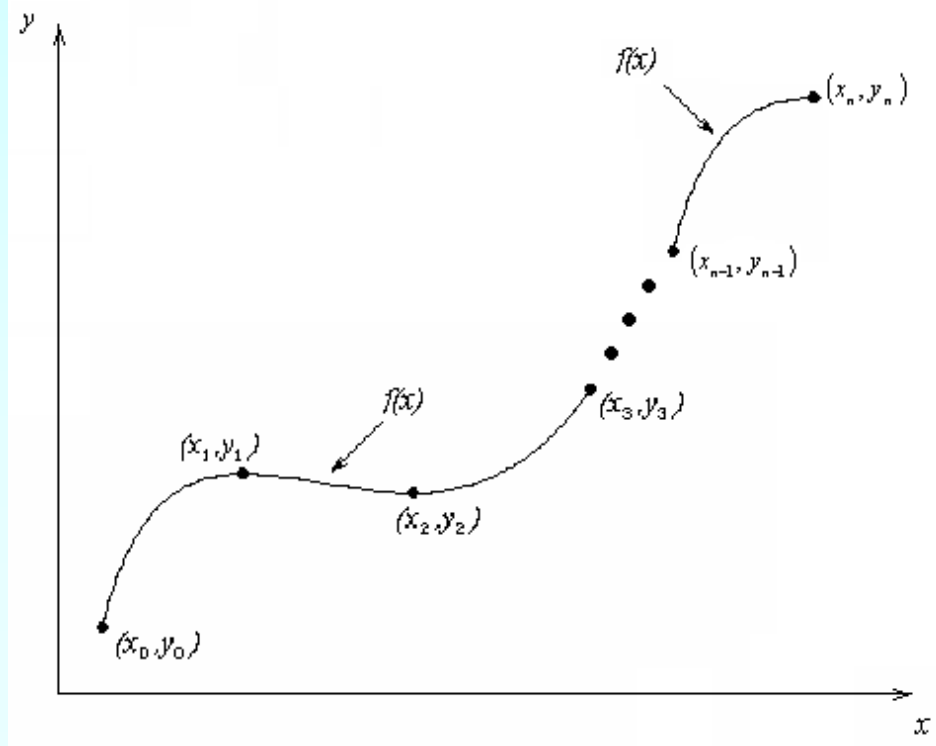


# Lagrange, Cubic Spline & Quadratic Interpolation

# What is Interpolation ?

Given  $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ , find the value of 'y' at a value of 'x' that is not given.



# Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

# Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

where ‘ $n$ ’ in  $f_n(x)$  stands for the  $n^{\text{th}}$  order polynomial that approximates the function  $y = f(x)$  given at  $(n + 1)$  data points as  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$  is a weighting function that includes a product of  $(n - 1)$  terms with terms of  $j = i$  omitted.

# Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at  $t=16$  seconds using the Lagrangian method for linear interpolation.

Table Velocity as a function of time

| $t$ (s) | $v(t)$ (m/s) |
|---------|--------------|
| 0       | 0            |
| 10      | 227.04       |
| 15      | 362.78       |
| 20      | 517.35       |
| 22.5    | 602.97       |
| 30      | 901.67       |

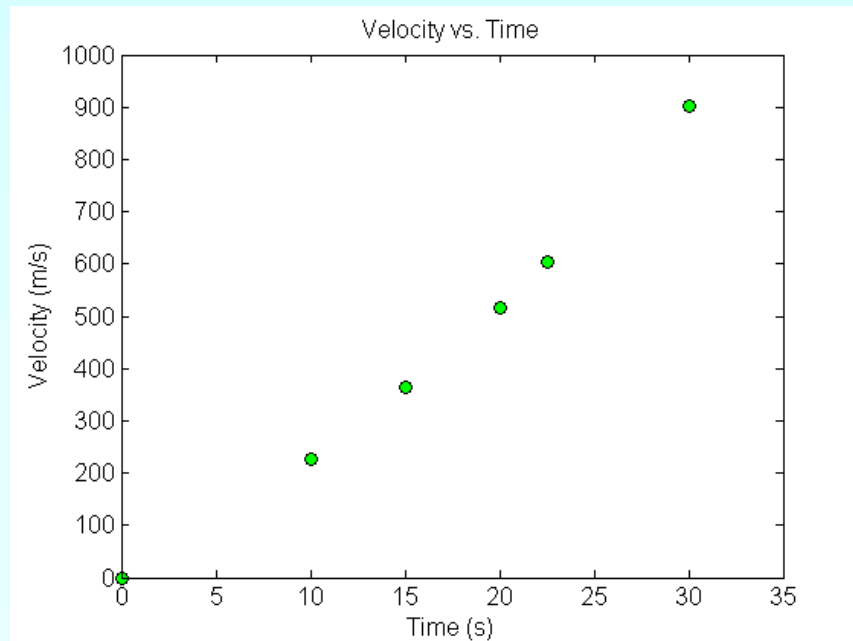


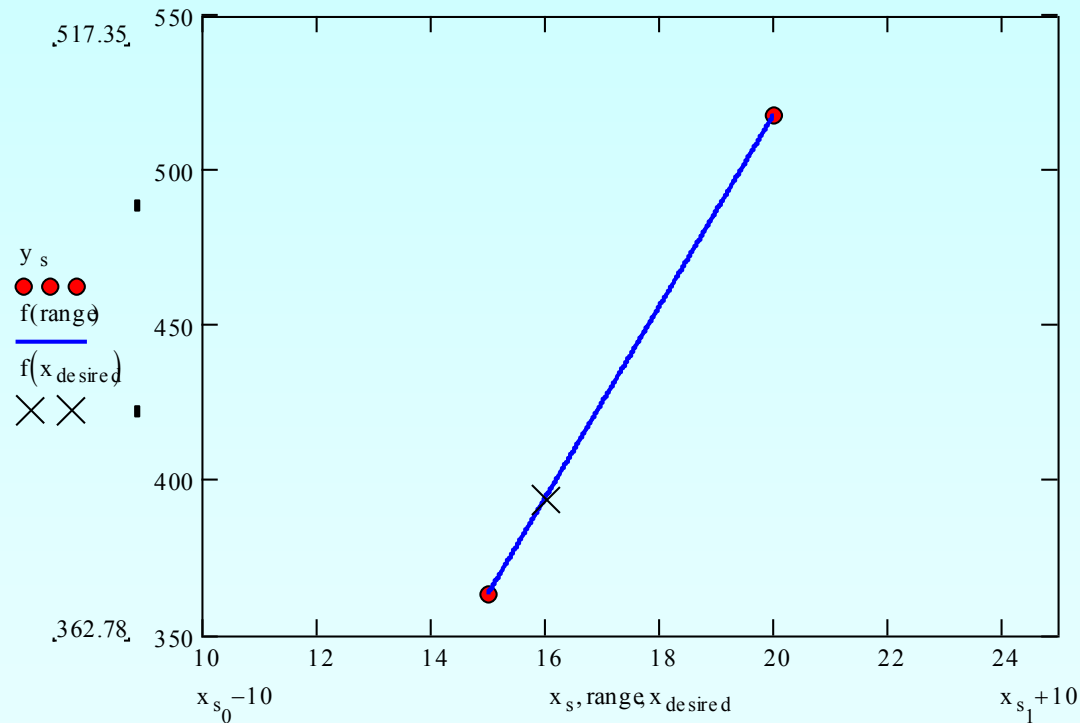
Figure. Velocity vs. time data for the rocket example

# Linear Interpolation

$$v(t) = \sum_{i=0}^1 L_i(t)v(t_i)$$
$$= L_0(t)v(t_0) + L_1(t)v(t_1)$$

$$t_0 = 15, v(t_0) = 362.78$$

$$t_1 = 20, v(t_1) = 517.35$$



# Linear Interpolation (contd)

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^1 \frac{t - t_j}{t_0 - t_j} = \frac{t - t_1}{t_0 - t_1}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^1 \frac{t - t_j}{t_1 - t_j} = \frac{t - t_0}{t_1 - t_0}$$

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1) = \frac{t - 20}{15 - 20} (362.78) + \frac{t - 15}{20 - 15} (517.35)$$

$$v(16) = \frac{16 - 20}{15 - 20} (362.78) + \frac{16 - 15}{20 - 15} (517.35)$$

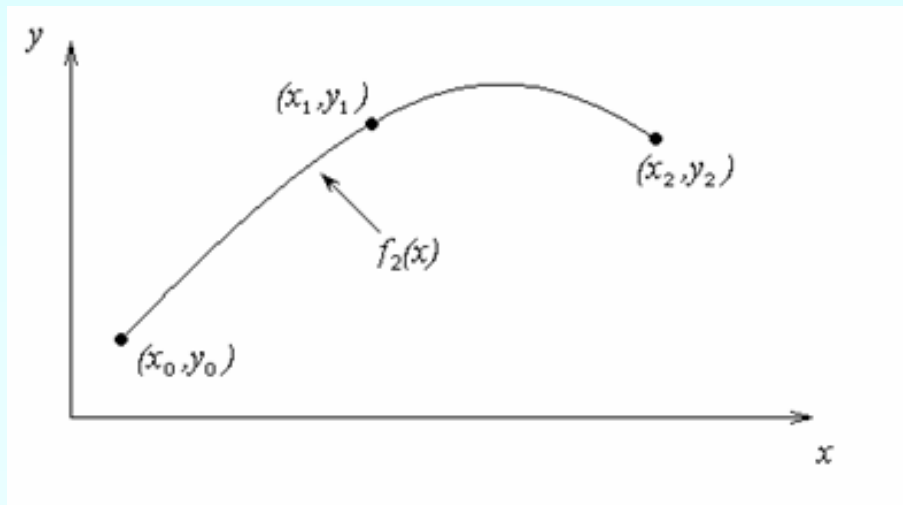
$$= 0.8(362.78) + 0.2(517.35)$$

$$= 393.7 \text{ m/s.}$$

# Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the velocity given by

$$\begin{aligned}v(t) &= \sum_{i=0}^2 L_i(t)v(t_i) \\ &= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)\end{aligned}$$





# Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at  $t=16$  seconds using the Lagrangian method for quadratic interpolation.

Table Velocity as a function of time

| $t$ (s) | $v(t)$ (m/s) |
|---------|--------------|
| 0       | 0            |
| 10      | 227.04       |
| 15      | 362.78       |
| 20      | 517.35       |
| 22.5    | 602.97       |
| 30      | 901.67       |

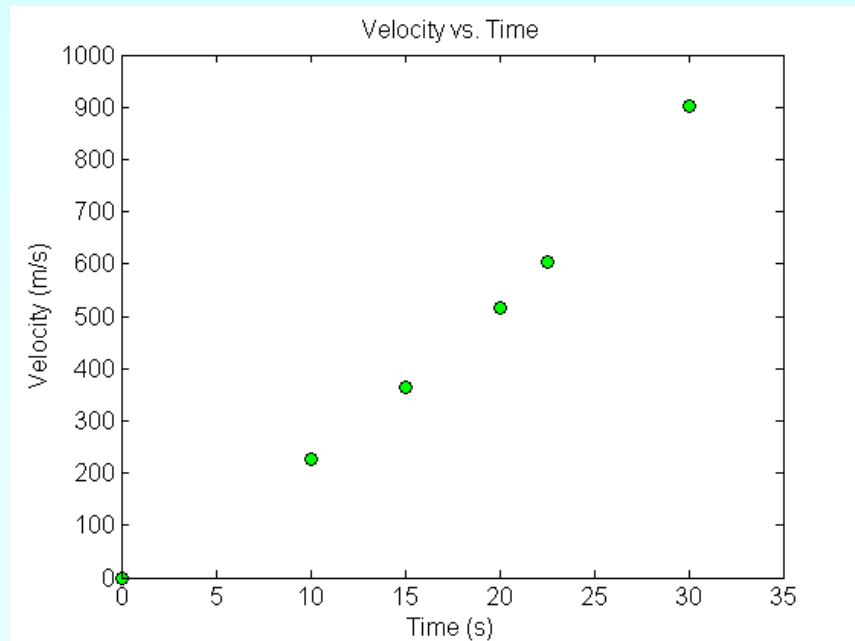


Figure. Velocity vs. time data for the rocket example

# Quadratic Interpolation (contd)

$$t_0 = 10, v(t_0) = 227.04$$

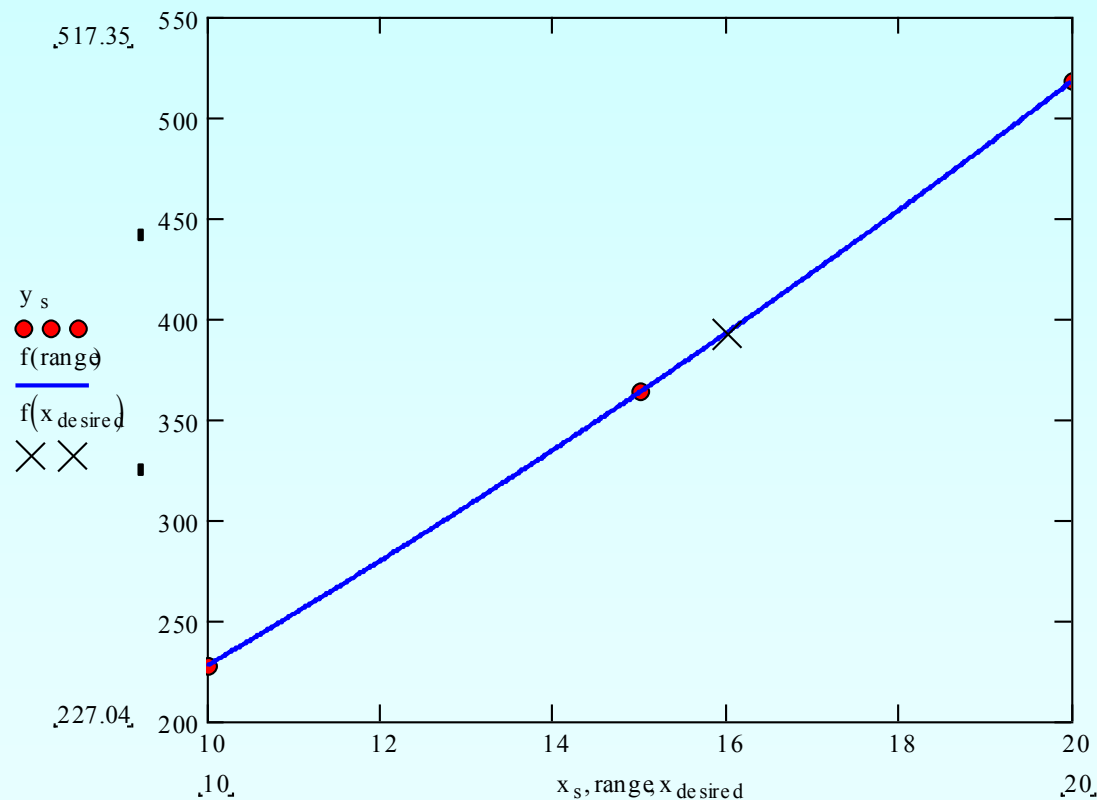
$$t_1 = 15, v(t_1) = 362.78$$

$$t_2 = 20, v(t_2) = 517.35$$

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^2 \frac{t-t_j}{t_0-t_j} = \left( \frac{t-t_1}{t_0-t_1} \right) \left( \frac{t-t_2}{t_0-t_2} \right)$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^2 \frac{t-t_j}{t_1-t_j} = \left( \frac{t-t_0}{t_1-t_0} \right) \left( \frac{t-t_2}{t_1-t_2} \right)$$

$$L_2(t) = \prod_{\substack{j=0 \\ j \neq 2}}^2 \frac{t-t_j}{t_2-t_j} = \left( \frac{t-t_0}{t_2-t_0} \right) \left( \frac{t-t_1}{t_2-t_1} \right)$$



# Quadratic Interpolation (contd)

$$\begin{aligned}v(t) &= \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)v(t_0) + \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)v(t_1) + \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)v(t_2) \\v(16) &= \left(\frac{16-15}{10-15}\right)\left(\frac{16-20}{10-20}\right)(227.04) + \left(\frac{16-10}{15-10}\right)\left(\frac{16-20}{15-20}\right)(362.78) + \left(\frac{16-10}{20-10}\right)\left(\frac{16-15}{20-15}\right)(517.35) \\&= (-0.08)(227.04) + (0.96)(362.78) + (0.12)(527.35) \\&= 392.19 \text{ m/s}\end{aligned}$$

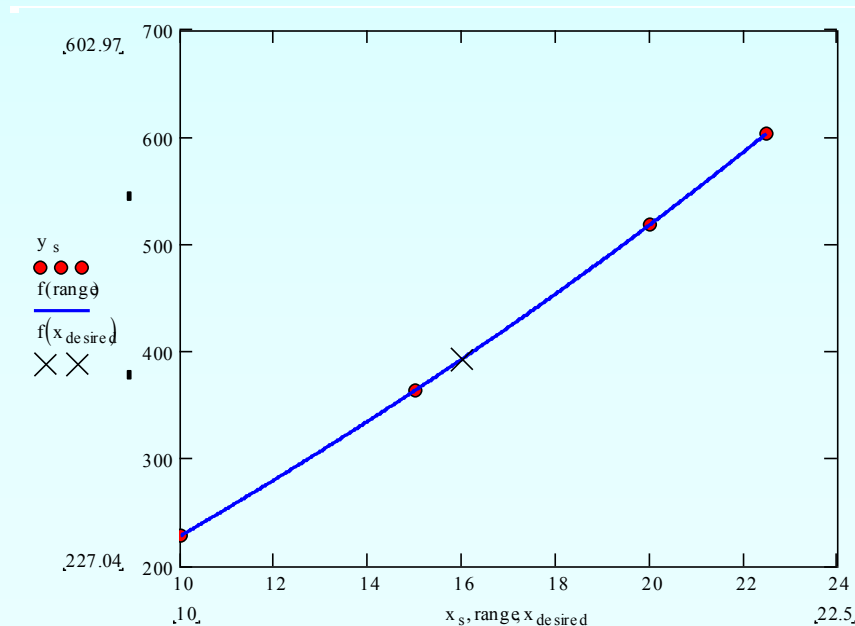
The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the first and second order polynomial is

$$\begin{aligned}|\epsilon_a| &= \left|\frac{392.19 - 393.70}{392.19}\right| \times 100 \\&= 0.38410\%\end{aligned}$$

# Cubic Interpolation

For the third order polynomial (also called cubic interpolation), we choose the velocity given by

$$\begin{aligned}v(t) &= \sum_{i=0}^3 L_i(t)v(t_i) \\ &= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2) + L_3(t)v(t_3)\end{aligned}$$



# Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at  $t=16$  seconds using the Lagrangian method for cubic interpolation.

Table Velocity as a function of time

| $t$ (s) | $v(t)$ (m/s) |
|---------|--------------|
| 0       | 0            |
| 10      | 227.04       |
| 15      | 362.78       |
| 20      | 517.35       |
| 22.5    | 602.97       |
| 30      | 901.67       |

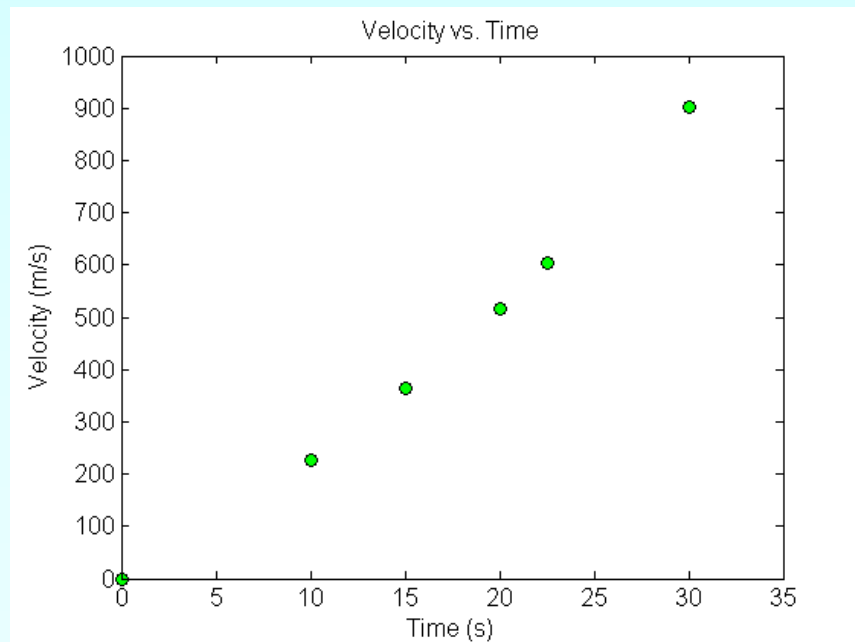


Figure. Velocity vs. time data for the rocket example

# Cubic Interpolation (contd)

$$t_0 = 10, v(t_0) = 227.04 \quad t_1 = 15, v(t_1) = 362.78$$

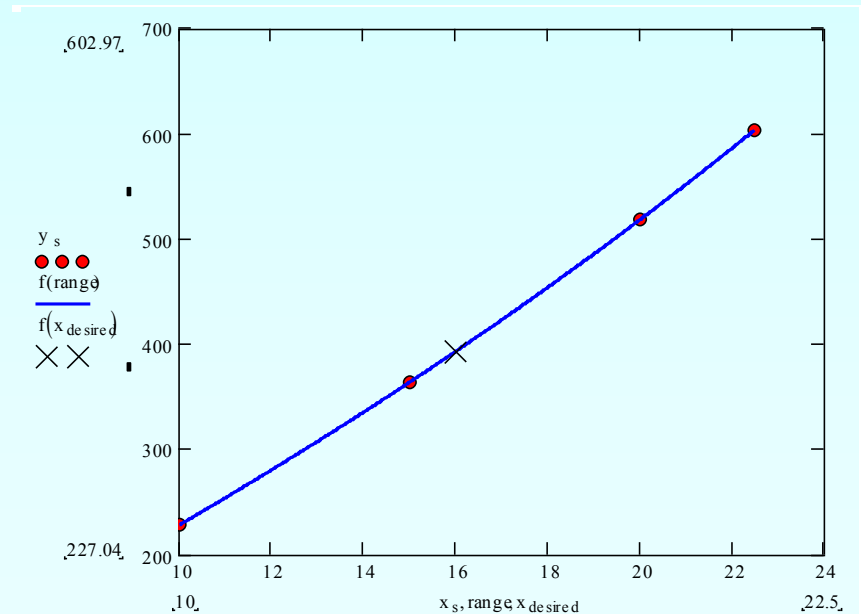
$$t_2 = 20, v(t_2) = 517.35 \quad t_3 = 22.5, v(t_3) = 602.97$$

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^3 \frac{t-t_j}{t_0-t_j} = \left( \frac{t-t_1}{t_0-t_1} \right) \left( \frac{t-t_2}{t_0-t_2} \right) \left( \frac{t-t_3}{t_0-t_3} \right);$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^3 \frac{t-t_j}{t_1-t_j} = \left( \frac{t-t_0}{t_1-t_0} \right) \left( \frac{t-t_2}{t_1-t_2} \right) \left( \frac{t-t_3}{t_1-t_3} \right)$$

$$L_2(t) = \prod_{\substack{j=0 \\ j \neq 2}}^3 \frac{t-t_j}{t_2-t_j} = \left( \frac{t-t_0}{t_2-t_0} \right) \left( \frac{t-t_1}{t_2-t_1} \right) \left( \frac{t-t_3}{t_2-t_3} \right);$$

$$L_3(t) = \prod_{\substack{j=0 \\ j \neq 3}}^3 \frac{t-t_j}{t_3-t_j} = \left( \frac{t-t_0}{t_3-t_0} \right) \left( \frac{t-t_1}{t_3-t_1} \right) \left( \frac{t-t_2}{t_3-t_2} \right)$$



# Cubic Interpolation (contd)

$$\begin{aligned}v(t) &= \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)v(t_1) + \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right)v(t_2) \\ &+ \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right)v(t_2) + \left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_2}{t_3-t_2}\right)v(t_3) \\ v(16) &= \left(\frac{16-15}{10-15}\right)\left(\frac{16-20}{10-20}\right)\left(\frac{16-22.5}{10-22.5}\right)(227.04) + \left(\frac{16-10}{15-10}\right)\left(\frac{16-20}{15-20}\right)\left(\frac{16-22.5}{15-22.5}\right)(362.78) \\ &+ \left(\frac{16-10}{20-10}\right)\left(\frac{16-15}{20-15}\right)\left(\frac{16-22.5}{20-22.5}\right)(517.35) + \left(\frac{16-10}{22.5-10}\right)\left(\frac{16-15}{22.5-15}\right)\left(\frac{16-20}{22.5-20}\right)(602.97) \\ &= (-0.0416)(227.04) + (0.832)(362.78) + (0.312)(517.35) + (-0.1024)(602.97) \\ &= 392.06 \text{ m/s}\end{aligned}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the first and second order polynomial is

$$\begin{aligned}|\epsilon_a| &= \left| \frac{392.06 - 392.19}{392.06} \right| \times 100 \\ &= 0.033269\%\end{aligned}$$

# Comparison Table

| <b>Order of Polynomial</b>          | <b>1</b> | <b>2</b> | <b>3</b>  |
|-------------------------------------|----------|----------|-----------|
| v(t=16) m/s                         | 393.69   | 392.19   | 392.06    |
| Absolute Relative Approximate Error | -----    | 0.38410% | 0.033269% |



# Distance from Velocity Profile

Find the distance covered by the rocket from  $t=11\text{s}$  to  $t=16\text{s}$  ?

$$v(t) = (t^3 - 57.5t^2 + 1087.5t - 6750)(-0.36326) + (t^3 - 52.5t^2 + 875t - 4500)(1.9348) \\ + (t^3 - 47.5t^2 + 712.5t - 3375)(-4.1388) + (t^3 - 45t^2 + 650t - 3000)(2.5727)$$

$$v(t) = -4.245 + 21.265t + 0.13195t^2 + 0.00544t^3, \quad 10 \leq t \leq 22.5$$

$$s(16) - s(11) = \int_{11}^{16} v(t) dt \\ \approx \int_{11}^{16} (-4.245 + 21.265t + 0.13195t^2 + 0.00544t^3) dt \\ = \left[ -4.245t + 21.265 \frac{t^2}{2} + 0.13195 \frac{t^3}{3} + 0.00544 \frac{t^4}{4} \right]_{11}^{16} \\ = 1605 \text{ m}$$

# Acceleration from Velocity Profile

Find the acceleration of the rocket at  $t=16\text{s}$  given that

$$v(t) = -4.245 + 21.265t + 0.13195t^2 + 0.00544t^3, \quad 10 \leq t \leq 22.5$$

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}(-4.245 + 21.265t + 0.13195t^2 + 0.00544t^3)$$

$$= 21.265 + 0.26390t + 0.01632t^2$$

$$a(16) = 21.265 + 0.26390(16) + 0.01632(16)^2$$

$$= 29.665 \text{ m/s}^2$$

# Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

[http://numericalmethods.eng.usf.edu/topics/lagrange\\_method.html](http://numericalmethods.eng.usf.edu/topics/lagrange_method.html)