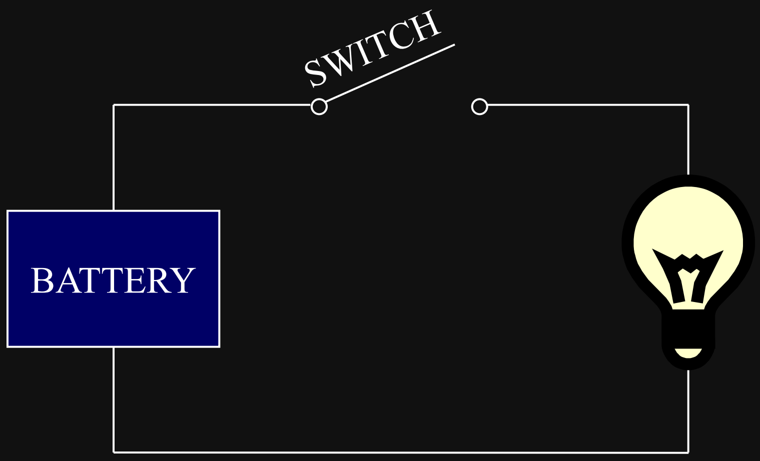
Automata theory and formal languages

What is automata theory

- Automata theory is the study of abstract computational devices
- Abstract devices are (simplified) models of real computations
- Computations happen everywhere: On your laptop, on your cell phone, in nature, …
- Why do we need abstract models?

A simple computer



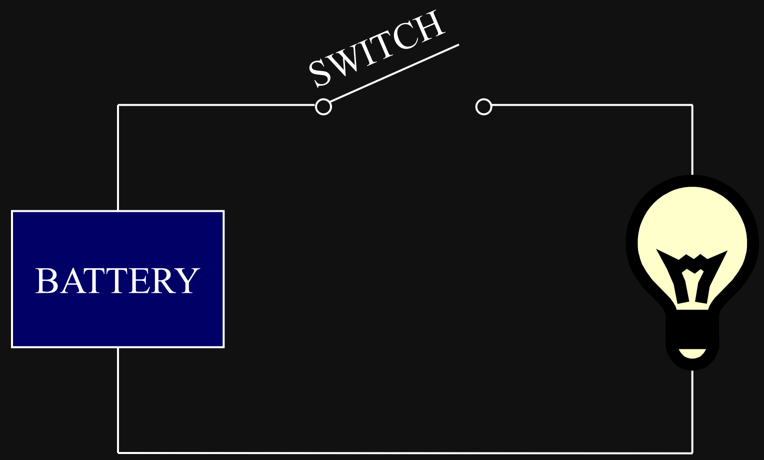
input: switch

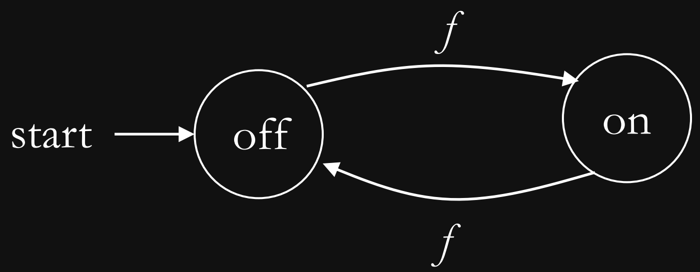
output: light bulb

actions: flip switch

states: on, off

A simple “computer”





input: switch

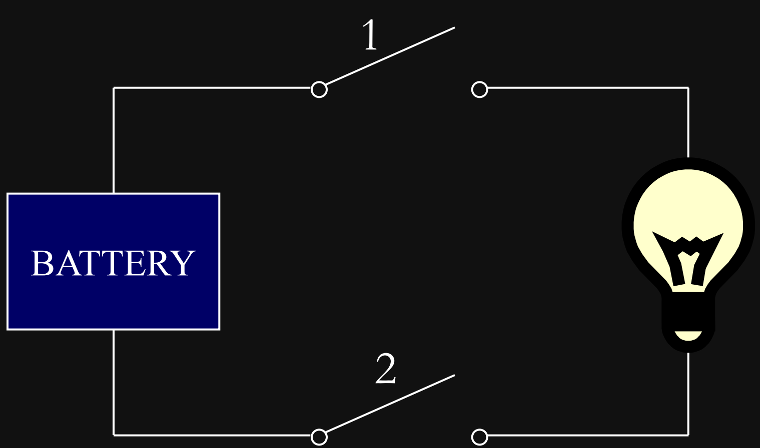
output: light bulb

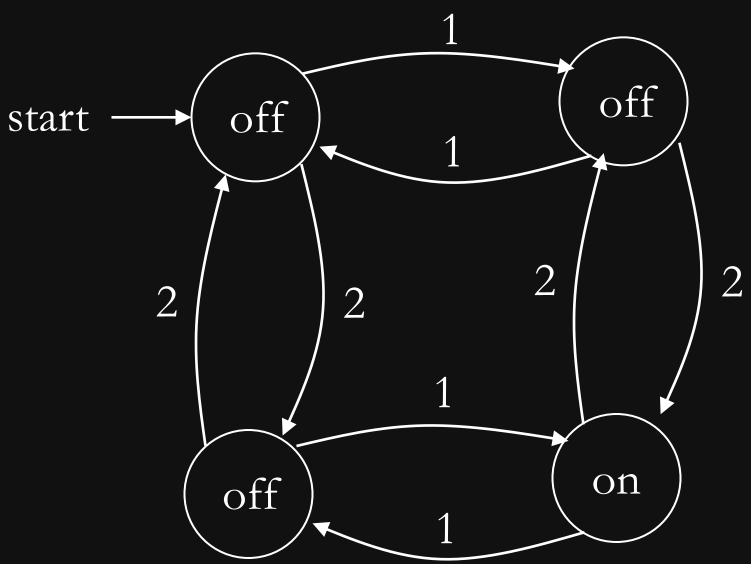
actions: f for “flip switch”

states: on, off

bulb is on if and only if there was an odd number of flips

Another “computer”





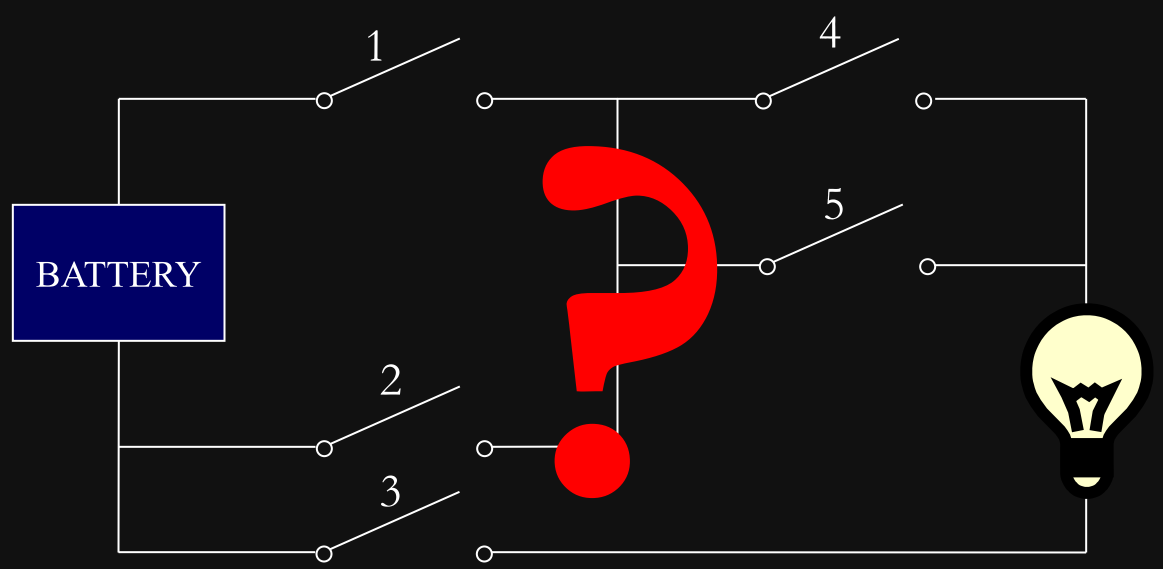
inputs: switches 1 and 2

actions: 1 for “flip switch 1”
2 for “flip switch 2”

states: on, off

bulb is on if and only if both switches were flipped an odd number of times

A design problem



Can you design a circuit where the light is on if and only if all the switches were flipped exactly the same number of times?

A design problem

- Such devices are difficult to reason about, because they can be designed in an infinite number of ways
- By representing them as abstract computational devices, or automata, we will learn how to answer such questions

These devices can model many things

- They can describe the operation of any “small computer”, like the control component of an alarm clock or a microwave
- They are also used in lexical analyzers to recognize well formed expressions in programming languages:

  `ab1` is a legal name of a variable in C
  `5u=` is not

Different kinds of automata

- This was only one example of a computational device, and there are others
- We will look at different devices, and look at the following questions:
  - What can a given type of device compute, and what are its limitations?
  - Is one type of device more powerful than another?

Some devices we will see

finite automata	Devices with a finite amount of memory. Used to model "small" computers.
push-down automata	Devices with infinite memory that can be accessed in a restricted way. Used to model parsers, etc.
Turing Machines	Devices with infinite memory. Used to model any computer.
time-bounded Turing Machines	Infinite memory, but bounded running time. Used to model any computer program that runs in a "reasonable" amount of time.

Some highlights of the course

- Finite automata
  - We will understand what kinds of things a device with finite memory can do, and what it cannot do
  - Introduce simulation: the ability of one device to "imitate" another device
  - Introduce nondeterminism: the ability of a device to make arbitrary choices
- Push-down automata
  - These devices are related to grammars, which describe the structure of programming (and natural) languages

Some highlights of the course

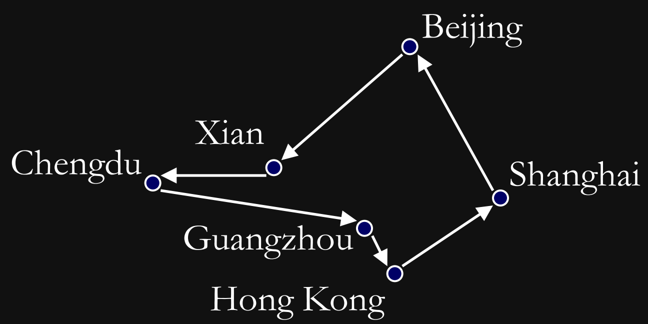
- Turing Machines
  - This is a general model of a computer, capturing anything we could ever hope to compute
  - Surprisingly, there are many things that we cannot compute, for example:

    > Write a program that, given the code of another program in C, tells if this program ever outputs the word "`hello`"

  - It seems that you should be able to tell just by looking at the program, but it is impossible to do!

Some highlights of the course

- Time-bounded Turing Machines
  - Many problems are possible to solve on a computer in principle, but take too much time in practice
  - Traveling salesman: Given a list of cities, find the shortest way to visit them and come back home



  - Easy in principle: Try the cities in every possible order
  - Hard in practice: For 100 cities, this would take 100+ years even on the fastest computer!

Preliminaries of automata theory

- How do we formalize the question

Can device A solve problem B?

- First, we need a formal way of describing the problems that we are interested in solving

Problems

- Examples of problems we will consider
  - Given a word s, does it contain the subword "fool"?
  - Given a number n, is it divisible by 7?
  - Given a pair of words s and t, are they the same?
  - Given an expression with brackets, e.g. $(()())$, does every left bracket match with a subsequent right bracket?
- All of these have "yes/no" answers.
- There are other types of problems, that ask "Find this" or "How many of that" but we won't look at those.

Alphabets and strings

- A common way to talk about words, number, pairs of words, etc. is by representing them as strings
- To define strings, we start with an alphabet

  An alphabet is a finite set of symbols.

- Examples

  $\Sigma_1 = \{a, b, c, d, \ldots, z\}$: the set of letters in English

  $\Sigma_2 = \{0, 1, \ldots, 9\}$: the set of (base 10) digits

  $\Sigma_3 = \{a, b, \ldots, z, \#\}$: the set of letters plus the special symbol $\#$

  $\Sigma_4 = \{ (,) \}$: the set of open and closed brackets

Strings

A string over alphabet Σ is a finite sequence of symbols in Σ.

- The empty string will be denoted by ε
- Examples

abfbz is a string over $\Sigma_1 = \{a, b, c, d, \ldots, z\}$

9021 is a string over $\Sigma_2 = \{0, 1, \ldots, 9\}$

ab#bc is a string over $\Sigma_3 = \{a, b, \ldots, z, \#\}$

))()(() is a string over $\Sigma_4 = \{\ (,\ )\ \}$

Languages

A language is a set of strings over an alphabet.

- Languages can be used to describe problems with “yes/no” answers, for example:

$L_1 =$ The set of all strings over Σ_1 that contain the substring “fool”

$L_2 =$ The set of all strings over Σ_2 that are divisible by 7

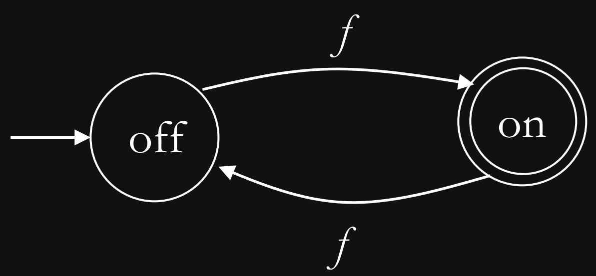
$= \{7, 14, 21, \ldots\}$

$L_3 =$ The set of all strings of the form $s\#s$ where s is any string over $\{a, b, \ldots, z\}$

$L_4 =$ The set of all strings over Σ_4 where every (can

Finite Automata

Example of a finite automaton

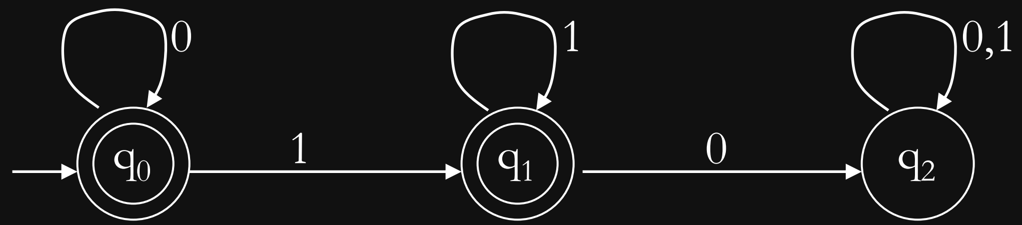


- There are states off and on, the automaton starts in off and tries to reach the "good state" on
- What sequences of fs lead to the good state?
- Answer: $\{f, fff, fffff, \ldots\} = \{f^n \colon n \text{ is odd}\}$
- This is an example of a deterministic finite automaton over alphabet $\{f\}$

Deterministic finite automata

- A deterministic finite automaton (DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where
  - Q is a finite set of states
  - Σ is an alphabet
  - $\delta: Q \times \Sigma \to Q$ is a transition function
  - $q_0 \in Q$ is the initial state
  - $F \subseteq Q$ is a set of accepting states (or final states).
- In diagrams, the accepting states will be denoted by double loops

Example



alphabet $\Sigma = \{0, 1\}$
start state $Q = \{q_0, q_1, q_2\}$
initial state q_0
accepting states $F = \{q_0, q_1\}$

transition function δ:

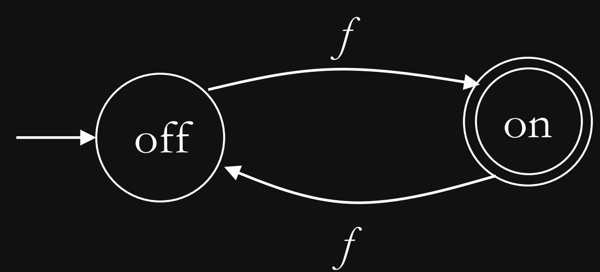
inputs

states	0	1
q_0	q_0	q_1
q_1	q_2	q_1
q_2	q_2	q_2

Language of a DFA

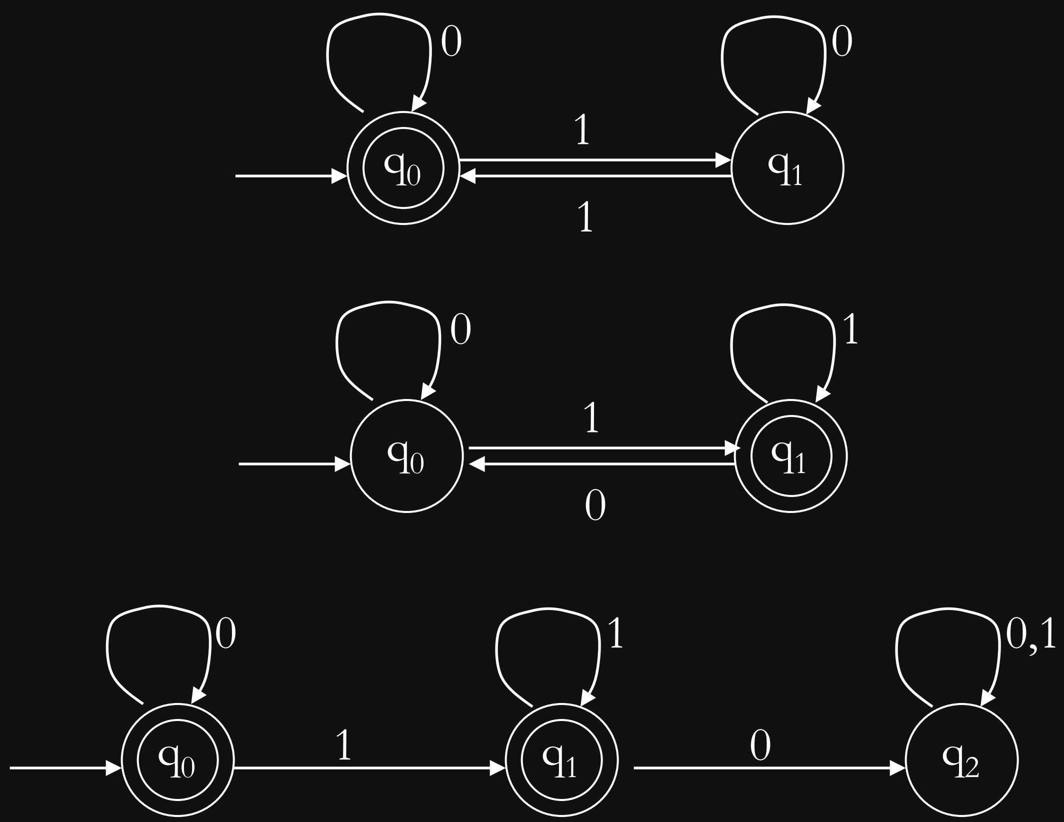
The language of a DFA $(Q, \Sigma, \delta, q_0, F)$ is the set of all strings over Σ that, starting from q_0 and following the transitions as the string is read left to right, will reach some accepting state.

M:



- Language of M is $\{f, fff, fffff, \ldots\} = \{f^n: n \text{ is odd}\}$

Examples



What are the languages of these DFAs?

Examples

- Construct a DFA that accepts the language

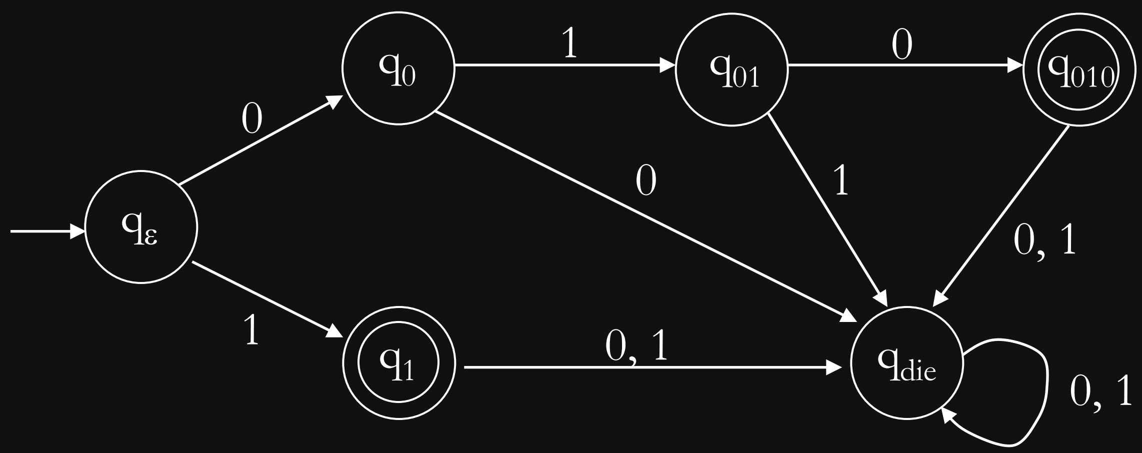
$$L = \{010, 1\} \qquad\qquad (\,\Sigma = \{0, 1\}\,)$$

Examples

- Construct a DFA that accepts the language

$$L = \{010, 1\} \qquad (\, \Sigma = \{0, 1\} \,)$$

- Answer



Examples

- Construct a DFA over alphabet $\{0, 1\}$ that accepts all strings that end in 101

Examples

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- Hint: The DFA must “remember” the last 3 bits of the string it is reading

Examples

- Construct a DFA over alphabet $\{0, 1\}$ that accepts all strings that end in 101
- Sketch of answer: