Lattices

- Lattice

  A lattice is a poset $(L, \leq)$ in which every subset $\{a, b\}$ consisting of two elements has a least upper bound and a greatest lower bound. we denote

  $LUB(\{a, b\})$ by $a \vee b$ (the join of a and b)

  $GLB(\{a, b\})$ by $a \wedge b$ (the meet of a and b)

Lattices

- Example 1

Let S be a set and let L=P(S). As we have seen, ⊆, containment, is a partial order on L. Let A and B belong to the poset (L, ⊆). Then

$$a \vee b = A \cup B \quad \& \quad a \wedge b = A \cap B$$

Why?

Assuming C is a upper bound of {a, b}, then

$$A \subseteq C \text{ and } B \subseteq C \quad \text{thus } A \cup B \subseteq C$$

Assuming C is a lower bound of {a, b}, then

$$C \subseteq A \text{ and } C \subseteq B \quad \text{thus } C \subseteq A \cap B$$

- Example 2

Consider the poset ($Z+$, $\leq$), where for a and b in $Z+$, $a \leq b$ if and only if $a \mid b$, then

$$a \vee b = LCM(a,b)$$

$$a \wedge b = GCD(a,b)$$

LCM: least common multiple

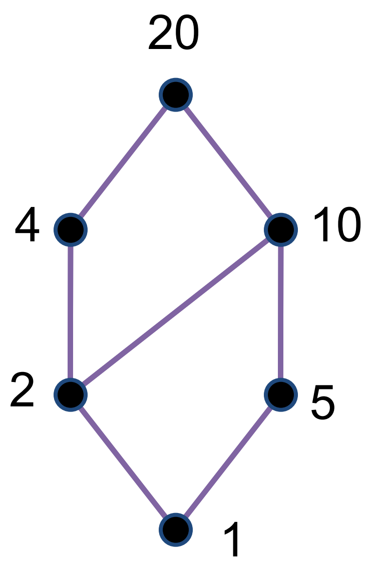
GCD: greatest common divisor

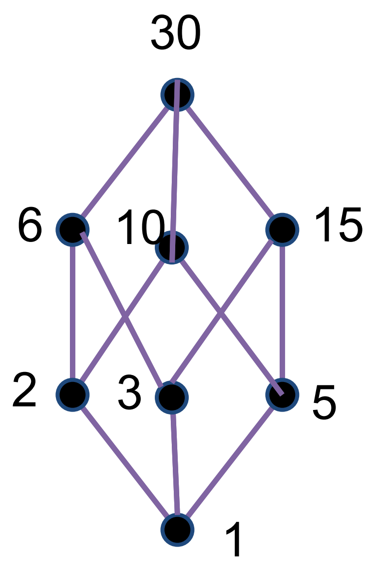
Lattices

- Example 3

 Let n be a positive integer and D_n be the set of all positive divisors of n. Then D_n is a lattice under the relation of divisibility. For instance,

 D_{20}= {1,2,4,5,10,20} D_{30}= {1,2,3,5,6,10,15,20}

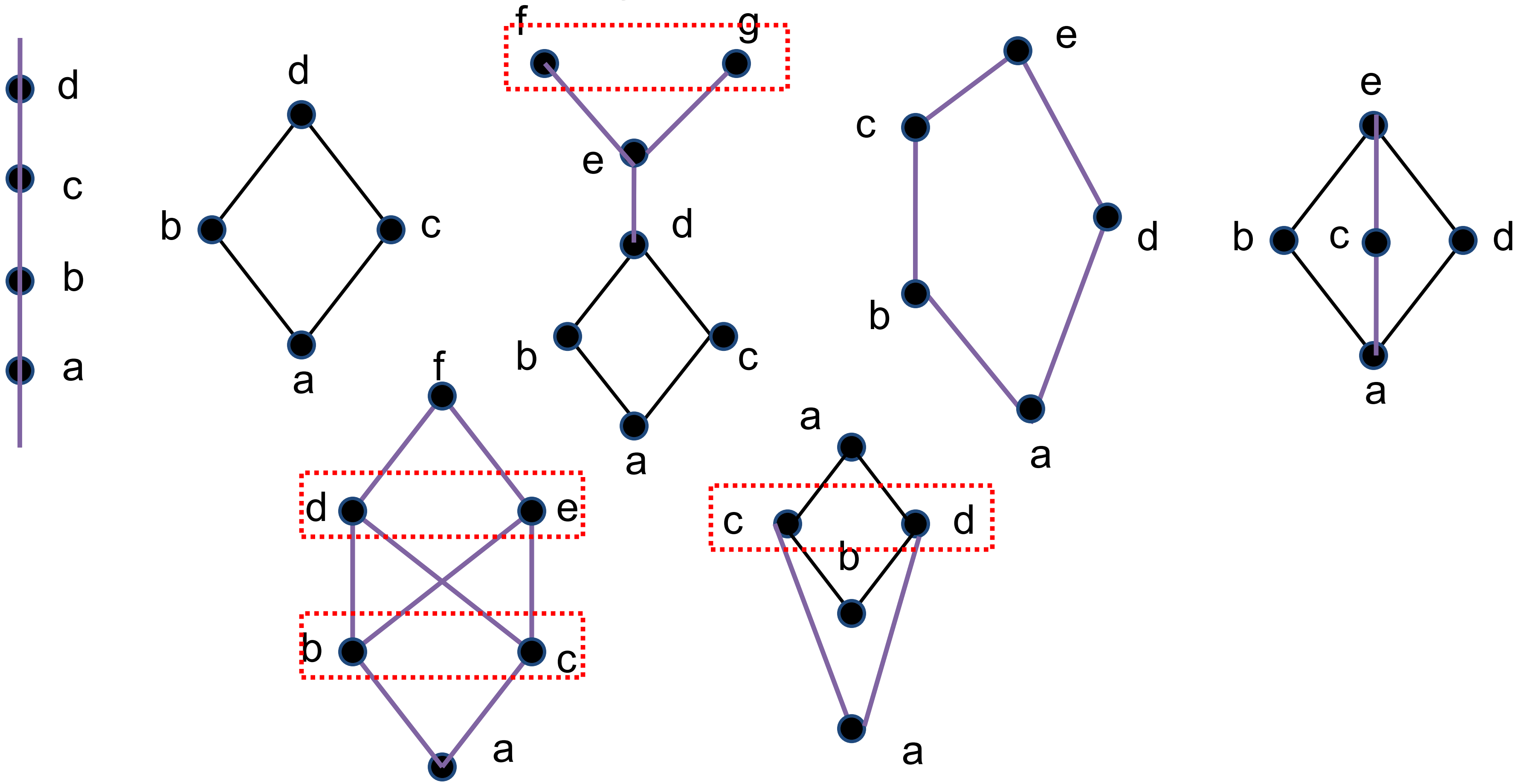




Lattices

- Example 4

  Which of the Hasse diagrams represent lattices?



Lattices

- Example 6

Let S be a set and L =P(S). Then (L, ⊆) is a lattice, and its dual lattice is (L, ⊇), where "⊆" is "contained in", and "⊇" is "contains". Then, in the poset (L, ⊇)

join: $A \vee B = A \cap B$,

meet: $A \wedge B = A \cup B$.

- Theorem 1

If $(L_1, \leq)$ and $(L_2, \leq)$ are lattices, then $(L, \leq)$ is a lattices, where $L = L_1 \times L_2$, and the partial order $\leq$ of L is the product partial order.

Proof: we denote

the join and meet in is L_1 by $\vee_1$ and $\wedge_1$

the join and meet in is L_2 by $\vee_2$ and $\wedge_2$

We know that L is a poset (Theorem 1 in p.219)

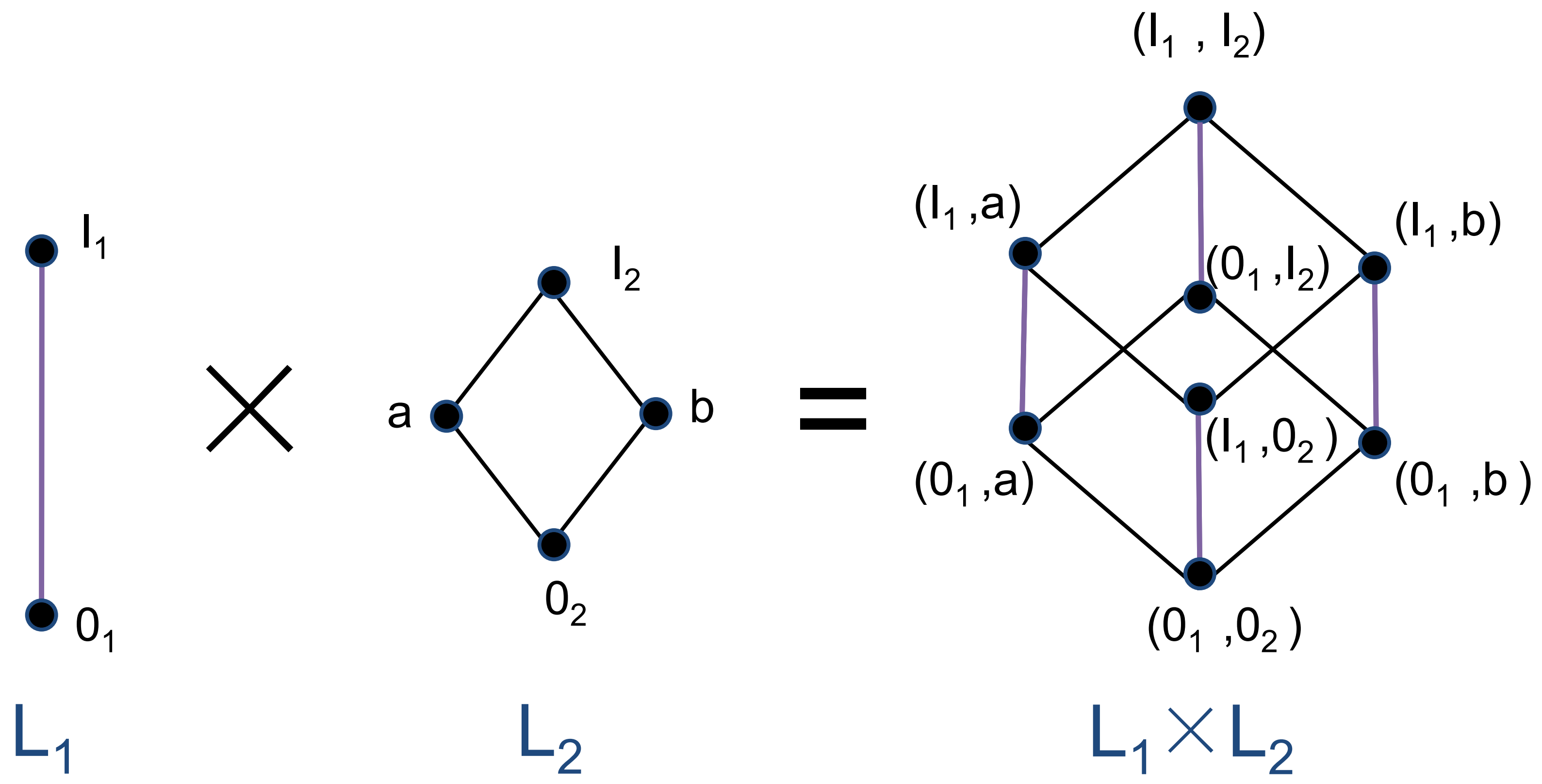
for (a_1,b_1) and (a_2,b_2) in L. then

$$(a_1,b_1) \vee (a_2,b_2) = (a_1 \vee_1 a_2,\ b_1 \vee_2 b_2) \text{ in } L$$

$$(a_1,b_1) \wedge (a_2,b_2) = (a_1 \wedge_1 a_2,\ b_1 \wedge_2 b_2) \text{ in } L$$

Lattices

- Example 7



Lattices

- Sublattice

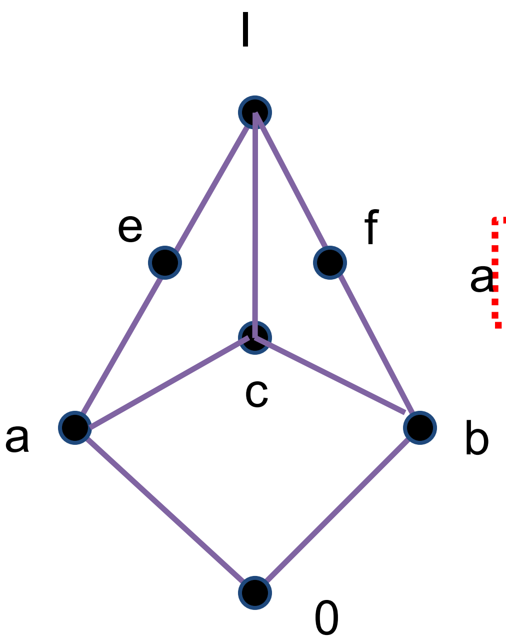
  Let (L, $\leq$) be a lattice. A nonempty subset S of L is called a sublattice of L if a ∨ b in S and a ∧ b in S whenever a and b in S

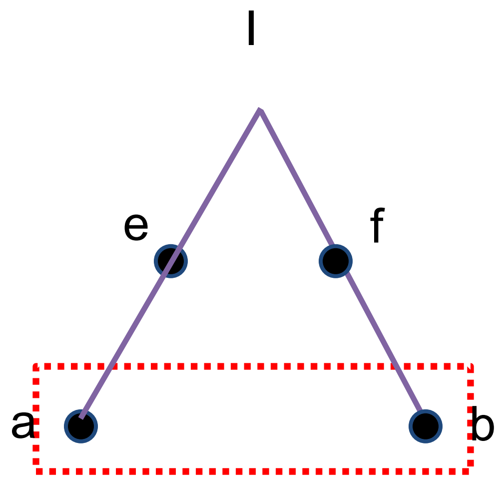
  For instance

  Example 3 is one of sublattices of Example 2

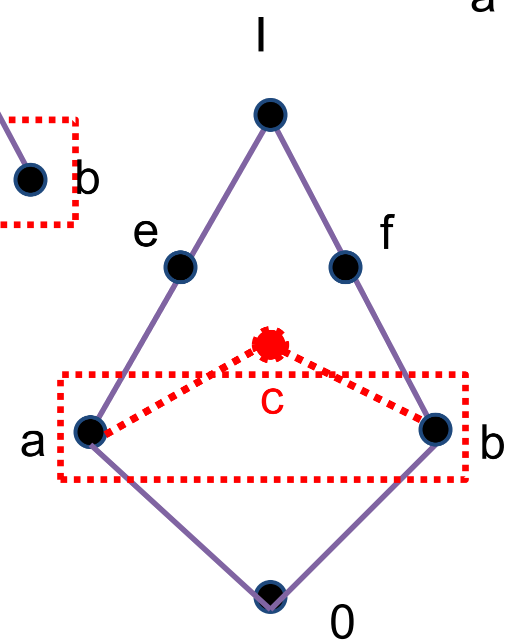
Lattices

- Example 9

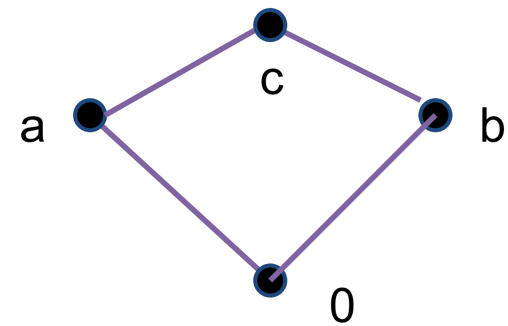




Not a lattice



a lattice, not a Sublattice



a sublattice

Lattices

- Isomorphic Lattices

  If f: $L_1 \rightarrow L_2$ is an isomorphism form the poset $(L_1, \leq_1)$ to the poset $(L_2, \leq_2)$, then L_1 is a lattice if and only if L_2 is a lattice. In fact, if a and b are elements of L_1, then

  $$f(a \vee b) = f(a) \vee f(b) \ \& \ f(a \wedge b) = f(a) \wedge f(b).$$

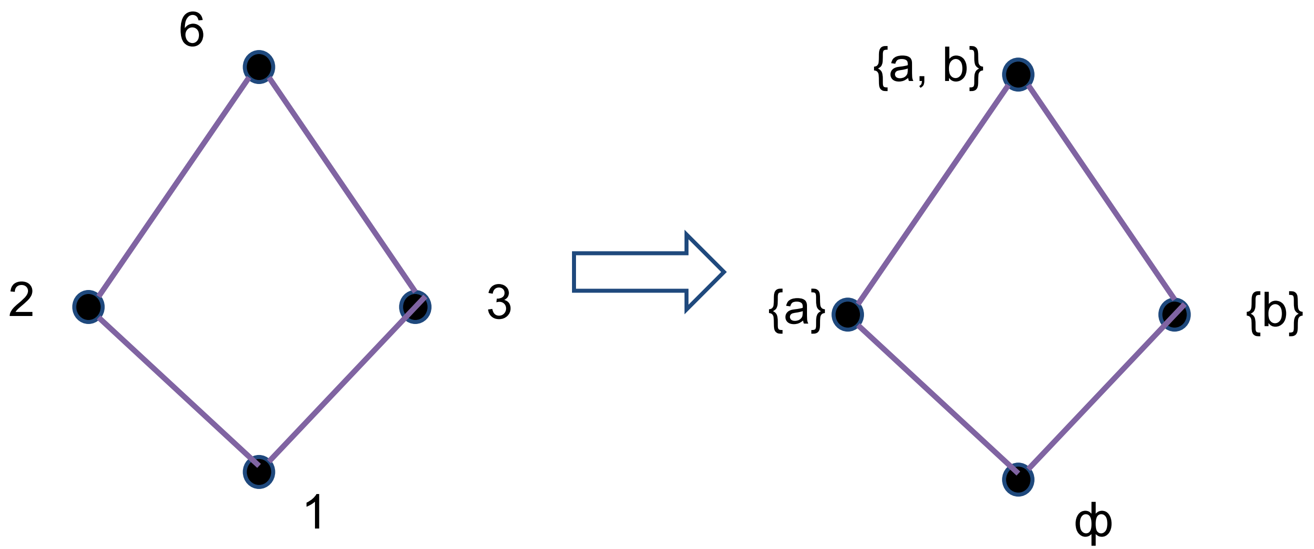
  If two lattices are isomorphic, as posets, we say they are isomorphic lattices.

- Example 10 (P.225 Ex.17)

  Let A={1,2,3,6} and let $\leq$ be the relation | .

  Let A'= {ϕ, {a}, {b}, {a, b}} and let $\leq'$ be set containment, $\subseteq$.

  If f(1)= ϕ, f(2)={a}, f(3)={b}, f(6)={a, b}, then f is an isomorphism. They have the same Hasse diagrams.



Lattices

- $a \vee b$ (LUB{a, b})

1. $a \leq a \vee b$ and $b \leq a \vee b$; $a \vee b$ is an upper bound of a and b
2. If $a \leq c$ and $b \leq c$, then $a \vee b \leq c$; $a \vee b$ is the least upper bound of a and b

- $a \wedge b$ (GLB{a, b})

3. $a \wedge b \leq a$ and $a \wedge b \leq b$; $a \wedge b$ is a lower bound of a and b
4. If $c \leq a$ and $c \leq b$, then $c \leq a \wedge b$; $a \wedge b$ is the greatest lower bound of a and b

- Theorem 2

  Let L be a lattice. Then for every a and b in L

  (a) $a \vee b = b$ if and only if $a \leq b$

  (b) $a \wedge b = a$ if and only if $a \leq b$

  (c) $a \wedge b = a$ if and only if $a \vee b = b$

Proof:

(a) if $a \vee b = b$, since $a \leq a \vee b$, thus $a \leq b$

if $a \leq b$, since $b \leq b$, thus b is a upper bound of a and b, by definition of least upper bound we have $a \vee b \leq b$. since $a \vee b$ is an upper bound of a and b, $b \leq a \vee b$, so $a \vee b = b$

(b) Similar to (a);   (c) the proof follows from (a) & (b)

Lattices

- Example 12

  Let L be a linearly ordered set. If a and b in L, then either $a \leq b$ or $b \leq a$. It follows form Theorem 2 that L is a lattice, since every pair of elements has a least upper bound and a greatest lower bound.

- Theorem 3

Let L be a lattice. Then

1. Idempotent properties: $a \vee a = a$; $a \wedge a = a$

2. Commutative properties: $a \vee b = b \vee a$; $a \wedge b = b \wedge a$

3. Associative properties: **(a)** $(a \vee b) \vee c = a \vee (b \vee c)$

(b) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

4. Absorption properties: **(a)** $a \vee (a \wedge b) = a$

(b) $a \wedge (a \vee b) = a$

Lattices

Proof: 3. (a) $(a \vee b) \vee c = a \vee (b \vee c)$

$a \leq a \vee (b \vee c)$ & $b \vee c \leq a \vee (b \vee c)$

$b \leq b \vee c$ & $c \leq b \vee c$ **(definition of LUB)**

$b \leq b \vee c$ & $c \leq b \vee c$ & $b \vee c \leq a \vee (b \vee c)$ ➔

$b \leq a \vee (b \vee c)$ & $c \leq a \vee (b \vee c)$ **(transitivity)**

$a \leq a \vee (b \vee c)$ & $b \leq a \vee (b \vee c)$ ➔ $a \vee (b \vee c)$ is a upper of a and b

then we have $a \vee b \leq a \vee (b \vee c)$ **(why?)**

$a \vee b \leq a \vee (b \vee c)$ & $c \leq a \vee (b \vee c)$ ➔

$a \vee (b \vee c)$ is a upper of $a \vee b$ and c

then we have $(a \vee b) \vee c \leq a \vee (b \vee c)$

Similarly, $a \vee (b \vee c) \leq (a \vee b) \vee c$

Therefore $(a \vee b) \vee c = a \vee (b \vee c)$ **(why?)**

Lattices

- $(a \vee b) \vee c = a \vee (b \vee c) = a \vee b \vee c$
- $(a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b \wedge c$
- $LUB(\{a_1, a_2, \ldots, a_n\}) = a_1 \vee a_2 \vee \ldots \vee a_n$
- $GLB(\{a_1, a_2, \ldots, a_n\}) = a_1 \wedge a_2 \wedge \ldots \wedge a_n$

Lattices

- Theorem 4

  Let L be a lattice. Then, for every a, b and c in L

  1. If $a \leq b$, then

     (a) $a \vee c \leq b \vee c$

     (b) $a \wedge c \leq b \wedge c$

  2. $a \leq c$ and $b \leq c$ if and only if $a \vee b \leq c$

  3. $c \leq a$ and $c \leq b$ if and only if $c \leq a \wedge b$

  4. If $a \leq b$ and $c \leq d$, then

     (a) $a \vee c \leq b \vee d$

     (b) $a \wedge c \leq b \wedge d$

- **Proof**

1. (a) If $a \leq b$, then $a \vee c \leq b \vee c$

$c \leq b \vee c$; $b \leq b \vee c$ **(definition of LUB)**

$a \leq b$; $b \leq b \vee c$ ➔ $a \leq b \vee c$ **(transitivity)**

therefore,

$b \vee c$ is a upper bound of a and c , which means

$a \vee c \leq b \vee c$ **(why?)**

The proofs for others left as exercises.

Lattices

- Bounded

  A lattice L is said to be bounded if it has a greatest element I and a least element 0

For instance:

Example 15: The lattice P(S) of all subsets of a set S, with the relation containment is bounded. The greatest element is S and the least element is empty set.

Example 13: The lattice Z^+ under the partial order of divisibility is not bounded, since it has a least element 1, but no greatest element.

Lattices

- If L is a bounded lattice, then for all a in A

$$0 \le a \le I$$

$$a \vee 0 = a, \qquad a \vee I = I$$

$$a \wedge 0 = 0\,, \qquad a \wedge I = a$$

Note: I (0) and a are comparable, for all a in A.

- Theorem 5

Let $L=\{a_1,a_2,\ldots,a_n\}$ be a finite lattice. Then L is bounded.

Proof:

The greatest element of L is $a_1 \vee a_2 \vee \ldots \vee a_n$, and the least element of L is $a_1 \wedge a_2 \wedge \ldots \wedge a_n$

- Distributive

A lattice L is called distributive if for any elements a, b and c in L we have the following distributive properties:

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

If L is not distributive, we say that L is nondistributive.

Note: the distributive property holds when

a. any two of the elements a, b and c are equal or

b. when any one of the elements is 0 or I.

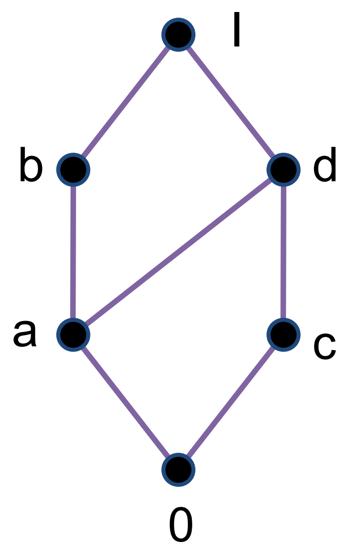
Lattices

- Example 16

  For a set S, the lattice P(S) is distributive, since union and intersection each satisfy the distributive property.

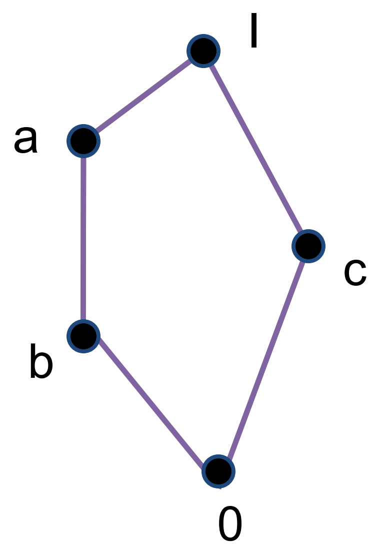
- Example 17

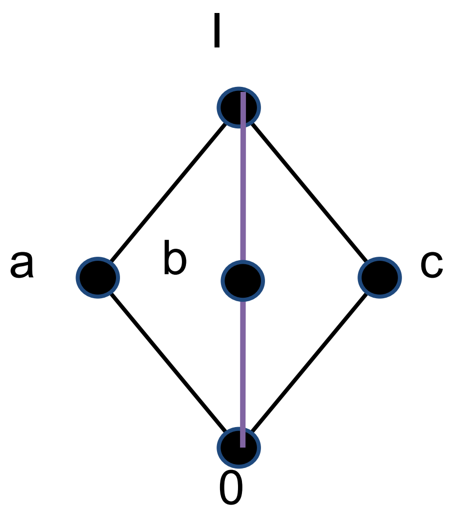
  The lattice whose Hasse diagram shown as follows is distributive.



- Example 18

  Show that the lattices as follows are nondistributive.





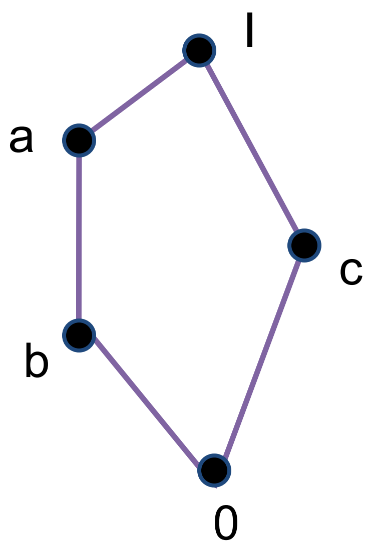
$a \wedge (b \vee c) = a \wedge I = a$
$(a \wedge b) \vee (a \wedge c) = b \vee 0 = b$

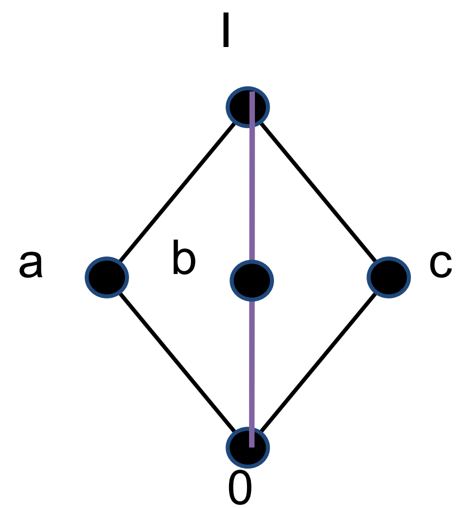
$a \wedge (b \vee c) = a \wedge I = a$
$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

Lattices

- Theorem 6

  A lattice L is nondistributive if and only if it contains a sublattice that is isomorphic to one of the lattices whose Hasse diagrams are as show.





Lattices

- Complement

  Let L be bounded lattice with greatest element I and least element 0, and let a in L. An element a’ in L is called a complement of a if

  $$a \vee a' = I \text{ and } a \wedge a' = 0$$

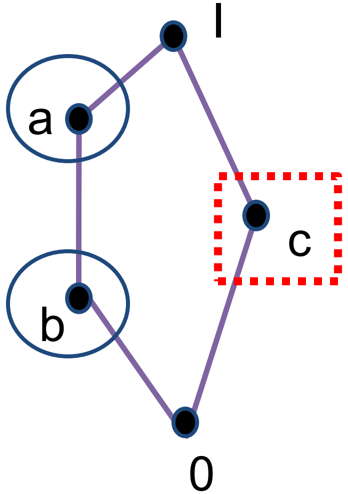
  Note that 0’=I and I’=0

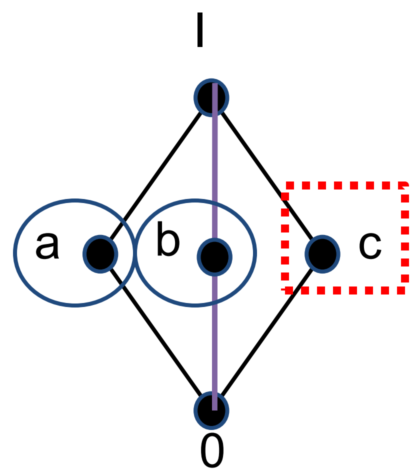
Lattices

- Example 19

  The lattice L=P(S) is such that every element has a complement, since if A in L, then its set complement $\overline{A}$ has the properties $A \vee \overline{A} = S$ and $A \wedge \overline{A} = \phi$. That is, the set complement is also the complement in L.

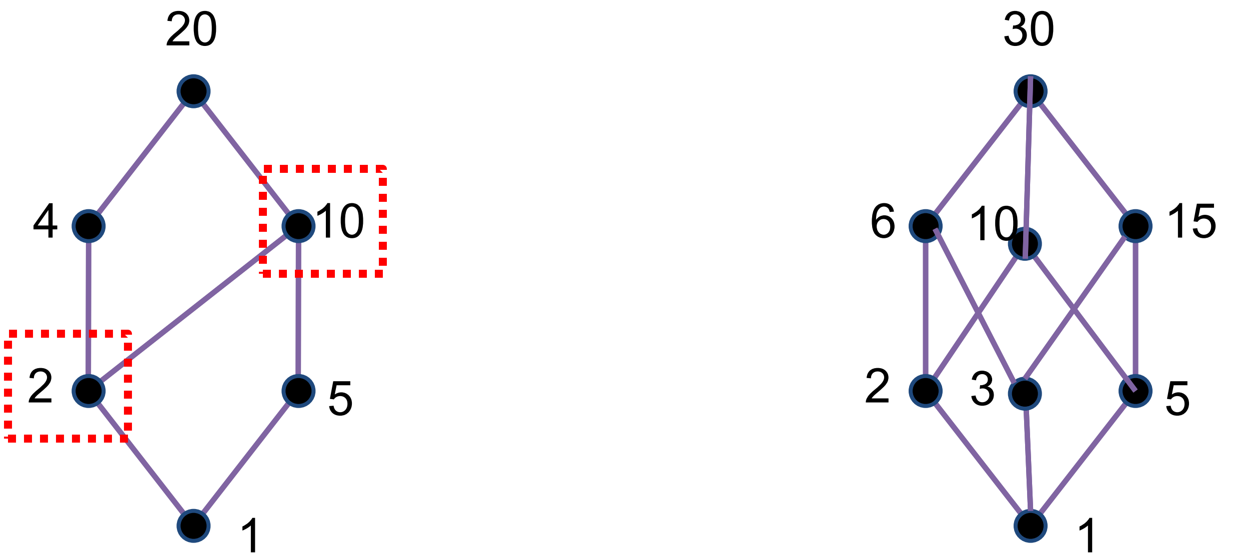
- Example 20





Lattices

- Example 21



D_{20} D_{30}

Lattices

- Theorem 7

  Let L be a bounded distributive lattice. If a complement exists, it is unique.

Proof: Let a' and a'' be complements of the element a in L, then

$$a \vee a' = I, \quad a \vee a'' = I; \qquad a \wedge a' = 0, \quad a \wedge a'' = 0$$

using the distributive laws, we obtain

$$a' = a' \vee 0 = a' \vee (a \wedge a'') = (a' \vee a) \wedge (a' \vee a'')$$
$$= I \wedge (a' \vee a'') = a' \vee a''$$

Also

$$a'' = a'' \vee 0 = a'' \vee (a \wedge a') = (a'' \vee a) \wedge (a'' \vee a')$$
$$= I \wedge (a' \vee a'') = a' \vee a''$$

Hence a'=a''

- Complemented

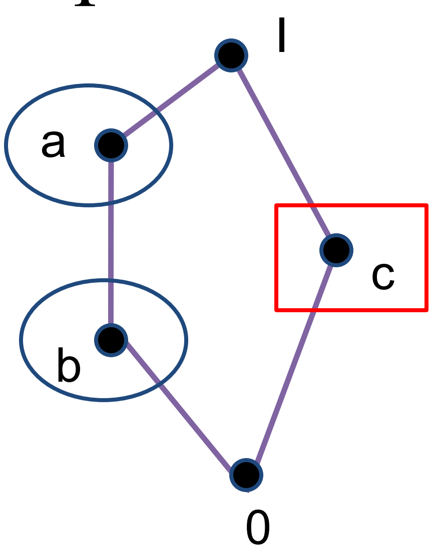
  A lattice L is called complemented if it is bounded and if every element in L has a complement.

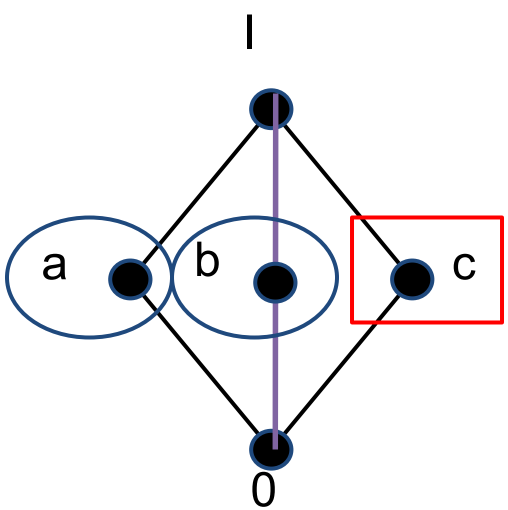
Lattices

- Example 22

The lattice L=P(S) is complemented. Observe that in this case each element of L has a unique complement, which can be seen directly or is implied by Theorem 7.

- Example 23



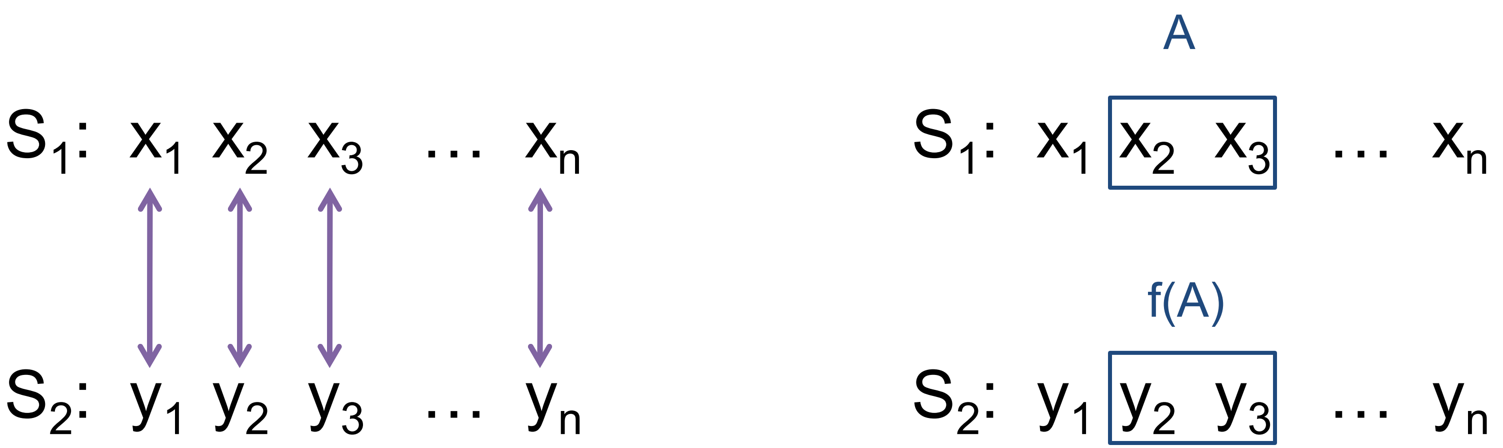


Finite Boolean Algebras

- Theorem 1

  If $S_1=\{x_1,x_2,\ldots,x_n\}$ and $S_2=\{y_1,y_2,\ldots,y_n\}$ are any two finite sets with n elements, then the lattices $(P(S_1), \subseteq)$ and $(P(S_2), \subseteq)$ are isomorphic. Consequently, the Hasse diagrams of these lattices may be drawn identically.

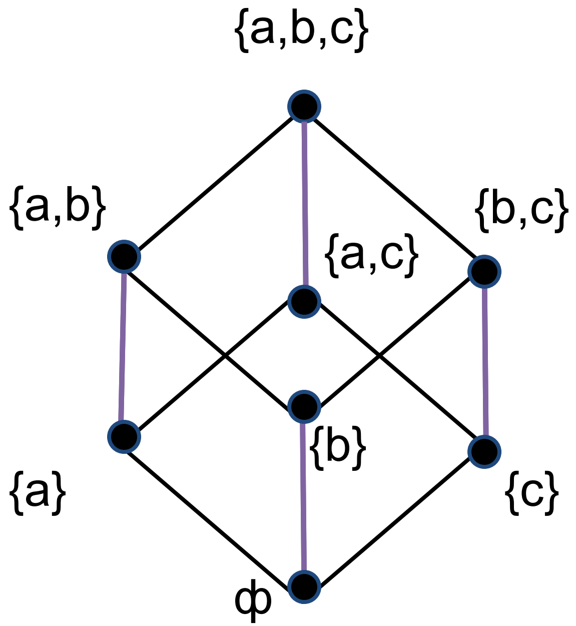
Arrange the elements in S1 and S2

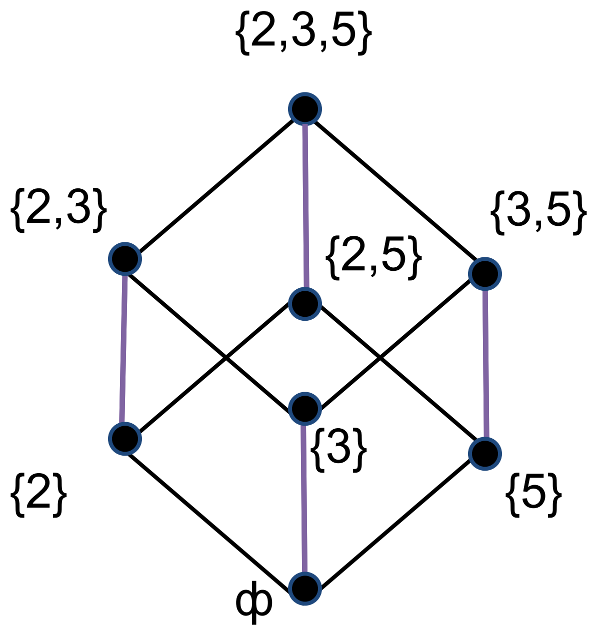


Finite Boolean Algebras

- Example 1:

  S={a, b, c} and T={2,3,5}. Consider the Hasse diagrams of the two lattices (P(S), ⊆) and (P(T), ⊆).





Note : the lattice depends only on the number of elements in set, not on the elements.

Finite Boolean Algebras

- Label the subsets

  Let a set $S=\{a_1,a_2,\ldots,a_n\}$, then P(S) has 2^n subsets. We label subsets by sequences of 0's and 1's of length n.

  For instance,

  $\{a_1,a_2\}$ → 1 1 0 0 …0

  $\{a_1,a_n\}$ → 1 0 0 0 …1

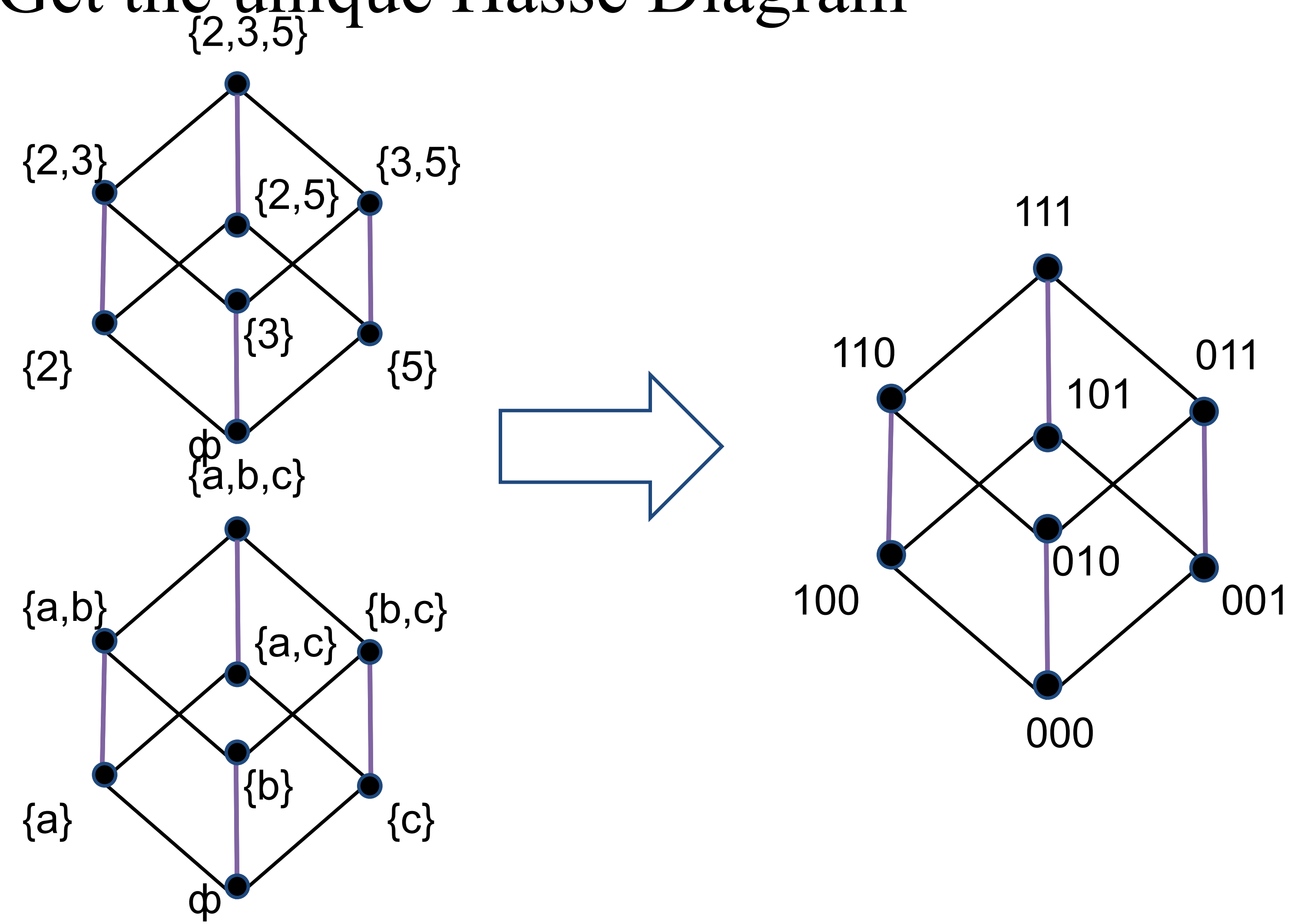
  ϕ → 0 0 0 0 …0

  $\{a_1,a_2,\ldots,a_n\}$ → 1 1 1 1 …1

  …

Finite Boolean Algebras

- Get the unique Hasse Diagram



Finite Boolean Algebras

- Lattice B_n

If the Hasse diagram of the lattice corresponding to a set with n elements is labeled by sequences of 0's and 1's of length n, the resulting lattice is named Bn. The properties of the partial order on B_n can be described directly as follows. If $x=a_1a_2\ldots a_n$ and $y=b_1b_2\ldots b_n$ are two element of B_n, then

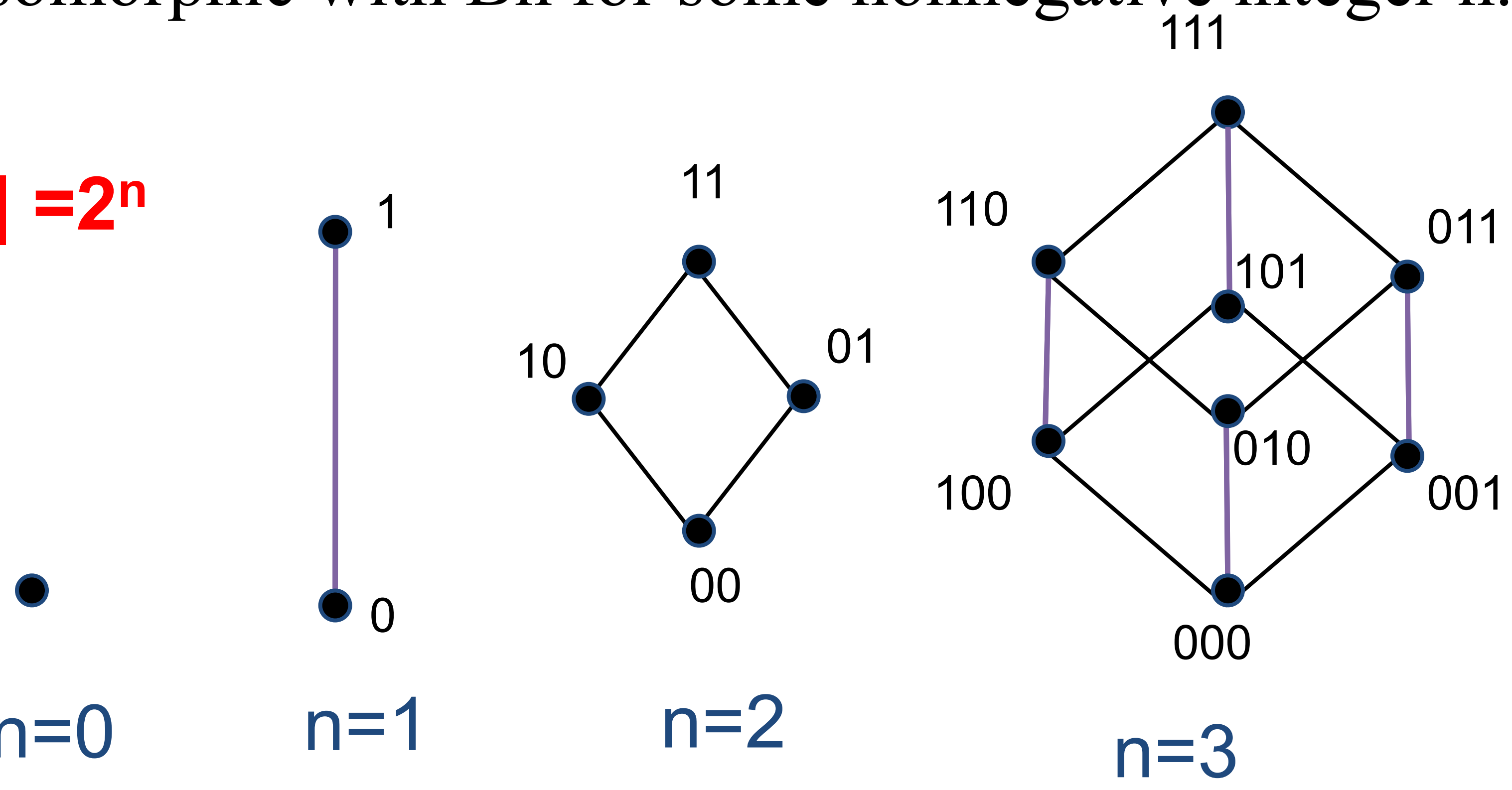
1. $x \le y$ iff $a_k \le b_k$ (as numbers 0 or 1) for $k=1,2,\ldots,n$
2. $x \wedge y=c_1c_2\ldots c_n$, where $c_k= \min\{a_k,b_k\}$
3. $x \vee y=c_1c_2\ldots c_n$, where $c_k= \max\{a_k,b_k\}$
4. x has a complement $x'=z_1z_2\ldots z_n$, where $z_k=1$ if $x_k=0$ and $z_k=0$ if $x_k=1$

Finite Boolean Algebras

- Boolean algebra

A finite lattice is called a Boolean algebra if it is isomorphic with Bn for some nonnegative integer n.

$|B_n| = 2^n$



Finite Boolean Algebras

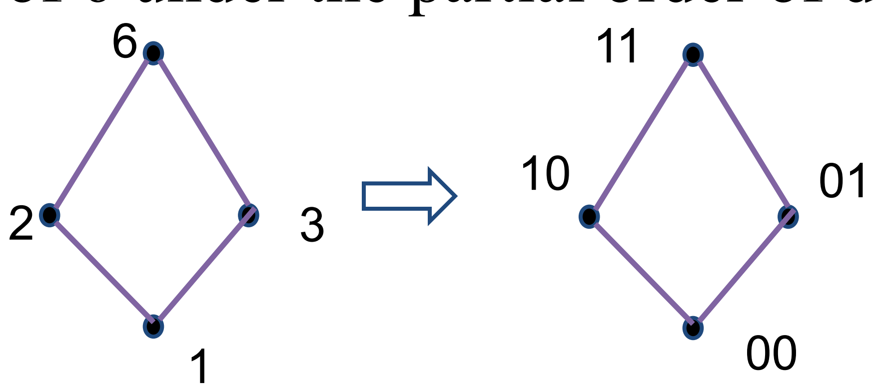
- (P(S), ⊆)

  Each x and y in B_n correspond to subsets A and B of S. Then $x \le y$, $x \wedge y$, $x \vee y$ and x' correspond to $A \subseteq B$, $A \cap B$, A U B and A. Therefore,

  (P(S), ⊆) is isomorphic with Bn, where n=|S|

- Example 3

  Consider the lattice D_6 consisting of all positive integer divisors of 6 under the partial order of divisibility.

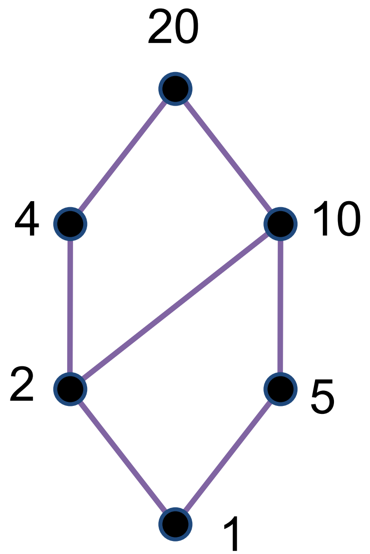


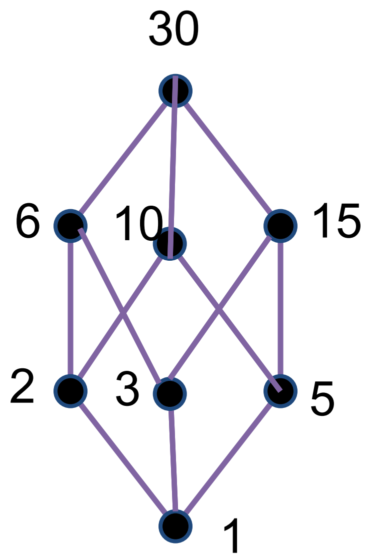
D_6 is a Boolean algebras

Finite Boolean Algebras

- Example 4

Consider the lattices D_{20} and D_{30} of all positive integer divisors of 20 and 30, respectively.





D_{20} is not a Boolean algebra (why? 6 is not 2^n)

D_{30} is a Boolean algebra, $D_{30} \rightarrow B_3$

Finite Boolean Algebras

- Theorem 2

Let $n=p_1p_2\ldots p_k$, where the p_i are **distinct** primes. The D_n is a Boolean algebra.

Proof:

Let $S=\{p_1 , p_2 , \ldots , p_k\}$. If $T \subseteq S$ and a_T is the product of the primes in T, then $a_T \mid n$. Any divisor of n must be of the form a_T for some subset T of S (let $a_\phi=1$) .

If V and T are subsets of S, $V \subseteq T$ if and only if $a_V \mid a_T$

$a_{V \cap T} = a_V \wedge a_T = GCD(a_V,a_T)$ &

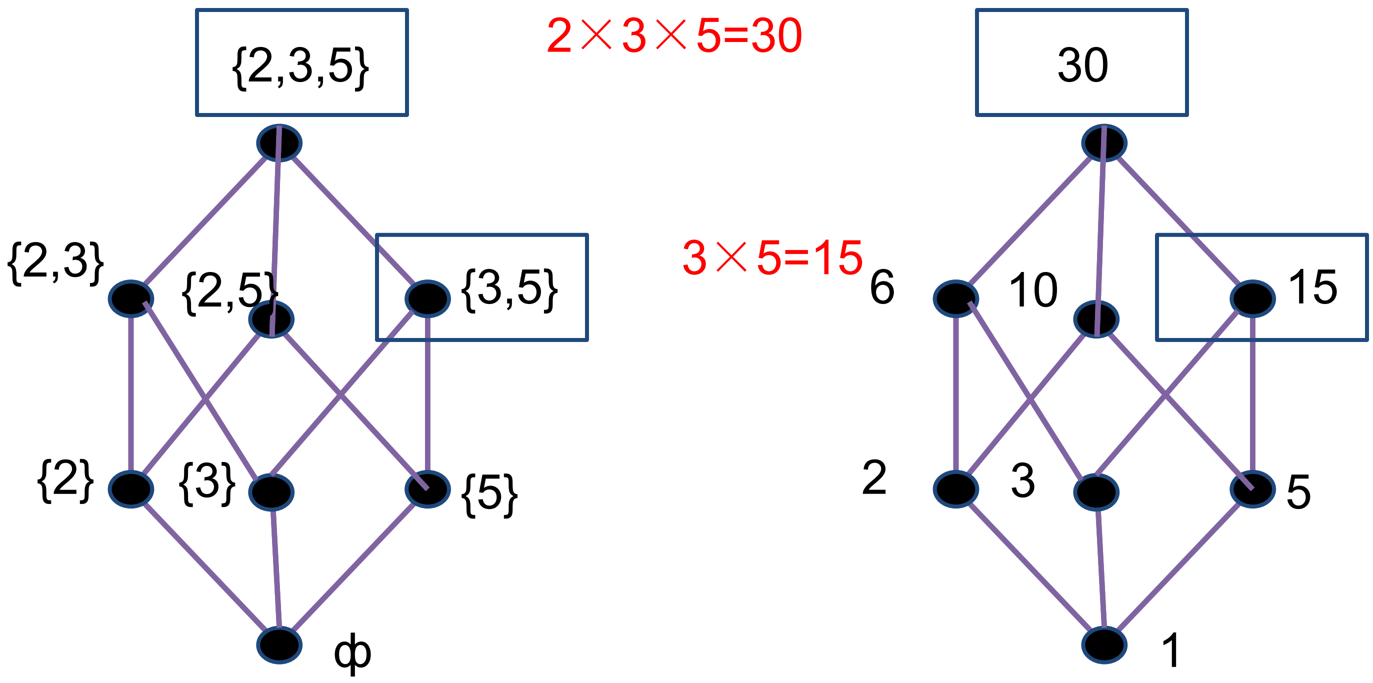
$a_{V \cup T} = a_V \vee a_T = LCM(a_V,a_T)$

Thus, the function $f: P(S) \rightarrow D_n$ given by $f(T)=a_T$ is a isomorphism form P(S) to D_n. Since P(S) is a Boolean algebra, so is D_n.

Finite Boolean Algebras

- Example

Let S={2,3,5}, show the Hasse diagrams of (P(S), ⊆) and D_{30} as follows.



Finite Boolean Algebras

- Example 5

  Since $210=2\times3\times5\times7$, $66=2\times3\times11$ and $646=2\times17\times19$, then D_{210}, D_{66} D_{646} are all Boolean algebras.

- Example 9

  Since $40=2^3\times5$, and $75=3\times5^2$, neither D_{40} and D_{75} are Boolean algebras.

  Note: If n is positive integer and $p^2 \mid n$, where p is a prime number, then Dn is not a Boolean algebra.

Finite Boolean Algebras

- Theorem 3 (Substitution rule for Boolean algebra)

  Any formula involving U or ∩ that holds for arbitrary subsets of a set S will continue to hold for arbitrary elements of a Boolean algebra L if is ∧ substituted for ∩ and ∨ for U.

Example 6 If L is any Boolean algebra and x,y and z are in L, then the following three properties hold.

1. $(x')' = x$ 2. $(x \wedge y)' = x' \vee y'$ 3. $(x \vee y)' = x' \wedge y'$

This is true by theorem 3,

1. $(A) = A$ 2. $(A \cap B) = A \cup B$ 3. $(A \cup B) = A \cap B$

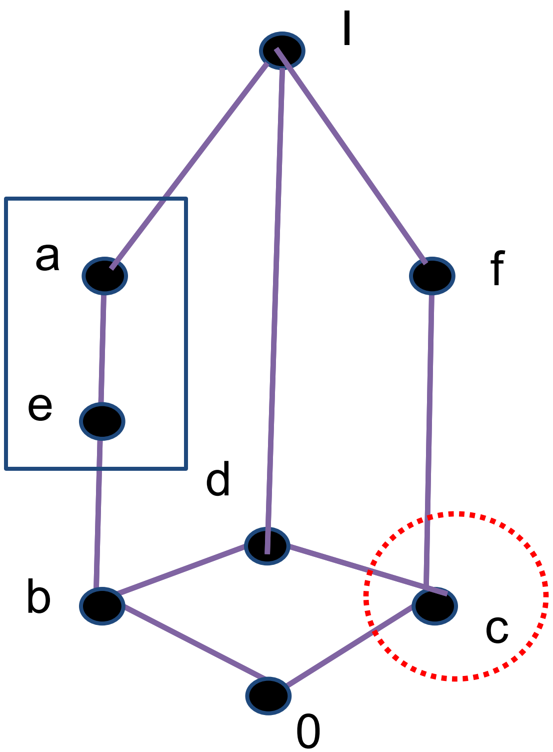
hold for arbitrary subsets A and B of a set S.

More properties can be found in p. 247, 1 ~12

Finite Boolean Algebras

- Example 7

Show the lattice whose Hasse diagram shown below is not a Boolean algebra.



a and e are both complements of c

However, based on the 11.

Every element x has a unique complement x'

Every element A has a unique complement $\bar{x}$

Theorem 3 (e.g. properties 1~14) is usually used to show that a lattice L is not a Boolean algebra.

Finite Boolean Algebras

Denote the Boolean algebra B_1 simply as B. Thus B contains only the two elements 0 and 1. It is a fact that any of the Boolean algebras B_n can be described in terms of B. The following theorem gives this description.

- Theorem 4

For any $n>=1$, B_n is the product $B \times B \times \ldots \times B$ of B, n factors, where $B \times B \times \ldots \times B$ is given the product partial order.