

Chapter 7

Limitations of Finite Automata

Limitations of FAs

Problem: Is there any set not regular ?

ans: yes!

example: $B = \{a^n b^n \mid n \geq 0\} = \{\epsilon, ab, aabb, aaabbb, \dots\}$

Intuition: Any machine accepting B must be able to remember the number of a 's it has scanned before encountering the first b , but this requires infinite amount of memory (states) and is beyond the capability of any FA, which has only a finite amount of memory (states).

The proof

Lemma 1: Let $M = (Q, \Sigma, \delta, s, F)$ be any DFA accepting B . Then for all non-negative numbers m, n , $m \neq n$ implies $\Delta(s, a^m) \neq \Delta(s, a^n)$.

pf: Assume $\Delta(s, a^m) = \Delta(s, a^n)$ from some $m \neq n$. Then $\Delta(s, a^m b^n)$
 $= \Delta(\Delta(s, a^m), b^n)$
 $= \Delta(\Delta(s, a^n), b^n) = \Delta(s, a^n b^n) \in F$

It implies $a^m b^n \in L(M) = B$. But $a^m b^n \notin B$ since $m \neq n$. Hence $\Delta(s, a^m) \neq \Delta(s, a^n)$ for all $m \neq n$.

Theorem: B is not regular.

Pf: Assume B is regular and accepted by some DFA M with k states. But by Lemma 1, M must have an infinite number of states (since all $\Delta(s, a^m) \in Q$ ($m = 0, 1, 2, \dots$) must be distinct.). This contradicts the requirement that the state set Q of M is finite.

Another nonregular set

- $C = \{a^{2^n} \mid n > 0\} = \{a, aa, aaaa, aaaaaaaaa, \dots\}$ is nonregular

pf: assume C is regular and is accepted by a DFA with k states.

Let $n > k$ and $x = a^{2^n} \in C$. Now consider the sequence of states:

$$\Delta(s, a), \Delta(s, aa), \dots, \Delta(s, a^n),$$

$$s - a - s_1 - a - s_2 - \dots - s_i - a - s_{i+1} - a \dots - s_{i+d} - a - \dots - s_n.$$

by pigeonhole principle, there are $0 < i < i+d \leq n$ s.t.

$$\Delta(s, a^i) = \Delta(s, a^{i+d}) \quad [= p]$$

$$\text{let } 2^n = i + d + m.$$

$$\Rightarrow \Delta(s, a^{2^n+d}) = \Delta(s, a^i a^d a^d a^m) = \Delta(s, a^i a^d a^m) = \Delta(s, a^{2^n}) \in F.$$

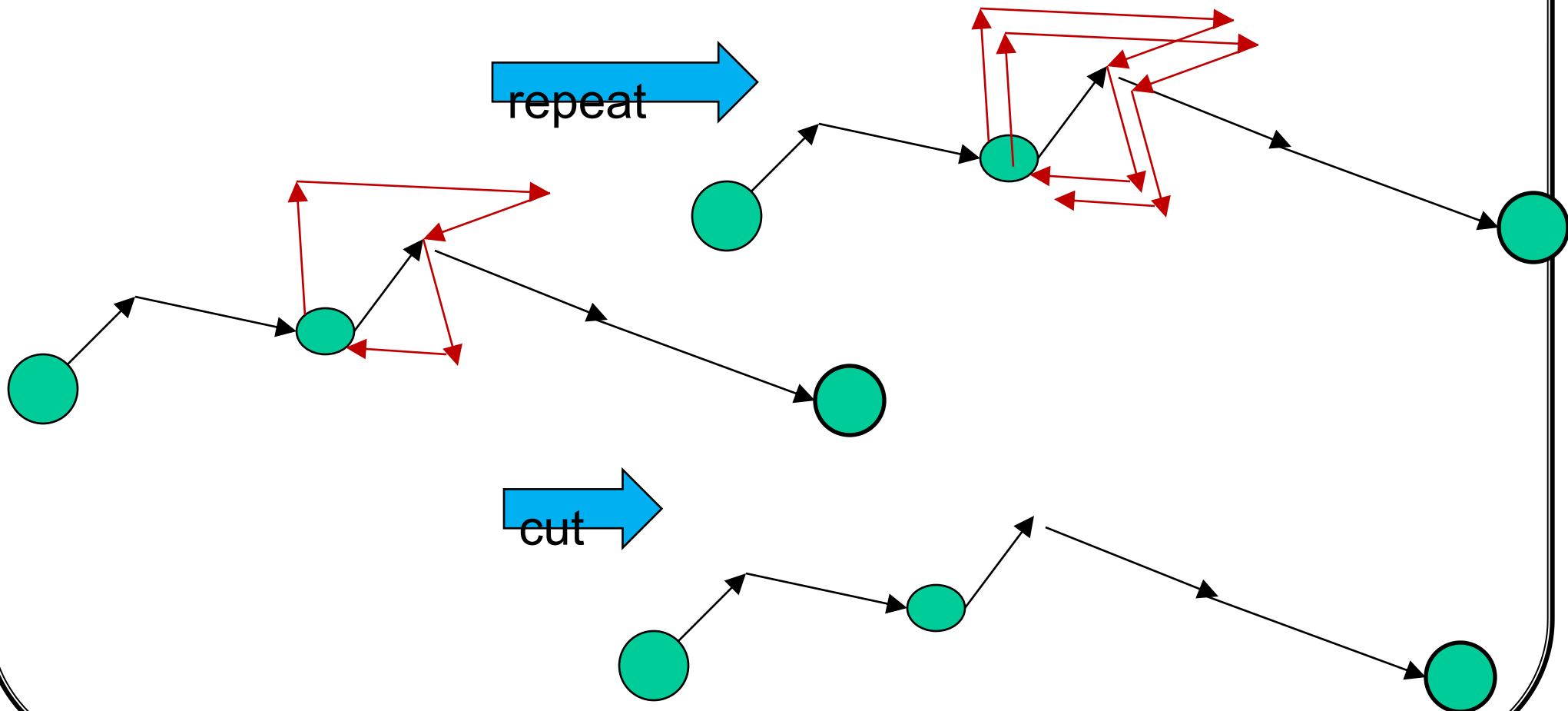
But since $2^n + d < 2^n + n < 2^n + 2^n = 2^{n+1}$, which is the next power of 2 $> 2^n$, Hence $a^{2^n+d} \notin C$

\Rightarrow the DFA also accepts a string $\notin C$, a contradiction!

Hence C is not regular.

Intuition behind the Pumping Lemma for FA

- For an FA to accept a long string s (\geq its number of states), the visited path for s must contain a cycle and hence can be cut or repeated to accept also many new strings.



The pumping lemma

Theorem 11.1: If A is a regular set, then

**(P): $\exists k > 0$ s.t. for any string $xyz \in A$ with $|y| \geq k$,
there exists a decomposition $y = uvw$ s.t.
 $v \neq \varepsilon$ and for all $i \geq 0$, the string $xuv^i wz \in A$.**

pf: Similar to the previous examples. Let $k = |Q|$ where Q is the set of states in a DFA accepting A . Also let s and F be the initial and set of final states of the FA, respectively. Now if there is a string $xyz \in A$ with $|y| \geq k$, consider the sequence of states:

$$\Delta(s, xy_0), \Delta(s, xy_1), \Delta(s, xy_2), \dots, \Delta(s, xy_k),$$

where y_j ($j = 0..k$) denote the prefix of y of the first j symbols. Since there are $k+1$ items in the sequence, each a state in Q , by pigeonhole principle, there must exist two items $\Delta(s, xy_m), \Delta(s, xy_n)$ corresponding to the same state. Without loss of generality, assume $m < n$. Now let $u = y_m, y_n = u v$ and $y = uvw$.

We thus have $\Delta(s, xuwz) = \Delta(s, xy_m wz) = \Delta(s, xy_n wz) = \Delta(s, xuvwz) \in F$

Likewise, for all $j > 1$, $\Delta(s, xuv^j wz) = \Delta(xuv v^{j-1} wz) = \Delta(xuv^{j-1} wz) = \dots = \Delta(xuv^{j-2} wz) = \dots = \Delta(s, xuvwz) \in F$. QED

The pumping lemma

Theorem 11.1: Let A be any language. If A is a regular, then

(P): $\exists k > 0$ s.t. for any string $xyz \in A$ with $|y| \geq k$,
 there exist a decomposition $y = uvw$ s.t.
 $v \neq \varepsilon$ and for all $i \geq 0$, the string $xuv^i wz \in A$.

Theorem 11.2 (pumping lemma, the **contrapositive** form)

If A is any language satisfying the property ($\sim P$):

$\forall k > 0 \exists xyz \in A$ s.t. $|y| \geq k$ and $\forall u, v, w$ with $uvw = y$ and $v \neq \varepsilon$,
 there exists an $i \geq 0$ s.t. $xuv^i vw \notin A$,

then A is not regular. [$\sim P$ means

for any $k > 0$, there is a substring of length $\geq k$ [of a member] of
 A , a cut or a certain duplicates of the middle of any 3-
 segment decomposition of which will produce a string $\notin A$.]

Game semantics for quantification

1. Two players:

- You (want to show a theorem T holds)
- Demon (the opponent want to show T does not hold)

● rules: If the game (or proposition) G is

- $\forall x:U, F \implies$ D pick a member a of U and continue the game $F(a)$.
- $\exists x:U, F \implies$ Y choose a member b of U and continue the game $F(b)$.
- if G has no quantification then end.

● Result:

- Y win if the resulting proposition holds
- D wins o/w

● T holds if Y has a winning strategy (always wins).

Examples

- Show that $(\forall x:\text{nat}, \exists y:\text{nat}, x < y)$.

pf:

D: choose any number k for x .

Y: let y be $k + 1$

Result: $k < k+1$, so Y wins.

Since Y always wins in this game. The result is proved.

The winning strategy is the function : $k \mapsto k+1$.

- Show that $(\forall x:\text{nat}, \exists y:\text{nat}, y < x)$.

pf: D: pick number 0 for x

Y: either fail or

pick a number m for y .

D wins since $\sim(0 < m)$.

Hence the statement is not proved.

Game-theoretical proof of nonregularity of a set

1. Two players:

- You (want to show that $\sim P$ holds and A is not regular)
- Demon (the opponent want to show that P holds)

2 The game proceeds as follows:

1. D picks a $k > 0$ (if A is regular, D's best strategy is to pick $k =$
#states of a FA accepting A)

2. Y picks x, y, x with $xyz \in A$ and $|y| \geq k$.

3. D picks u, v, w s.t. $y = uvw$ and $v \neq \varepsilon$.

4. Y picks $i \geq 0$

3. Finally Y wins if $xuv^i wz \notin A$ and D wins if $xuv^i wz \in A$.

4. By Theorem 11.2, A is not regular if there is a winning strategy according to which Y always win.

Note: P is a necessary but not a sufficient condition for the regularity of A (i.e., there is nonregular set A satisfying P).

Using the pumping lemma

- **Ex1: Show the set $A = \{a^n b^m \mid n \geq m\}$ is not regular.**

the proof:

- 1. D gives k [for any $k > 0$]
- 2. Y pick $x = a^k, y = b^k, z = \varepsilon$ [$\exists xyz \in A$ with $|y| \geq k$]
- $\implies xyz = a^k b^k \in A$
- 3. D decompose $y = uvw$ with [for all uvw with $uvw=y$ and
- $|u|=j, |v|=m > 0$ and $|w|=n \quad v \neq \varepsilon$]
- 4. Y take $i = 2$. [$\exists i \geq 0$ s.t. $xuv^i wz \notin A$]
- $\implies xuv^2 wz = a^k b^j b^{2m} b^n = a^k b^{k+m} \notin A$
- \implies Y wins. Hence A is not regular.

- **Ex2: $C = \{a^{n!} \mid n \geq 0\}$ is not regular.**

pf: similar to Ex1. Left as an exercise.

hint: for any $k > 0$ D chooses, let $xyz = a^{kxk!} a^{k!} \varepsilon$ and let $i = 0$.

Other techniques:

- Using closure property of regular sets.

Ex3: $D = \{ x \in \{a,b\}^* \mid \#a(x) = \#b(x) \}$

$= \{ \varepsilon, ab, ba, aabb, abab, baba, bbaa, abba, baab, \dots \}$

is not regular. (Why ?)

if regular $\Rightarrow D \cap a^*b^* = \{a^n b^n \mid n \geq 0\} = B$ is regular.

But B is not regular, D thus is not regular.

- [H2E2:] A: any language; if A is regular, then $\text{rev}(A) =_{\text{def}} \{x_n x_{n-1} \dots x_1 \mid x_1 x_2 \dots x_n \in A\}$ is regular.

- Ex4: $A = \{a^n b^m \mid m \geq n\}$ is not regular.

pf: If A is regular $\Rightarrow \text{rev}(A)$ and $h(\text{rev}(A)) = \{a^n b^m \mid n \geq m\}$ is regular, where $h(a) = b$ and $h(b) = a$.

$\Rightarrow A \cap h(\text{rev}(A)) = \{a^n b^n \mid n \geq 0\}$ is regular, a contradiction!