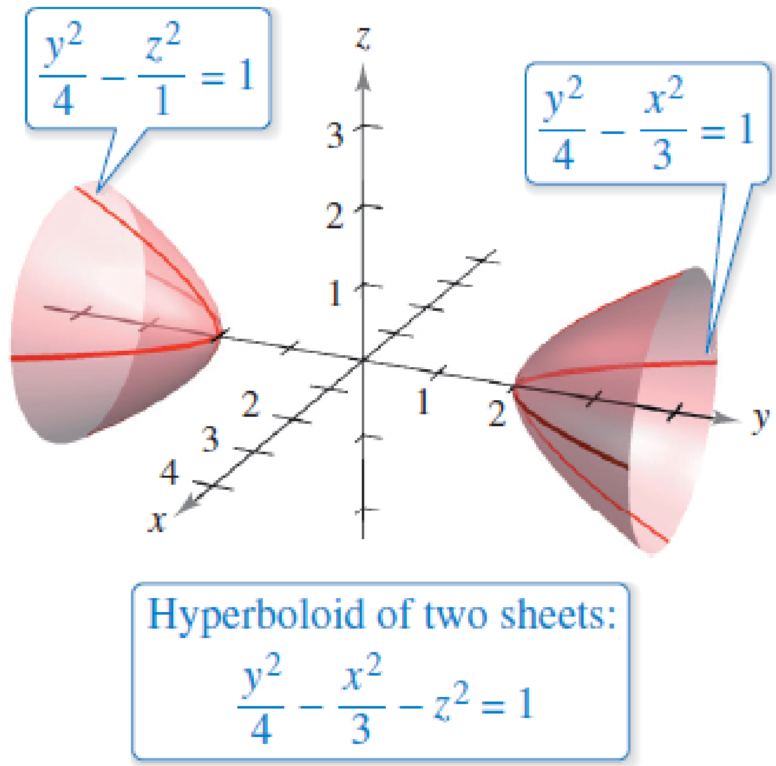


Figure 11.59

Surfaces of Revolution

Surfaces of Revolution

The fifth special type of surface you will study is called a **surface of revolution.**

You will now look at a procedure for finding its *equation.*

Consider the graph of the **radius function**

$y = r(z)$ Generating curve

in the yz-plane.

Surfaces of Revolution

If this graph is revolved about the z-axis, it forms a surface of revolution, as shown in Figure 11.62.

The trace of the surface in the plane $z = z_0$ is a circle whose radius is $r(z_0)$ and whose equation is

$x^2 + y^2 = [r(z_0)]^2$. Circular trace in plane: $z = z_0$

Replacing z_0 with z produces an equation that is valid for all values of z.

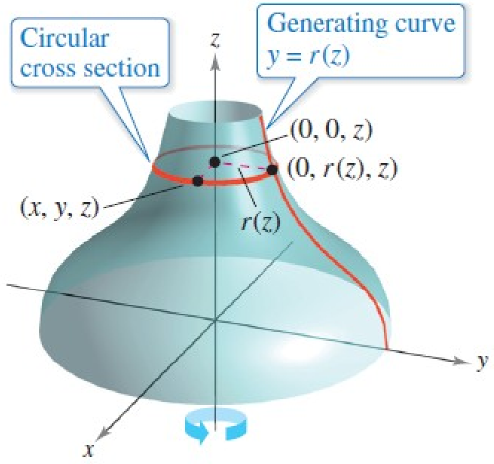


Figure 11.62

Surfaces of Revolution

In a similar manner, you can obtain equations for surfaces of revolution for the other two axes, and the results are summarized as follows.

Surface of Revolution

If the graph of a radius function r is revolved about one of the coordinate axes, then the equation of the resulting surface of revolution has one of the forms listed below.

1. Revolved about the x-axis: $y^2 + z^2 = [r(x)]^2$
2. Revolved about the y-axis: $x^2 + z^2 = [r(y)]^2$
3. Revolved about the z-axis: $x^2 + y^2 = [r(z)]^2$

Example 5 – *Finding an Equation for a Surface of Revolution*

Find an equation for the surface of revolution formed by revolving (a) the graph of $y = 1 / z$ and (b) the graph of $9x^2 = y^3$ about the y-axis.

Example 5 (a) – *Solution*

a. An equation for the surface of revolution formed by revolving the graph of

$$y = \frac{1}{z} \qquad \text{Radius function}$$

about the *z*-axis is

$$x^2 + y^2 = [r(z)]^2 \qquad \text{Revolved about the } z\text{-axis}$$

$$x^2 + y^2 = \left(\frac{1}{z}\right)^2. \qquad \text{Substitute } 1/z \text{ for } r(z).$$

Example 5 (b) – *Solution*

cont'd

b. To find an equation for the surface formed by revolving the graph of $9x^2 = y^3$ about the y-axis, solve for x in terms of y to obtain

$$x = \frac{1}{3}y^{3/2} = r(y).$$ Radius function

So, the equation for this surface is

$$x^2 + z^2 = [r(y)]^2$$ Revolved about the y-axis

$$x^2 + z^2 = \left(\frac{1}{3}y^{3/2}\right)^2$$ Substitute $\frac{1}{3}y^{3/2}$ for $r(y)$.

$$x^2 + z^2 = \frac{1}{9}y^3.$$ Equation of surface

The graph is shown in Figure 11.63.

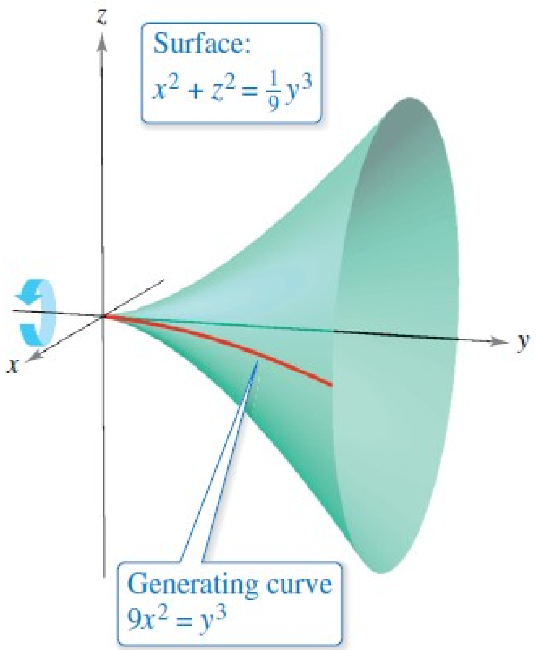


Figure 11.63