Introduction to Spectral method.

- Finite difference method – approximate a function locally using lower order interpolating polynomials.
- Spectral method – approximate a function using global higher order interpolating polynomials.
- Using spectral method, a higher order approximation can be made with moderate computational resources.

Definitions:

$I := [a, b] \in \mathbf{R}$, an interval.

$f, g : I \to \mathbf{R}$, smooth functions.

$w : I \to \mathbf{R}$, a weight function.

($\forall x \in I$, $w(x) \geq 0$ and $\{x \,|\, w(x) = 0\}$ are descrete points).

$$(f, g) := \int_a^b f(x)g(x)w(x)dx.$$

$\mathbf{\Pi}_N$: a family of all polynomials of degree N or less than N.

$\{\phi_n \,|\, n = 0, \cdots, N\}$: a set of orthogonal basis of $\mathbf{\Pi}_N$ with respect to the weight $w(x)$,

$$(\phi_n, \phi_m) := \int_a^b \phi_n(x)\phi_m(x)w(x)dx \begin{cases} = 0 \text{ for } n \neq m, \\ \neq 0 \text{ for } n = m. \end{cases}$$

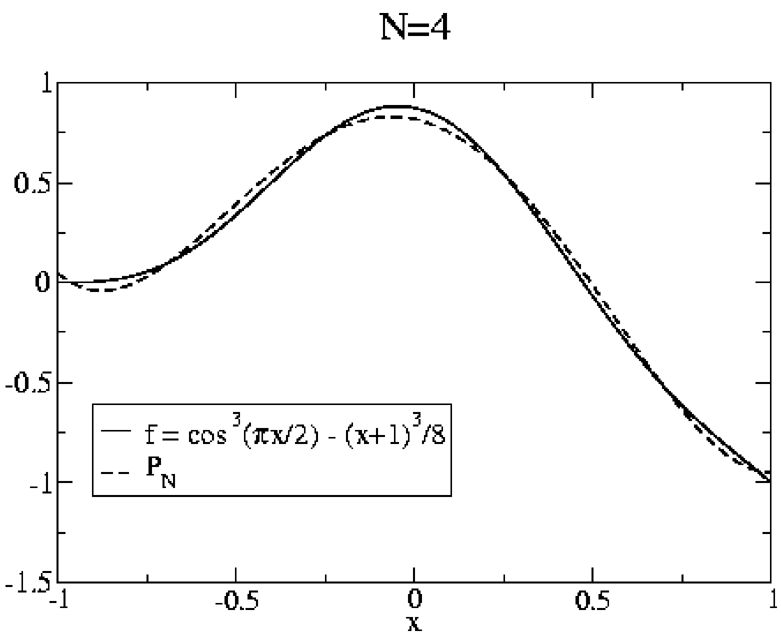
- In spectral methods, a function f(x) is approximated by its projection to the polynomial basis

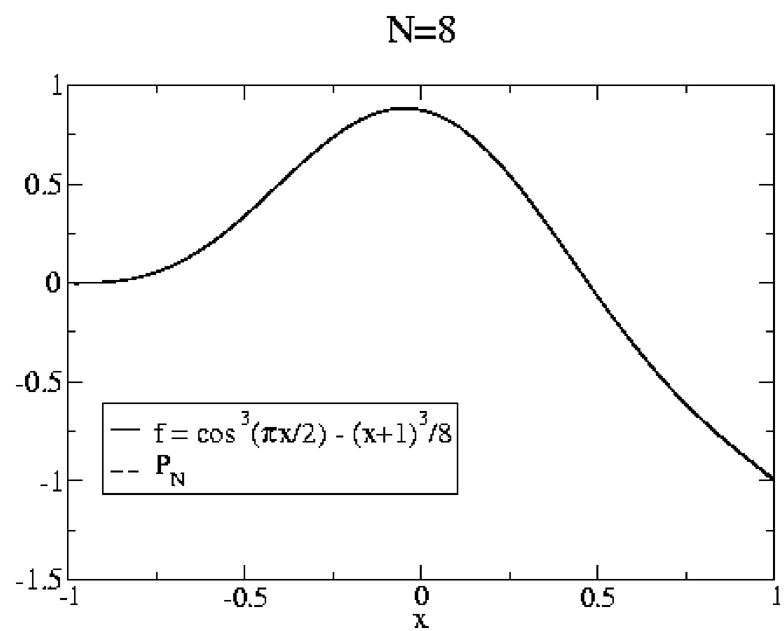
$$P_N f(x) := \sum_{n=0}^{N} \hat{f}_n \, \phi_n(x), \ \text{where } \hat{f}_n := \frac{\int_a^b f(x)\, \phi_n(x) w(x) dx}{\int_a^b \phi_n(x)\, \phi_n(x) w(x) dx} = \frac{(f, \phi_n)}{(\phi_n, \phi_n)}$$

- Difference between f(x) and the approximation $P_N f(x)$ is called the truncation error. For a well behaved function f(x), the truncation error goes to zero as increasing N.

$$\lim_{N \to \infty} ||f(x) - P_N f(x)|| = 0$$

Ex) an approximation for a function $u(x) = \cos^3(\pi x/2) - (x+1)^3/8$





Approximation $P_N f(x) := \sum_{n=0}^{N} \widehat{f}_n \phi_n(x)$ will be good, if the integrals

$$(f, \phi_n) = \int_a^b f(x)\,\phi_n(x) w(x) dx, \qquad (\phi_n, \phi_n) = \int_a^b \phi_n(x)\,\phi_n(x) w(x) dx$$

are evaluated accurately.

- Gaussian integration (quadrature) formula is used to achieve high precision.
- Gauss formula is less convenient since it doesn't include end points of I = [a,b].

Recall: (Gauss formula, Gaussian quadrature)

Let $w(x)$: weight function on $[a, b]$. ϕ_k: k-th degree polynomials.

$\{\phi_0, \cdots, \phi_{N+1}\} \subset \mathbf{\Pi}_{N+1}$: Orthogonal family of polynomials.

Writing the roots of ϕ_{N+1} by $x_0, \cdots x_N$, and define

$$L_i(x) := \prod_{j=0, j\neq, i}^{N} \frac{x - x_j}{x_i - x_j}, \quad \text{for } i = 0, \cdots N,$$

the corresponding Gaussian quadrature formula is given by

$$I(f) = \int_a^b f(x) w(x) dx \approx I_N(f) = \sum_{i=0}^{N} w_i f(x_i), \text{ where, } w_i := \int_a^b L_i(x) w(x) dx.$$

The formula $I_N(f)$ has dgree of precision $D = 2N + 1$, that is,

$\forall f(x) \in \mathbf{\Pi}_{2N+1}, \quad I(f) \equiv I_N(f).$

Gauss-Lobatto formula.

⋆ Gauss Lobatto formula uses function values at the both end points

$$I(f) = \int_a^b f(x)w(x)dx \approx I_N(f) = w_0 f(a) + w_N f(b) + \sum_{i=1}^{N-1} w_i f(x_i),$$

then optimize the values of weights $\{w_i\}$, $i = 0, \cdots, N$, and the abscissas $\{x_i\}$, $i = 1, \cdots, N-1$.

- Since we have two less free parameters compare to the Gauss formula, the degree of precision for the Gauss-Lobatto formula is D = 2N – 1.
- Since N – 1 roots are used for { x_i }, the basis is $\{\phi_0, \cdots, \phi_{N-1}\} \subset \mathbf{\Pi}_{N-1}$:
- For I = [-1,1] and w(x) = 1, x_i are roots of $\phi_{N-1} = P'_N(x)=0$.

⋆ Gauss Radau formula uses a function value at one of the end points.

$\int_a^b f(x)w(x)dx = w_0 f(a) + \sum_{i=2}^{N} w_i f(x_i)$, then optimize the values of weights $\{w_i\}$, $i = 0, \cdots, N$, and the abscissas $\{x_i\}$, $i = 1, \cdots, N$. The degree of precision D=2N.

$$\forall f(x) \in \mathbf{\Pi}_{2n+k}, \quad I(f) \equiv I_N(f) = \sum_{i=0}^{N} w_i f(x_i) \quad \begin{cases} k = 1 & \text{Gauss,} \\ k = 0 & \text{Gauss-Radau,} \\ k = -1 & \text{Gauss-Lobatto.} \end{cases}$$

``Exact'' spectral expansion differs from numerically evaluated expansion.

$$P_N f(x) := \sum_{n=0}^{N} \widehat{f}_n \phi_n(x), \qquad \widehat{f}_n := \frac{\int_a^b f(x)\, \phi_n(x) w(x) dx}{\int_a^b \phi_n(x)\, \phi_n(x) w(x) dx} = \frac{(f, \phi_n)}{(\phi_n, \phi_n)}$$

$$I_N f(x) := \sum_{n=0}^{N} \widetilde{f}_n \phi_n(x), \qquad \widetilde{f}_n := \frac{1}{\gamma_n} \sum_{i=0}^{N} w_i f(x_i)\, \phi_n(x_i) = \frac{(f, \phi_n)_N}{(\phi_n, \phi_n)_N},$$

$$\gamma_n := \sum_{i=0}^{N} w_i [\, \phi_n(x_i)\,]^2 =: (\phi_n, \phi_n)_N.$$

$\widehat{f}$ and $\widetilde{f}$ are different. [Aliasing error] $:= |I_N f - P_N f|$

- The Interpolant of f(x), $I_N f$, is called the spectral approximation of f(x).
- Abscissas used in the Gauss quadrature formula $\{x_i\}$ are also called collocation points.

Exc 6-1) Show that the value of interpolant agrees with the function value at each collocation points,

$$I_N f(x_i) = f(x_i) \text{ at each collocation point } \{x_0, \cdots, x_N\}.$$

- A set of function values at collocation points $\{f(x_0), \cdots, f(x_N)\}$ is called configuration space.
- A set of coefficients of the spectral expansion $\{\tilde{f}_0, \cdots, \tilde{f}_N\}$ is called coefficient space.

The map between configuration space and coefficient space is a bijection (one to one and onto).

$$\tilde{f}_n := \frac{1}{\gamma_n} \sum_{i=0}^{N} w_i f(x_i)\, \phi_n(x_i), \qquad \text{configuration space} \rightarrow \text{coeffcient space}$$

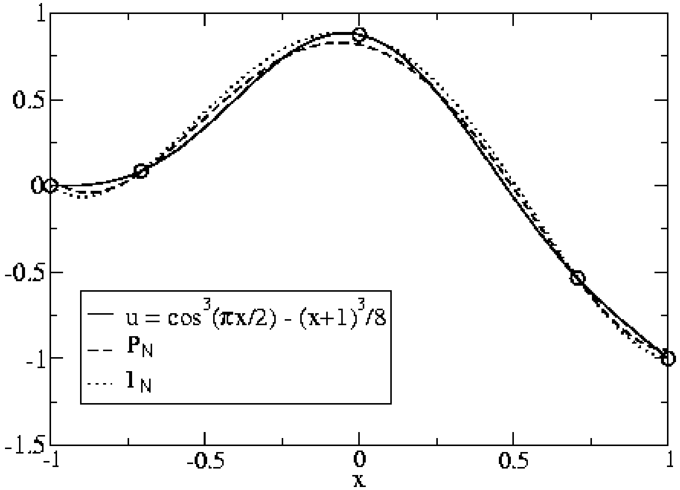
$$I_N f(x_n) := \sum_{n=0}^{N} \tilde{f}_n\, \phi_n(x_n), \qquad \text{coeffcient space} \rightarrow \text{configuration space}$$

Ex) a derivative is calculated using a spectral expansion in the coefficient space.

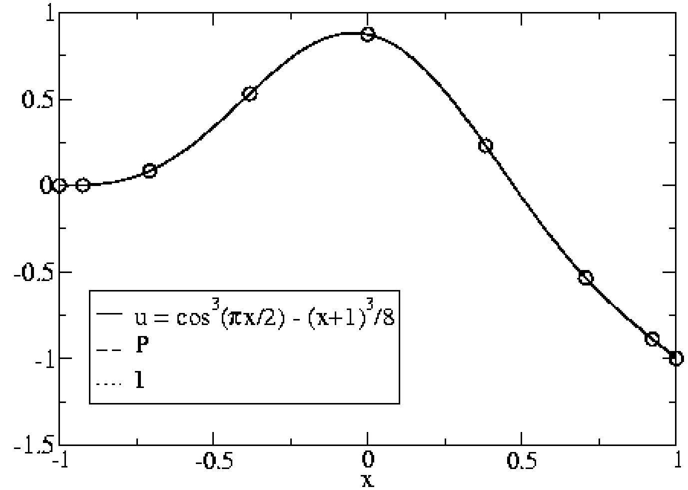
$$\frac{df}{dx} \approx \frac{d}{dx}[\, I_N f(x)\,] = \sum_{n=0}^{N} \tilde{f}_n \frac{d\phi_n}{dx}(x) \neq I_N \frac{df}{dx}(x) = \sum_{n=0}^{N} \widetilde{\left(\frac{df}{dx}\right)}_n \phi_n(x) \approx \frac{df}{dx}.$$

Difference in P_N f (analytic) and I_N f (interpolant).

N=4

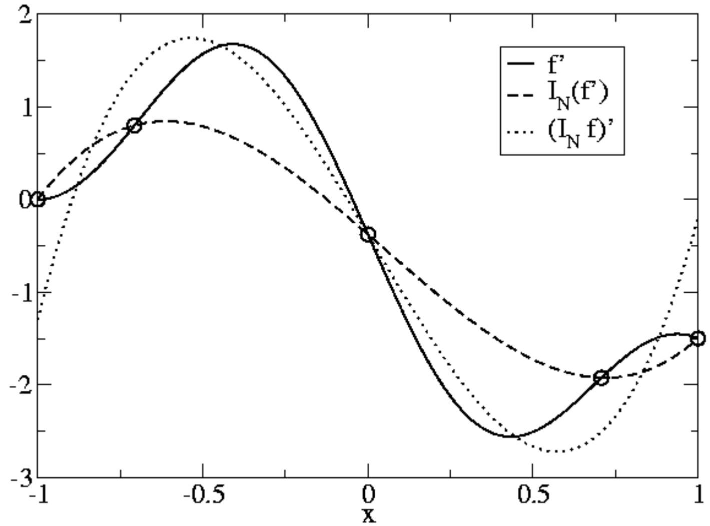


N=8

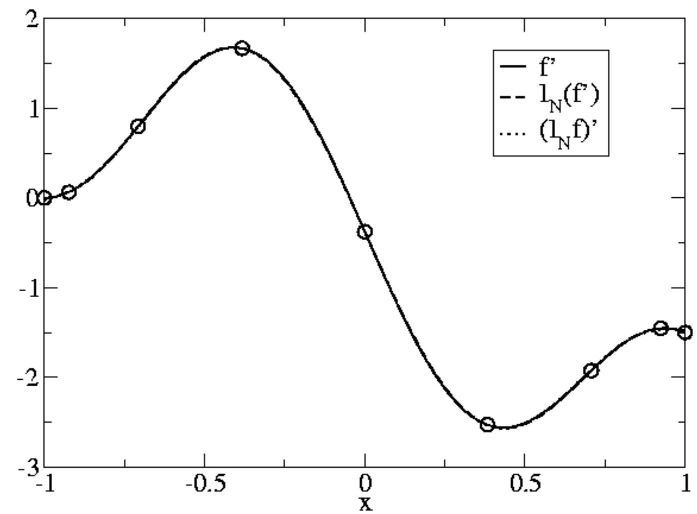


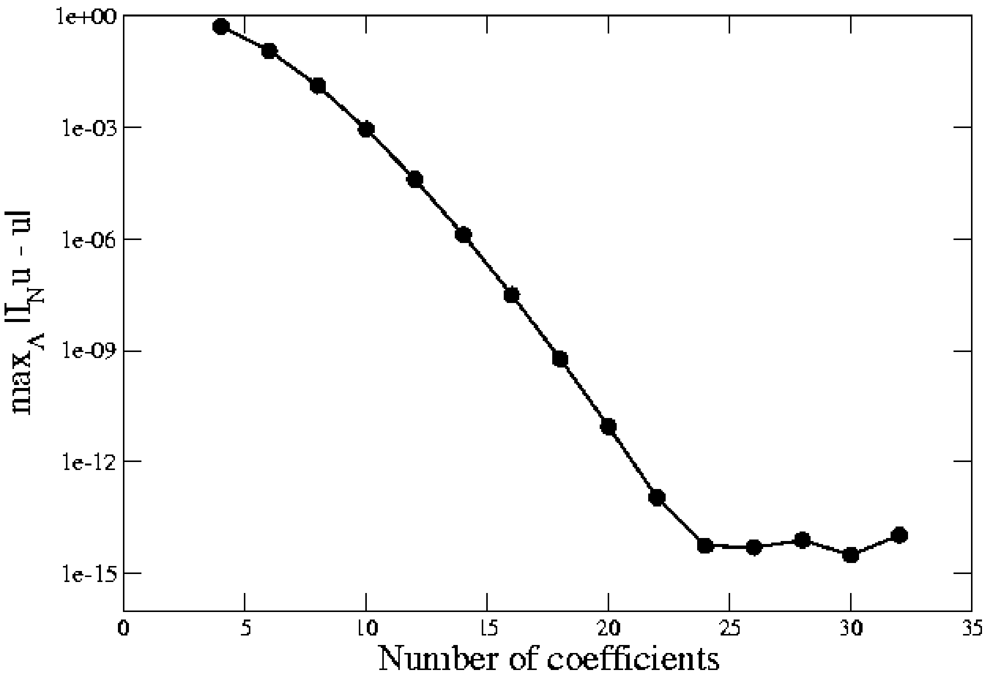
N=4

$$[\, I_N f(x)\,]' \neq I_N f'(x)$$



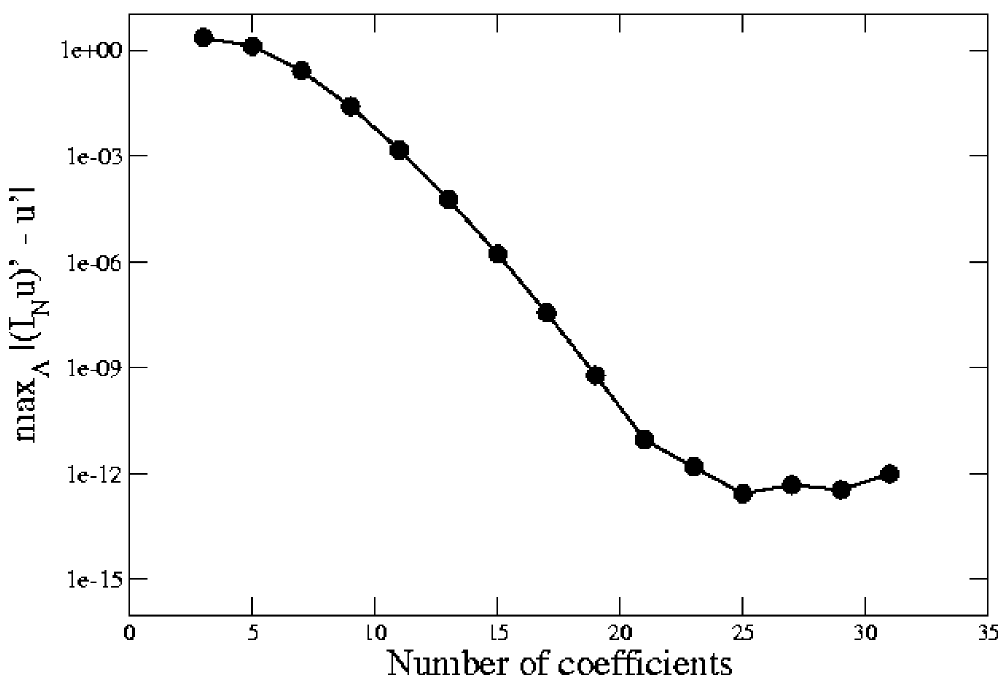
N=8





Error in interpolant.

Error in derivative.



Choice for the polynomials:

1) Legendre polynomials. $\phi_n(x) = P_n(x)$. Interval $I = [-1,1]$, and weight $w(x) = 1$.

$$\tilde{f}_n := \frac{1}{\gamma_n} \sum_{i=0}^{N} w_i f(x_i)\, P_n(x_i),$$

$$I_N f(x) := \sum_{n=0}^{N} \tilde{f}_n\, P_n(x),$$

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2}{2n+1} \delta_{nm}.$$

	Abscissas $\{x_i\}_{i=0,\cdots,N}$	weight $\{w_i\}_{i=0,\cdots,N}$
Gauss-Legendre	Roots of $P_{N+1}(x) = 0$	$w_i = \frac{2}{1-x_i^2}[\, P'_{N+1}(x_i)\,]$
Gauss-Radau -Legendre	$x_0 = -1$ and the Roots of $P_N(x) + P_{N+1}(x) = 0$	$w_0 = \frac{2}{(N+1)^2}$, and $w_i \; \frac{1}{(N+1)^2}$
Gauss-Lobatto -Legendre	$x_0 = -1$, $x_N = 1$ and the Roots of $P'_N(x) = 0$	$w_i = \frac{2}{N(N+1)} \frac{1}{[\, P_N(x_i)\,]^2}$

Some linear operations to the Legendre interpolant.

For some linear operators L acting on the interpolant $L[I_N f(x)] := \sum_{n=0}^{N} a_n P_n(x)$, the coeffient a_n can be explicitly written by $\tilde{f}_n$ of $I_N f(x) := \sum_{n=0}^{N} \tilde{f}_n P_n(x)$.

(1) For L the multiplication of x, $L[I_N f(x)] := \sum_{n=0}^{N} \tilde{f}_n \, x P_n(x)$,

$$a_n = \frac{n}{2n-1}\tilde{f}_{n-1} + \frac{n+1}{2n+3}\tilde{f}_{n+1}, \quad (n \geq 1).$$

(2) For L the derivative, $L[I_N f(x)] := \sum_{n=0}^{N} \tilde{f}_n \, P_n'(x)$,

$$a_n = (2n+1) \sum_{p=n+1,\, p+n=odd}^{N} \tilde{f}_p.$$

(3) For L the second derivative, $L[I_N f(x)] := \sum_{n=0}^{N} \tilde{f}_n \, P_n''(x)$,

$$a_n = (n+1/2) \sum_{p=n+2,\ p+n=even}^{N} [\,p(p+1) - n(n+1)\,]\tilde{f}_p.$$

Exc 6-2) Show the above relations using recursion relations for $P_n(x)$.

2) Chebyshev polynomials. $\phi_n(x) = T_n(x)$. Interval $I = [-1,1]$, and weight $w(x) = \dfrac{1}{\sqrt{1-x^2}}$

$$\tilde{f}_n := \frac{1}{\gamma_n} \sum_{i=0}^{N} w_i f(x_i) T_n(x_i),$$

$$I_N f(x) := \sum_{n=0}^{N} \tilde{f}_n T_n(x),$$

$$\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{2}(1+\delta_{0n})\delta_{nm}.$$

	Abscissas $\{x_i\}_{i=0,\cdots,N}$	weight $\{w_i\}_{i=0,\cdots,N}$
Gauss-Chebyshev	$x_i = \cos \dfrac{(2i+1)\pi}{2N+2}$	$w_i = \dfrac{\pi}{N+1}$
Gauss-Radau -Chebyshev	$x_i = \cos \dfrac{2\pi i}{2N+1}$	$w_0 = \dfrac{\pi}{2N+1}$, and $w_i \ \dfrac{2\pi}{2N+1}$
Gauss-Lobatto -Chebyshev	$x_i = \cos \dfrac{\pi i}{N}$	$w_0 = w_N = \dfrac{\pi}{2N}$ $w_i = \dfrac{\pi}{N}$

Some linear operations to the Chebyshev interpolant.

For some linear operators L acting on the interpolant

$$L[I_N f(x)] := \sum_{n=0}^{N} a_n T_n(x), \qquad I_N f(x) := \sum_{n=0}^{N} \tilde{f}_n T_n(x).$$

(1) For L the multiplication of x, $L[I_N f(x)] := \sum_{n=0}^{N} \tilde{f}_n \, x T_n(x)$,

$$a_n = \frac{1}{2}[(1+\delta_{0\,n-1})\tilde{f}_{n-1} + \tilde{f}_{n+1}], \quad (n \geq 1).$$

(2) For L the derivative, $L[I_N f(x)] := \sum_{n=0}^{N} \tilde{f}_n \, T_n^{'}(x)$,

$$a_n = \frac{2}{1+\delta_{0\,n}} \sum_{p=n+1,\, p+n=odd}^{N} p\, \tilde{f}_p.$$

(3) For L the second derivative, $L[I_N f(x)] := \sum_{n=0}^{N} \tilde{f}_n \, T_n^{''}(x)$,

$$a_n = \frac{1}{1+\delta_{0\,n}} \sum_{p=n+2,\; p+n=even}^{N} p(p^2-n^2)\tilde{f}_p.$$

Exc 6-3) Show the above relations using recursion relations for $T_n(x)$.

Convergence property

For a function $f(x) \in C^m$, the truncation error is bounded as follows.

⋆ For Legendre : $||I_N f - f||_{L^2} \le \frac{C}{N^{m-1/2}} \sum_{k=0}^{m} ||f^{(k)}||_{L^2}$.

⋆ For Chebyshev : $||I_N f - f||_{L^2_w} \le \frac{C}{N^m} \sum_{k=0}^{m} ||f^{(k)}||_{L^2_w}$.

$$||I_N f - f||_\infty \le \frac{C}{N^{m-1/2}} \sum_{k=0}^{m} ||f^{(k)}||_\infty .$$

For C^1 – functions, the error decays faster than any power of N.
(evanescent error)

Differential equation solver.

Consider a system differential equations of the following form.

$$\begin{aligned} Lf(x) &= S(x) \quad \text{for} \quad x \in U \\ Bf(x) &= 0 \quad \text{for} \quad x \in \partial U \end{aligned}$$

L and B are linear differential operators.

Numerically constructed function $f_{\text{num}}(x)$ is called admissible solution, if

1) $Bf_{\text{num}}(x) = 0$ at $x \in \partial U$ i.e. satisfies boundary condition exactly, and

2) Residual $R(x) := Lf_{\text{num}}(x) - S(x)$ at $\forall x \in U$ is small.

Weighted residual method requires that, for N+1 test functions $\xi_n(x)$

$$(\xi_n, R)_N = 0 \text{ for } \forall n = 0, \cdots, N.$$

(Or its continuum version $(\xi_n, R) = 0$ for $\forall n = 0, \cdots, N$.)

For the spectral method, $f_{\text{num}}(x) \to I_N f(x)$. Threfore For a system,

$$\begin{aligned} L(I_N f(x)) &= S(x), \quad x \in U, \\ B(I_N f(x)) &= 0, \quad x \in \partial U, \end{aligned}$$

we impose $(\xi_n, L(I_N f) - S)_N = 0, \quad \text{for} \quad \forall n = 0, \cdots, N.$

Recall: Notation for the spectral expansion.

$$I_N f(x) := \sum_{n=0}^{N} \tilde{f}_n \, \phi_n(x), \qquad \tilde{f}_n := \frac{1}{\gamma_n} \sum_{i=0}^{N} w_i f(x_i)\, \phi_n(x_i) = \frac{(f, \phi_n)_N}{(\phi_n, \phi_n)_N},$$

$$(f, \phi_n)_N := \sum_{i=0}^{N} w_i f(x_i)\, \phi_n(x_i)$$

$$\gamma_n := \sum_{i=0}^{N} w_i [\, \phi_n(x_i)\,]^2 =: (\phi_n, \phi_n)_N.$$

Gauss type quadrature formula (including Radau, Lobatto) is used.

Continuum.

$$P_N f(x) := \sum_{n=0}^{N} \hat{f}_n \, \phi_n(x), \qquad \hat{f}_n := \frac{\int_a^b f(x)\, \phi_n(x) w(x) dx}{\int_a^b \phi_n(x)\, \phi_n(x) w(x) dx} = \frac{(f, \phi_n)}{(\phi_n, \phi_n)}$$

$$(f, \phi_n) := \int_a^b f(x)\, \phi_n(x) w(x) dx$$

Three types of solvers.

- Depending on the choice of the spectral basis ϕ_n and the test function ξ_n, one can generate various different types of spectral solvers.
- A manner of imposing boundary conditions also depend on the choice.

(i) The Tau-method.

Choose ϕ_n as one of the orthogonal basis such as $P_n(x)$, $T_n(x)$.

Choose the test function ξ_n the same as the spectral basis ϕ_n .

(ii) The collocation method.

Choose ϕ_n as one of the orthogonal basis such as $P_n(x)$, $T_n(x)$.

Choose the test function $\xi_n = \delta (x - x_n)$ fpr any spectral basis ϕ_n.

(iii) The Galerkin method.

Choose the spectral basis ϕ_n and the test function ξ_n as some linear combinations of orthogonal polynomial basis G_n that satisfies the boundary condition. The basis G_n is called Galerkin basis.

(G_n is not orthogonal in general.)

(i) The Tau-method.

Choose the test function ξ_n the same as the spectral basis ϕ_n . Then solve

$$(\phi_n, L(I_N f) - S)_N = 0, \quad n = 0, \cdots, N \qquad \cdots (*).$$

(Note: here we have N+1 equations for N+1 unknowns.)

- Linear operator, L, acting on the interpolant $I_N f(x) = \sum_{m=0}^{N} \tilde{f}_m \phi_m(x)$ can be replaced by a matrix L_{nm} .

$$L(I_N f)(x) = \sum_{m=0}^{N} \tilde{f}_m \, L\phi_m(x) = \sum_{m=0}^{N} \sum_{p=0}^{N} L_{pm} \, \tilde{f}_m \, \phi_p(x)$$

$$(\phi_n, L(I_N f))_N = \sum_{m=0}^{N} \sum_{p=0}^{N} L_{pm} \, \tilde{f}_m \, (\phi_n, \phi_p)_N = \gamma_n \sum_{m=0}^{N} L_{nm} \, \tilde{f}_m$$

$$(\phi_n, S)_N = \gamma_n \, \tilde{S}_n, \qquad (\phi_n, \phi_p)_N = \gamma_n \, \delta_{np}, \quad n = 0, \cdots, N.$$

Therefore $(*)$ becomes

$$\sum_{m=0}^{N} L_{nm} \, \tilde{f}_m = \tilde{S}_n, \quad n = 0, \cdots, N$$

- A few of these equations with the largest n are replaced by the boundary condition. (The number is that of the boundary condition.)

(i) The Tau-method (continued).

Boundary condition: suppose operator on the boundary B is linear,

$$B(I_N f)(x) = \sum_{m=0}^{N} \tilde{f}_m \, B\, \phi_m(x) = \sum_{m=0}^{N} \sum_{p=0}^{N} B_{pm} \, \tilde{f}_m \, \phi_p(x)$$

ex) Dirichlet boundary $Bf(x)|_{x=0} = f(a) - g = 0$

$$\sum_{m=0}^{N} \tilde{f}_m \, \phi_m(a) = g.$$

A test problem.

Consider 2 point boundary value problem of the second order ODE,

$$\frac{d^2 f}{dx^2} - 4\frac{df}{dx} + 4f = \exp[x] + C$$

with $x \in [-1, 1]$, $C = -4e/(1+e^2)$, and boundary conditions, $f(-1) = 0$, and $f(1) = 0$.

- This boundary value problem has unique exact solution, $f_{\text{sol}} = \exp[x] - \frac{\sinh(1)}{\sinh(2)}\exp(2x) + \frac{C}{4}$.

The linear operator $L := \frac{d^2}{dx^2} - 4\frac{d}{dx} + 4\,\text{Id}$ becomes a matrix when it operate to an Interpolant.

Example: Apply Tau-method to the test problem with the Chebyshev basis.

When the spectral basis is the Chebyshev polynomials,

$$L(I_N f)(x) = \sum_{m=0}^{N} \tilde{f}_m \, L T_m(x)$$

$$= \sum_{m=0}^{N} \sum_{p=0}^{N} L_{pm} \, \tilde{f}_m \, T_p(x)$$

For $N = 4$, $L_{ij} = \begin{pmatrix} 4 & -4 & 4 & -12 & 32 \\ 0 & 4 & -16 & 24 & -32 \\ 0 & 0 & 4 & -24 & 48 \\ 0 & 0 & 0 & 4 & -32 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$.

Example: Apply Tau-method to the test problem with the Chebyshev (Continued)

The spectral expansion of the R.H.S

$S(x) = \exp[x] - 4e/(1+e^2)$ becomes For $N=4$, $\tilde{S}_n = \begin{pmatrix} -0.03 \\ 1.13 \\ 0.27 \\ 0.0449 \\ 0.00547 \end{pmatrix}$.

$$\begin{pmatrix} 4 & -4 & 4 & -12 & 32 \\ 0 & 4 & -16 & 24 & -32 \\ 0 & 0 & 4 & -24 & 48 \\ 0 & 0 & 0 & 4 & -32 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \widehat{f}_0 \\ \widehat{f}_1 \\ \widehat{f}_2 \\ \widehat{f}_3 \\ \widehat{f}_4 \end{pmatrix} = \begin{pmatrix} -0.03 \\ 1.13 \\ 0.27 \\ 0.0449 \\ 0.0055 \end{pmatrix} \leftarrow \sum_{m=1}^{4} L_{nm}\tilde{f}_m = \tilde{S}_n \quad \cdots (**) \quad n = 0, \cdots, 4$$

Boundary conditions $f(-1) = 0$, and $f(1) = 0$

$$B(I_N f)(-1) = \sum_{m=0}^{N} \tilde{f}_m T_m(-1) = \sum_{m=0}^{N} (-1)^m \tilde{f}_m = 0$$

$$T_n(-1) = (-1)^n$$

$$B(I_N f)(1) = \sum_{m=0}^{N} \tilde{f}_m T_m(1) = \sum_{m=0}^{N} \tilde{f}_m = 0$$

$$T_n(1) = 1$$

Replace two largest componets (n = 4 and 3) of $(**)$ with the two boundary conditions.

$$\begin{pmatrix} 4 & -4 & 4 & -12 & 32 \\ 0 & 4 & -16 & 24 & -32 \\ 0 & 0 & 4 & -24 & 48 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \widehat{f}_0 \\ \widehat{f}_1 \\ \widehat{f}_2 \\ \widehat{f}_3 \\ \widehat{f}_4 \end{pmatrix} = \begin{pmatrix} -0.03 \\ 1.13 \\ 0.27 \\ 0 \\ 0 \end{pmatrix}$$

Done!

(ii) The collocation method.

Choose ϕ_n as one of the orthogonal basis such as $P_n(x)$, $T_n(x)$.

Choose the test function $\xi_n = \delta(x - x_n)$ fpr any spectral basis ϕ_n.

Then solve, $(\delta(x - x_n), L(I_N f) - S) = 0, \quad n = 0, \cdots, N.$

This is rewritten $L(I_N f)(x_n) = S(x_n)$, or,

$$\sum_{m=0}^{N} \sum_{p=0}^{N} L_{pm}\, \phi_p(x_n) \tilde{f}_m = S(x_n), \quad n = 0, \cdots, N$$

Note the difference from the Tau method.

LHS double sum. RHS not a spectral coefficients

The boundary points are also taken as the collocation points. (Lobatto)

The equations at the boundaries are replaced by the boundary conditions.

Ex). A test problem with Chebyshev basis.

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 4 & -6.83 & 15.3 & -26.1 & 28 \\ 4 & -4 & 0 & 12 & -12 \\ 4 & -1.17 & -7.31 & 2.14 & 28 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \\ \tilde{f}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -0.80 \\ -0.30 \\ 0.73 \\ 0 \end{pmatrix}$$

Exc 6-4) Make a spectral code to solve the same test problem using the collocation method. Try both of Chebyshev and Legendre basis.

Estimate the norm $\|I_N f - f\|$ for the different N.

(iii) The Galerkin method.

Choose the spectral basis ϕ_n and the test function ξ_n as some linear combinations of orthogonal polynomial basis G_n that satisfies the boundary condition. The basis G_n is called Galerkin basis.

- The Galerkin basis is not orthogonal in general.
- It is usually better to construct G_n that relates to a certain orthogonal basis ϕ_n in a simple manner (no general recipe for the construction.)

Ex)
$$G_{2k}(x) = T_{2k+2}(x) - T_0(x)$$
$$G_{2k+1}(x) = T_{2k+3}(x) - T_1(x).$$

- Highest order of the basis should be N – 1 to maintain a consistent degree of approximation. (so the highest basis appears is $T_N(x)$.)

Ex) Consider the case with two point boundary value problem.

Number of collocation points is N + 1.

Since two boundary condition is imposed on the Galerkin basis $\{G_n\}$
$\{G_n\}$: N – 1 are basis, n= 0, …, N – 2 .

Assume that $\{G_n\}$ can be constructed from a linear combination of the orthogonal basis $\{\phi_n\}$. Then we may introduce a matrix M_{mn} such that

$$G_n(x) = \sum_{m=0}^{N} M_{mn}\,\phi_m(x), \quad \text{where} \quad M_{mn} \quad \text{is } (N+1)\times(N-1) \text{ matrix}$$

The interpolant is defined by $I_N f(x) = \sum_{n=0}^{N-2} \tilde{f}_n^G\, G_n(x).$

Taking the test function ξ_n the same as Galerkin basis G_n ,

$$(G_n, L(I_N f) - S)_N = 0, \quad n = 0, \cdots, N-2, \quad \text{are solved for } \tilde{f}_n^G.$$

Exc 6-5) Show that this equation is wrtten

$$\sum_{m=0}^{N-2} \tilde{f}_m^G \sum_{p=0}^{N} \sum_{k=0}^{N} M_{kn}\, M_{pm}\, L_{kp}\, (\phi_k, \phi_k)_N = \sum_{m=0}^{N} M_{mn}\, \tilde{S}_m\, (\phi_m, \phi_m)_N$$

Finally, using transformation matrix M_{mn} again, we spectral coefficients

$$I_N f(x) = \sum_{n=0}^{N-2} \tilde{f}_n^G\, G_n(x) = \sum_{m=0}^{N} \left(\sum_{n=0}^{N-2} M_{mn}\, \tilde{f}_n^G \right) \phi_m(x) = \sum_{m=0}^{N} \tilde{f}_m\, \phi_m(x).$$

A comparison of erros of the different method.