SET THEORY

Set Theory - Definitions and notation

A ***set*** is an unordered collection of objects referred to as elements.

A set is said to contain its elements.

Different ways of describing a set.

1 – Explicitly: listing the elements of a set

$\{1, 2, 3\}$ is the set containing "1" and "2" and "3." list the members between braces.

$\{1, 1, 2, 3, 3\} = \{1, 2, 3\}$ since repetition is irrelevant.

$\{1, 2, 3\} = \{3, 2, 1\}$ since sets are unordered.

$\{1,2,3, \ldots, 99\}$ is the set of positive integers less than 100; use ellipses when the general pattern of the elements is obvious.

$\{1, 2, 3, \ldots\}$ is a way we denote an infinite set (in this case, the natural numbers).

$\varnothing = \{\}$ is the empty set, or the set containing no elements.

Note: $\varnothing \neq \{\varnothing\}$

Set Theory - Definitions and notation

2 – Implicitly: by using a set builder notations, stating the property or properties of the elements of the set.

$S = \{m| \ 2 \leq m \leq 100, \ m \text{ is integer}\}$

S is
the set of
all m
such that
m is between 2 and 100
and
m is integer.

Set Theory - Ways to define sets

: and | are read "such that" or "where"

Explicitly: {John, Paul, George, Ringo}

Implicitly: {1,2,3,…}, or {2,3,5,7,11,13,17,…}

Set builder: { x : x is prime }, { x | x is odd }. In general { x : P(x) is true }, where P(x) is some description of the set.

Let D(x,y) denote "x is divisible by y."

Give another name for

$$\{ x : \forall y\, ((y > 1) \wedge (y < x)) \rightarrow \neg D(x,y) \}.$$

Primes

Can we use **any** predicate P to define a set S = { x : P(x) }?

"the set of all sets that do not contain themselves as members"

Set Theory - Russell's Paradox

Can we use **any** predicate P to define a set
$S = \{ x : P(x) \}$?

No!

Define $S = \{ x : x \text{ is a set where } x \notin x \}$

Then, if $S \in S$, then by defn of S, $S \notin S$.

So S must not be in S, right?

But, if $S \notin S$, then by defn of S, $S \in S$.

ARRRGH!

There is a town with a barber who shaves all the people (and only the people) who don't shave themselves.

Who shaves the barber?

Set Theory - Russell's Paradox

There is a town with a barber who shaves all the people (and only the people) who don't shave themselves.

Who shaves the barber?

Does the barber shave himself?
If the barber does not shave himself, he must abide by the rule and shave himself.
If he does shave himself, according to the rule he will not shave himself.

$$(\exists x)\ (barber(x) \land (\forall y)(\neg shaves(y, y) \leftrightarrow shaves(x, y))$$

This sentence is unsatisfiable (a contradiction) because of the universal quantifier.
The universal quantifier y will include every single element in the domain, including our infamous barber x. So when the value x is assigned to y, the sentence can be rewritten to:

$$\{\neg shaves(x,x) \leftrightarrow shaves(x,x)\} \equiv$$

$$\{(shaves(x,x) \lor shaves(x,x)) \land (\neg shaves(x,x) \lor \neg shaves(x,x))\} \equiv$$

$$\{shaves(x,x) \land (\neg shaves(x,x)\}$$

Contradiction!

Set Theory - Definitions and notation

Important Sets

N = {0,1,2,3,…}, the set of **natural numbers**, non negative integers, (occasionally IN)

Z = { …, -2, -1, 0, 1, 2,3, …), the set of **integers**

$\mathbf{Z}^+$ = {1,2,3,…} set of **positive integers**

Q = {p/q | p $\in$ Z, q $\in$Z, and q$\neq$0}, set of **rational numbers**

R, the set of real numbers

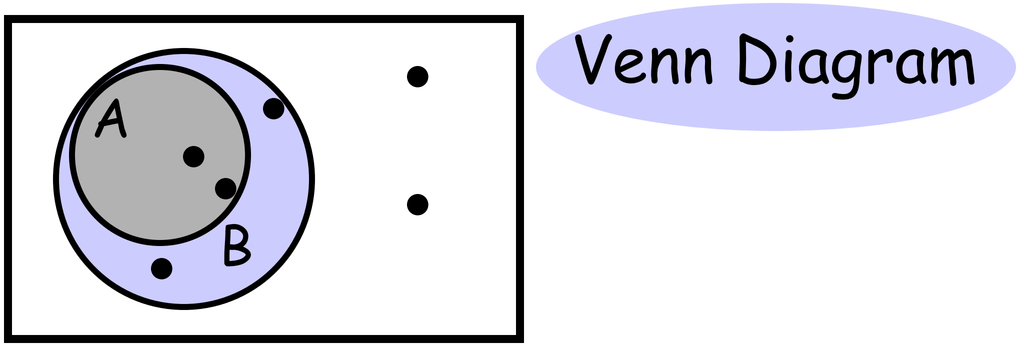
Note: Real number are the numbers that can be represented by an infinite decimal representation, such as 3.4871773339…. The real numbers include both **rationa**l, and **irrational** numbers such as π and the $\sqrt{2}$ and can be represented as points along an infinitely long number line.

Set Theory - Definitions and notation

$x \in S$ means "x is an element of set S."
$x \notin S$ means "x is not an element of set S."

$A \subseteq B$ means "A is a subset of B."

or, "B contains A."
or, "every element of A is also in B."
or, $\forall x\, ((x \in A) \rightarrow (x \in B))$.



Set Theory - Definitions and notation

$A \subseteq B$ means "A is a subset of B."
$A \supseteq B$ means "A is a superset of B."

$A = B$ if and only if A and B have exactly the same elements.

iff, $A \subseteq B$ and $B \subseteq A$
iff, $A \subseteq B$ and $A \supseteq B$
iff, $\forall x \, ((x \in A) \leftrightarrow (x \in B))$.

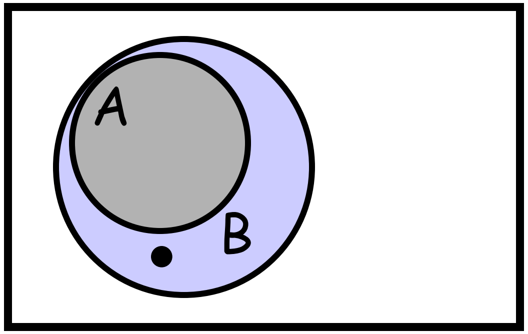
So to show equality of sets A and B, show:

- $A \subseteq B$
- $B \subseteq A$

Set Theory - Definitions and notation

A $\subset$ B means “A is a proper subset of B.”

- A $\subseteq$ B, and A $\neq$ B.
- $\forall x\ ((x \in A) \rightarrow (x \in B)) \wedge \neg\forall x\ ((x \in B) \rightarrow (x \in A))$
- $\forall x\ ((x \in A) \rightarrow (x \in B)) \wedge \exists x\ \neg(\neg(x \in B) \vee (x \in A))$
- $\forall x\ ((x \in A) \rightarrow (x \in B)) \wedge \exists x\ ((x \in B) \wedge \neg(x \in A))$
- $\forall x\ ((x \in A) \rightarrow (x \in B)) \wedge \exists x\ ((x \in B) \wedge (x \notin A))$



Set Theory - Definitions and notation

Quick examples:

$\{1,2,3\} \subseteq \{1,2,3,4,5\}$

$\{1,2,3\} \subset \{1,2,3,4,5\}$

Is $\emptyset \subseteq \{1,2,3\}$? Yes! $\forall x\ (x \in \emptyset) \rightarrow (x \in \{1,2,3\})$ holds (for all over empty domain)

Is $\emptyset \in \{1,2,3\}$? No!

Is $\emptyset \subseteq \{\emptyset,1,2,3\}$? Yes!

Is $\emptyset \in \{\emptyset,1,2,3\}$? Yes!

Set Theory - Definitions and notation

A few more:

Is $\{a\} \subseteq \{a\}$? Yes

Is $\{a\} \in \{a,\{a\}\}$? Yes

Is $\{a\} \subseteq \{a,\{a\}\}$? Yes

Is $\{a\} \in \{a\}$? No

Set Theory - Cardinality

If S is finite, then the *cardinality* of S, |S|, is the number of distinct elements in S.

If S = {1,2,3}, |S| = 3.

If S = {3,3,3,3,3}, |S| = 1.

If S = ∅, |S| = 0.

If S = { ∅, {∅}, {∅,{∅}} }, |S| = 3.

If S = {0,1,2,3,…}, |S| is infinite.

Set Theory - Power sets Multisets

If S is a set, then the ***power set*** of S is

$$2^S = \{ x : x \subseteq S \}.$$

aka P(S)

We say, "P(S) is the set of all subsets of S."

If $S = \{a\}$, $2^S = \{\emptyset, \{a\}\}$.

If $S = \{a,b\}$, $2^S = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$.

If $S = \emptyset$, $2^S = \{\emptyset\}$.

If $S = \{\emptyset,\{\emptyset\}\}$, $2^S = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset,\{\emptyset\}\}\}$.

Fact: if S is finite, $|2^S| = 2^{|S|}$. (if $|S| = n$, $|2^S| = 2^n$)

Why?

Set Theory – Ordered Tuples
Cartesian Product

When order matters, we use ordered n-tuples

The ***Cartesian Product*** of two sets A and B is:

$$A \times B = \{ \langle a,b \rangle : a \in A \wedge b \in B \}$$

If A = {Charlie, Lucy, Linus}, and B = {Brown, VanPelt}, then

A x B = {<Charlie, Brown>, <Lucy, Brown>, <Linus, Brown>, <Charlie, VanPelt>, <Lucy, VanPelt>, <Linus, VanPelt>}

We'll use these special sets soon!

$$A_1 \times A_2 \times \ldots \times A_n = \{\langle a_1, a_2, \ldots, a_n \rangle : a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

Size?

n^n if $(\forall i)\ |A_i| = n$

A,B finite → |AxB| = ?

a) AxB
b) |A|+|B|
c) |A+B|
d) |A||B|

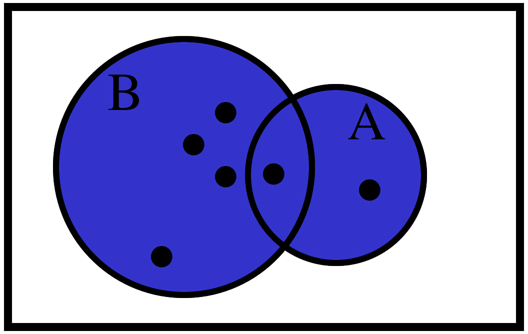
Set Theory - Operators

The *union* of two sets A and B is:

$$A \cup B = \{ x : x \in A \vee x \in B \}$$

If A = {Charlie, Lucy, Linus}, and B = {Lucy, Desi}, then

A ∪ B = {Charlie, Lucy, Linus, Desi}



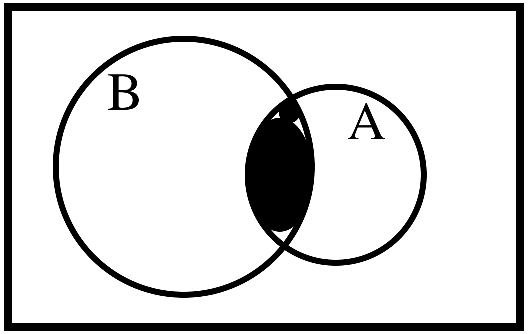
Set Theory - Operators

The *intersection* of two sets A and B is:

$$A \cap B = \{ x : x \in A \wedge x \in B\}$$

If A = {Charlie, Lucy, Linus}, and B = {Lucy, Desi}, then

$A \cap B = \{\text{Lucy}\}$



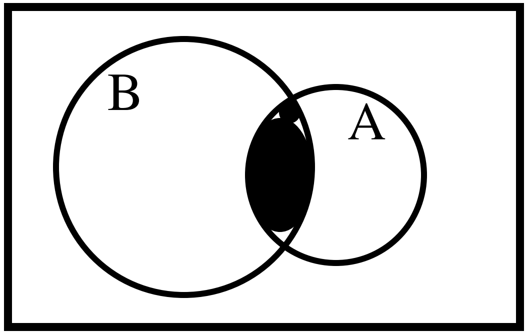
Set Theory - Operators

The *intersection* of two sets A and B is:

$$A \cap B = \{ x : x \in A \wedge x \in B\}$$

If A = {x : x is a US president}, and B = {x : x is deceased}, then

$A \cap B$ = {x : x is a deceased US president}



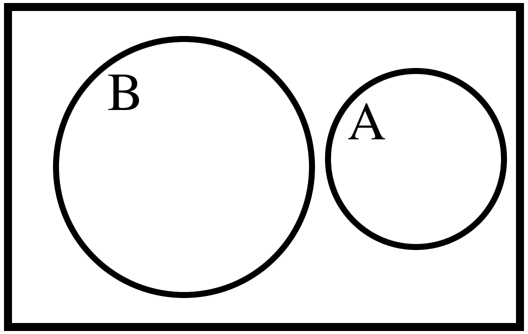
Set Theory - Operators

The *intersection* of two sets A and B is:

$$A \cap B = \{ x : x \in A \wedge x \in B \}$$

If A = {x : x is a US president}, and B = {x : x is in this room}, then

$A \cap B$ = {x : x is a US president in this room} = $\varnothing$



Sets whose intersection is empty are called *disjoint* sets

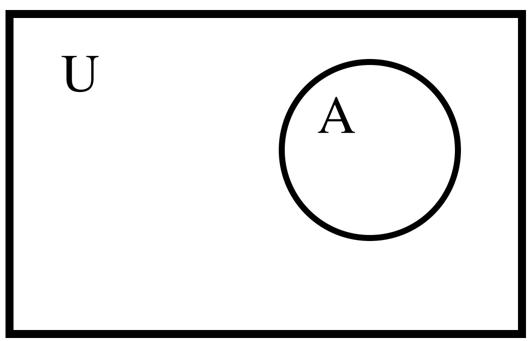
Set Theory - Operators

The *complement* of a set A is:

$$\overline{A} = \{ x : x \notin A \}$$

If A = {x : x is bored}, then

$\overline{A}$ = {x : x is not bored}



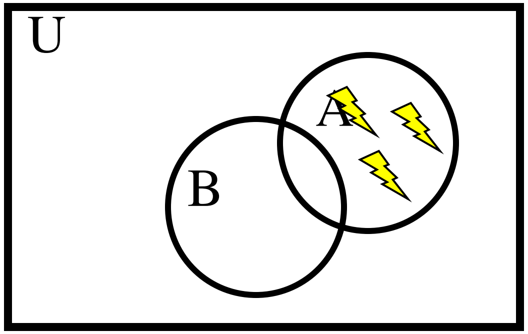
$\overline{\varnothing} = U$
and
$\overline{U} = \varnothing$

I.e., $\overline{A} = U - A$, where U is the universal set.

"A set fixed within the framework of a theory and consisting of all objects considered in the theory. "

Set Theory - Operators

The *set difference*, A - B, is:



$$A - B = \{ x : x \in A \land x \notin B \}$$

$$A - B = A \cap \overline{B}$$

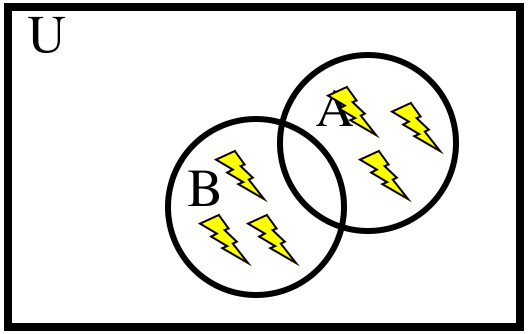
Set Theory - Operators

The *symmetric difference*, A ⊕ B, is:

$$A \oplus B = \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

$$= (A - B) \cup (B - A)$$

like "exclusive or"



Set Theory - Operators

Theorem: $A \oplus B = (A - B) \cup (B - A)$

Proof: $A \oplus B = \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$

$= \{ x : (x \in A - B) \vee (x \in B - A)\}$

$= \{ x : x \in ((A - B) \cup (B - A))\}$

$= (A - B) \cup (B - A)$

.

Directly from defns.
Semantically clear.

Set Theory - Identities

Identity

$$A \cap U = A$$
$$A \cup \varnothing = A$$

Domination

$$A \cup U = U$$
$$A \cap \varnothing = A$$

Idempotent

$$A \cup A = A$$
$$A \cap A = A$$

Set Theory – Identities, cont.

Complement Laws

$$A \cup \overline{A} = U$$

$$A \cap \overline{A} = \varnothing$$

Double complement

$$\overline{\overline{A}} = A$$

Set Theory - Identities, cont.

Commutativity

$A \cup B = B \cup A$

$A \cap B = B \cap A$

Associativity

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Distributivity

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

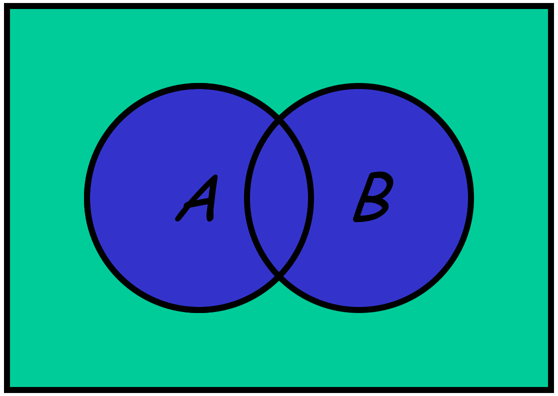
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

DeMorgan's I

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

DeMorgan's II

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$



Proof by "diagram" (useful!), but we aim for a more formal proof.

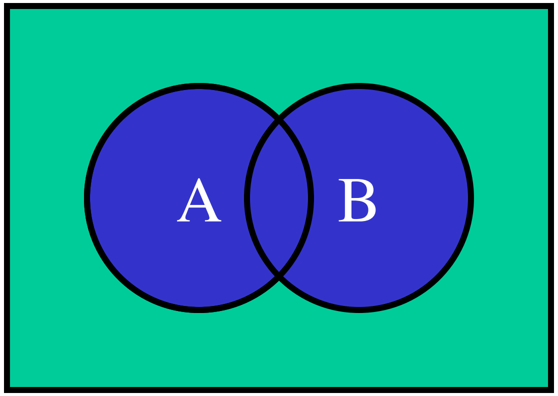
Proving identities

Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ (De Morgan)

1. $(\subseteq)$ $(x \in \overline{A \cup B})$
$\Rightarrow (x \notin (A \cup B))$
$\Rightarrow (x \notin A \text{ and } x \notin B)$
$\Rightarrow (x \in \overline{A} \cap \overline{B})$

2. $(\supseteq)$ $(x \in (\overline{A} \cap \overline{B}))$
$\Rightarrow (x \notin A \text{ and } x \notin B)$
$\Rightarrow (x \notin A \cup B)$
$\Rightarrow (x \in \overline{A \cup B})$

Alt. proof



Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ using a membership table.

0 : x is not in the specified set

1 : otherwise

A	B	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$	$A \cup B$	$\overline{A \cup B}$
1	1	0	0	0	1	0
1	0	0	1	0	1	0
0	1	1	0	0	1	0
0	0	1	1	1	0	1

Haven't we seen this before?

General connection via Boolean algebras

Proof using logically equivalent set definitions.

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

$$\overline{(A \cup B)} = \{x : \neg(x \in A \vee x \in B)\}$$

$$= \{x : \neg(x \in A) \wedge \neg(x \in B)\}$$

$$= \{x : (x \in \overline{A}) \wedge (x \in \overline{B})\}$$

$$= \overline{A} \cap \overline{B}$$

Example

$X \cap (Y - Z) = (X \cap Y) - (X \cap Z)$. True or False?
Prove your response.

$(X \cap Y) - (X \cap Z) = (X \cap Y) \cap (X \cap Z)'$

What kind of law?

$= (X \cap Y) \cap (X' \cup Z')$

Distributive law

$= (X \cap Y \cap X') \cup (X \cap Y \cap Z')$

$= \varnothing \cup (X \cap Y \cap Z')$

$= (X \cap Y \cap Z')$

$= X \cap (Y - Z)$

Note: $\overline{(X \cap Z)} = (X \cap Z)'$ (just different notation)

TABLE 1 Set Identities.

Identity	*Name*
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement laws

Example

Pv that if (A - B) U (B - A) = (A U B) then $\underline{A \cap B = \varnothing}$

Suppose to the contrary, that $A \cap B \neq \varnothing$, and that $x \in A \cap B$.

Then x cannot be in A-B and x cannot be in B-A.

Then x is not in (A - B) U (B - A).

Do you see the contradiction yet?

But x is in A U B since $(A \cap B) \subseteq (A \cup B)$.

Thus, $(A - B) \cup (B - A) \neq (A \cup B)$.

Contradiction.

Thus, $A \cap B = \varnothing$.

a) A = B
b) $A \cap B = \varnothing$
c) $A-B = B-A = \varnothing$

Trying to prove p --> q

Assume p and not q, and find a contradiction.

Our contradiction was that sets weren't equal.

Set Theory - Generalized Union/Intersection

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

Set Theory - Generalized Union/Intersection

Ex. Suppose that:

$$A_i = \{1,2,3,\ldots,i\} \; i = 1,2,3,\ldots$$

$$\bigcup_{i=1}^{\infty} A_i = ? \qquad \bigcup_{i=1}^{\infty} A_i = Z^+$$

$$\bigcap_{i=1}^{\infty} A_i = ? \qquad \bigcap_{i=1}^{\infty} A_i = \{1\}$$

Example:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

Ex. Let U = **N**, and define:

$$A_i = \{x : \exists k > 1, x = ki, k \in \mathrm{N}\} \qquad \text{i=1,2,…,N}$$

A_1 = {2,3,4,…}

A_2 = {4,6,8,…}

A_3 = {6,9,12,…}

$$\overline{\bigcup_{i=2}^{\infty} A_i} = \;?$$

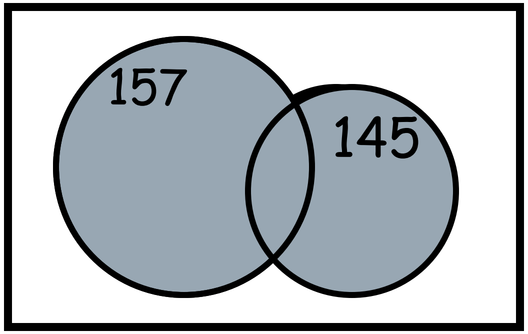
primes

a) Primes
b) Composites
c) $\varnothing$
d) N
e) I have no clue.

Note: i starts at 2

Union is all the composite numbers.

Set Theory - Inclusion/Exclusion



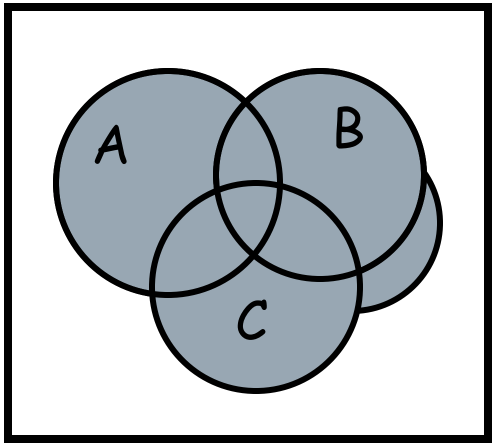
Example:
There are 217 cs majors.
157 are taking CS23021.
145 are taking cs23022.
98 are taking both.

How many are taking neither?

$$217 - (157 + 145 - 98) = 13$$

Generalized Inclusion/Exclusion

Suppose we have:



What about 4 sets?

And I want to know |A U B U C|

|A U B U C| = |A| + |B| + |C|

- |A ∩ B| - |A ∩ C| - |B ∩ C|

+ |A ∩ B ∩ C|

Generalized Inclusion/Exclusion

For sets A_1, A_2,…A_n we have:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{1\le i\le n}\left|A_i\right| - \sum_{1\le i<j\le n}\left|A_i \cap A_j\right| + \ldots + (-1)^{(n-1)}\left|\bigcap_{i=1}^{n} A_i\right|$$

Set Theory - Sets as bit strings

Let $U = \{x_1, x_2, \ldots, x_n\}$, and let $A \subseteq U$.

Then the *characteristic vector* of A is the n-vector whose elements, x_i, are 1 if $x_i \in A$, and 0 otherwise.

Ex. If $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, and $A = \{x_1, x_3, x_5, x_6\}$, then the characteristic vector of A is

(101011)

Sets as bit strings

Ex. If $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$,
$A = \{x_1, x_3, x_5, x_6\}$, and
$B = \{x_2, x_3, x_6\}$,

Then we have a quick way of finding the characteristic vectors of $A \cup B$ and $A \cap B$.

A	1	0	1	0	1	1
B	0	1	1	0	0	1
$A \cup B$	1	1	1	0	1	1
$A \cap B$	0	0	1	0	0	1

Bit-wise OR

Bit-wise AND