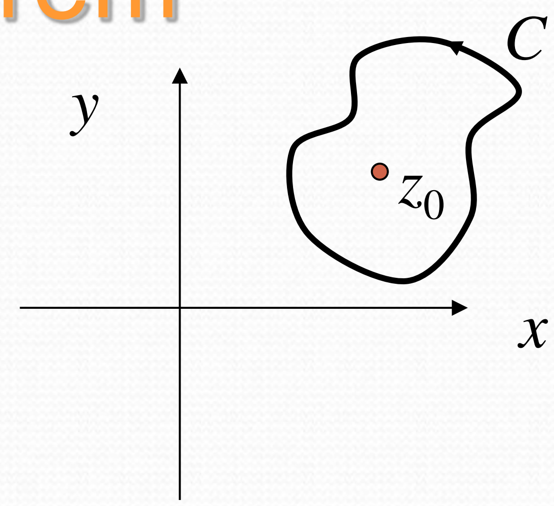
Residue Theorem

The Residue Theorem

Consider a line integral about a path enclosing an isolated singular point:



$$\oint_C f(z)\,dz$$

Expand ***f(z)*** in a Laurent series, deform the contour ***C*** to a circle of radius ***r*** centered at z_0, and evaluate the integral:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Let $z - z_0 \equiv re^{i\theta}, \quad dz = rie^{i\theta}d\theta,$

$$\Rightarrow \oint_C f(z)\,dz = i\sum_{n=-\infty}^{\infty} a_n r^{n+1} \underbrace{\int_0^{2\pi} e^{i(n+1)\theta}d\theta}_{=\,0,\; n\neq -1}$$

$$= \; 2\pi i\, a_{-1}$$

The Residue Theorem (cont.)

The value a_{-1} corresponding to an isolated singular point z_0 is called the "residue of $f(z)$ at z_0".

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res} f(z_0)$$

Note path orientation is assumed counterclockwise

The Residue Theorem (cont.)

Extend the theorem to multiple isolated singularities in the usual way:

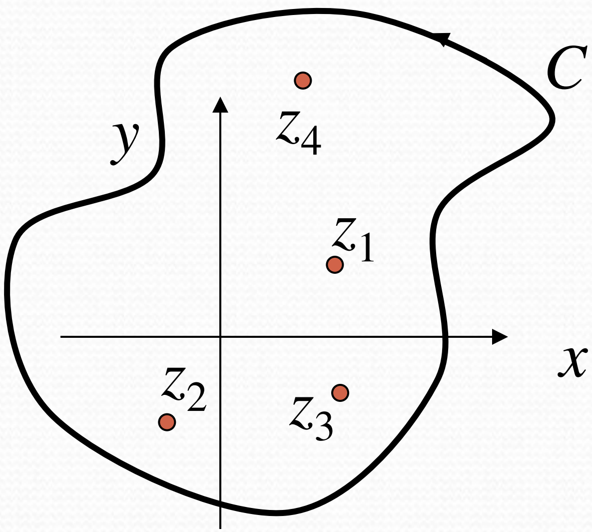
$$\oint_{C-\sum_n C_n} f(z)\,dz = \oint_C f(z)\,dz - \sum_n \oint_{C_n} f(z)\,dz = 0$$

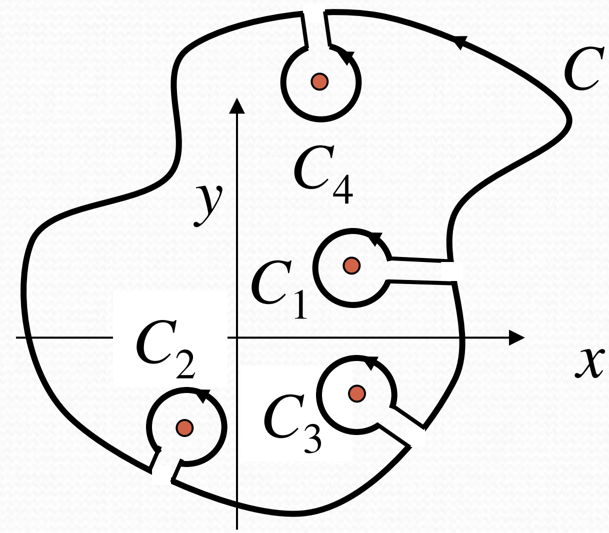
$$\Rightarrow \oint_C f(z)\,dz = \sum_n \oint_{C_n} f(z)\,dz$$

$$\oint_C f(z)\,dz = 2\pi i \sum_n \operatorname{Res} f(z_n)$$

$$= 2\pi i \times \text{ sum of residues enclosed by } C$$

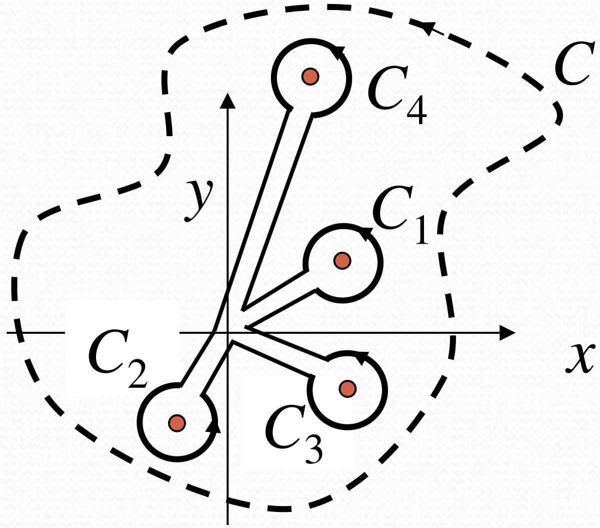
Isolated singularities at z_n





The Residue Theorem (cont.)

Alternatively, shrink the path to a point, leaving only the singularities encircled:



Isolated singularities at z_n

$$\oint_C f(z)\,dz = 2\pi i \sum_n \operatorname{Res} f(z_n)$$

$$= 2\pi i \times \text{ sum of residues enclosed by } C$$

The Residue Theorem (cont.)

Note the residue theorem subsumes several of our earlier results and theorems, e.g. :

Cauchy's Formula:

$\oint_C f(z)\,dz = 0$ if $f(z)$ analytic (no isolated singularities) in C

Cauchy Integral Formula:

$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)$ if $f(z)$ analytic (no isolated singularities) in C

Evaluating Residues

- Construct Laurent series about each singularity z_n, identify coefficient $a_{-1,n} = \text{Res}\, f(z_n)$. Sometimes this can be a tedious approach.
- For a *simple* pole at $z = z_0$,

$$\text{Res}\, f(z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

$$= \lim_{z \to z_0} (z - z_0) \left(\frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + \cdots \right)$$

$$= a_{-1}$$

$$\text{Res}\, f(z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Evaluating Residues (cont.)

Example:

- $f(z) = \dfrac{2z}{(z+2)(z^2+1)}$ has *simple* poles at $z = -2, +i, -i$

$\Rightarrow$

$$\text{Res}\, f(-2) = \lim_{z \to -2} (z+2) \frac{2z}{(z+2)(z^2+1)} = -\frac{4}{5}$$

$$\text{Res}\, f(i) = \lim_{z \to i} (z-i) \frac{2z}{(z+2)(z+i)(z-i)} = \frac{2i}{(2+i)\,2i} = \frac{2-i}{5}$$

$$\text{Res}\, f(-i) = \lim_{z \to -i} (z+i) \frac{2z}{(z+2)(z-i)(z+i)} = \frac{(-2i)}{(2-i)(-2i)} = \frac{2+i}{5}$$

Evaluating Residues (cont.)

Example:

$f(z)=\dfrac{1}{\sin z}$ has simple poles at $z=n\pi,\ n=0,\pm1,\pm2,\cdots$

$\Rightarrow$

$$\operatorname{Res} f(n\pi) = \lim_{z\to n\pi}(z-n\pi)\times\frac{1}{\sin z} \overset{\text{L'Hospital's rule}}{=} \lim_{z\to n\pi}\frac{1}{\cos z}=(-1)^n$$

Alternatively,

$$\operatorname{Res} f(n\pi) = \lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin z} = \lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin(z-n\pi+n\pi)}$$

$$= \lim_{z\to n\pi}(z-n\pi)\frac{1}{\sin(z-n\pi)\cos n\pi+\cos(z-n\pi)\sin n\pi}$$

$$= \lim_{z\to n\pi}\frac{(z-n\pi)}{\sin(z-n\pi)\cos n\pi} = (-1)^n$$ since we already know $\lim_{z\to 0}\dfrac{z}{\sin z}\to 1$

Evaluating Residues (cont.)

- The above rule can be specialized to functions of the form $g(z)/f(z)$ where $g(z)$ is analytic, non-vanishing, and $f(z)$ has a simple zero, all at z_0 :

$$\operatorname{Res}\frac{g(z_0)}{f(z_0)} = \lim_{z\to z_0}(z-z_0)\frac{g(z)}{f(z)} = \lim_{z\to z_0}(z-z_0)\frac{g(z)}{f(z)-f(z_0)}$$

$$= \frac{g(z_0)}{\lim\limits_{z\to z_0}\dfrac{f(z)-f(z_0)}{(z-z_0)}} = \frac{g(z_0)}{f'(z_0)}$$

$$\Rightarrow \operatorname{Res}\frac{g(z_0)}{f(z_0)} = \frac{g(z_0)}{f'(z_0)}$$

Example:

$\tan z = \dfrac{\sin z}{\cos z}$ has simple poles at $z = \dfrac{(2n+1)\pi}{2}$,

for $n = 0, \pm 1, \pm 2, \cdots$

$$\Rightarrow \operatorname{Res}\tan\frac{(2n+1)\pi}{2} = \frac{\sin\frac{(2n+1)\pi}{2}}{-\sin\frac{(2n+1)\pi}{2}} = -1$$

Evaluating Residues (cont.)

- For a pole of finite order m at $z = z_0$:

At the pole, the Laurent series is

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Hence the following series is a Taylor series about the pole,

$$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + \cdots$$

for which the formula for the coefficent of $(z-z_0)^{m-1}$ is

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Big|_{z=z_0}$$

$$\text{Res}\, f(z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Big|_{z=z_0} \qquad (m \text{ finite})$$

Evaluating Residues (cont.)

$$\text{Res}\, f(z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Big|_{z=z_0} \qquad (m \text{ finite})$$

Example:

$$f(z) = \frac{2z^2+z}{(z+2)^3}$$ has a pole of order 3 at $z=-2$

$$\Rightarrow \text{Res}\, f(-2) = \frac{1}{2!}\frac{d^2}{dz^2}\left[\cancel{(z+2)^3}\frac{2z^2+z}{\cancel{(z+2)^3}}\right]\Big|_{z=-2}$$

$$= \frac{1}{2!}\frac{d^2}{dz^2}\left[2z^2+z\right]\Big|_{z=-2} = \frac{4}{2} = 2$$

Evaluating Residues (cont.)

Example: $f(z)=\dfrac{1}{\sin^2 z}$ has poles of order 2 at $z=n\pi,\ n=0,\pm1,\pm2,\cdots$

$$\Rightarrow \operatorname{Res} f(n\pi) = \frac{d}{dz}\left[\frac{(z-n\pi)^2}{\sin^2 z}\right]\Bigg|_{z=n\pi} = \frac{d}{dz}\left[\frac{(z-n\pi)^2}{\sin^2(z-n\pi+n\pi)}\right]\Bigg|_{z=n\pi}$$

$$= \frac{d}{dz}\left[\frac{(z-n\pi)^2}{\left[\sin(z-n\pi)\cos n\pi+\cos(z-n\pi)\sin n\pi\right]^2}\right]\Bigg|_{z=n\pi} = \frac{d}{dz}\left[\frac{(z-n\pi)^2}{\sin^2(z-n\pi)}\right]\Bigg|_{z=n\pi}$$

Let $u=z-n\pi$

$$= \frac{d}{du}\left[\frac{u^2}{\sin^2 u}\right]\Bigg|_{u=0} = \left[\frac{2u\left(\sin^{\cancel{2}} u - u\,\cancel{\sin u}\,\cos u\right)}{\sin^{\cancel{4}\,3} u}\right]\Bigg|_{u=0}$$

L'hospital's rule

$$= \left[\frac{2\left(\sin u - u\cos u\right)+2u^2\sin u}{3\sin^2 u\cos u}\right]\Bigg|_{u=0}$$

L'hospital's rule

$$= \left[\frac{6u\sin u+2u^2\cos u}{3\sin u\left(2\cos^2 u-\sin^2 u\right)}\right]\Bigg|_{u=0}$$

L'hospital's rule

$$= \left[\frac{6\sin u+10u\cos u-2u^2\sin u}{3\cos u\left(2\cos^2 u-\sin^2 u\right)-18\sin^2 u\cos u}\right]\Bigg|_{u=0} = 0$$

Evaluating Residues (cont.)

The previous result might be "more easily" obtained as follows:

$$f(z) = \frac{1}{\sin^2 z} = \frac{1}{\sin^2(z - n\pi + n\pi)} = \frac{1}{\sin^2(z - n\pi)} = \left.\frac{1}{\sin^2 u}\right|_{u = z - n\pi \to 0}$$

$$= \frac{1}{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right)^2} = \frac{1}{u^2\left(1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \cdots\right)^2}$$

Geometric Series

$$= \frac{1}{u^2}\left[1 + \left(\frac{u^2}{3!} - \frac{u^4}{5!} + \frac{u^6}{7!}\cdots\right) + \left(\frac{u^2}{3!} - \frac{u^4}{5!} + \frac{u^6}{7!}\cdots\right)^2 + \cdots\right]^2$$

$$= \frac{1}{u^2}\left[1 + \frac{u^2}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)u^4 + \cdots\right]^2 = \frac{1}{u^2} + \frac{2}{3!} + \cdots$$

missing $\frac{a_{-1}}{u}$ term $\Rightarrow \text{Res}\, f(n\pi) = 0$

Evaluating Residues (cont.)

Example:

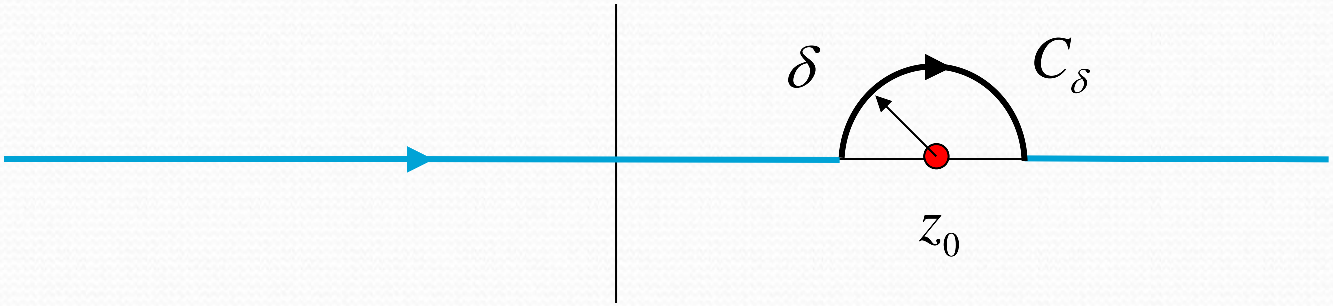
$$f(z) = e^{\frac{1}{z}}$$ (isolated essential singularity at $z = 0$)

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \ldots$$

Residue: $a_{-1} = 1$

Evaluating Residues (cont.)

Extension for simple poles (going halfway around)



As $\delta \to 0$:

$$\int_{C_\delta} f(z)\,dz = -\pi i \operatorname{Res} f(z_0)$$

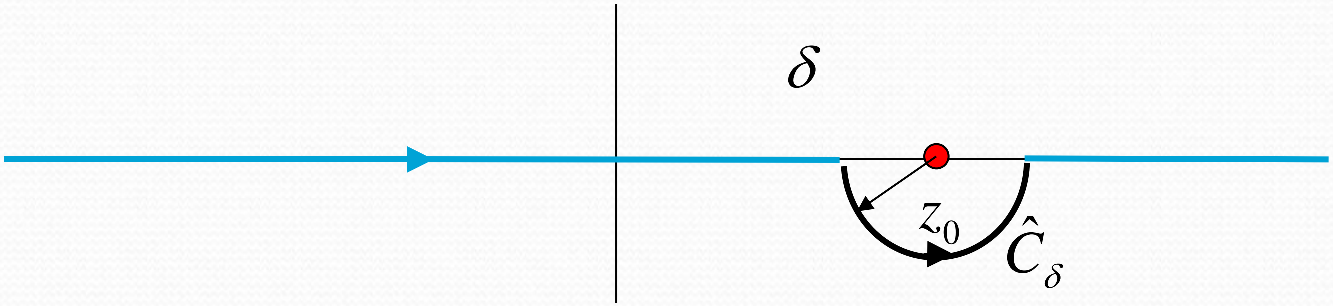
Proof:

$$z - z_0 = \delta e^{i\theta}$$

$$\int_{C_\delta} \frac{a_{-1}}{z - z_0}\,dz = a_{-1} \int_{C_\delta} \frac{dz}{z - z_0} = a_{-1} \int_{\pi}^{0} \frac{i\delta e^{i\theta}}{\delta e^{i\theta}}\,d\theta = a_{-1}(-\pi i)$$

Evaluating Residues (cont.)

What is the result if the contour goes around the simple pole on the opposite side?



Hint: The contour $\hat{C}_\delta - C_\delta$ encircles the pole in the counterclockwise direction so that

$$\oint_{\hat{C}_\delta - C_\delta} f(z)\,dz = 2\pi i \operatorname{Res} f(z_0)$$

$$\Rightarrow \int_{\hat{C}_\delta} f(z)\,dz = \pi i \operatorname{Res} f(z_0)$$

Evaluation of residues at multiple poles

If *f(z)* has a pole of order n at *z* = *a* and no other singularity, *f(z)* is:

$$f(z) = \frac{F(z)}{(z-a)^n}$$

where *n* is a finite integer, and *F(z)* is analytic at *z* = *a*.

F(z) can be expanded by the Taylor series:

$$F(z) = F(a) + (z-a)F'(a) + \frac{(z-a)^2}{2!}F''(a) + \ldots + \frac{(z-a)^n}{n!}F^n(a) + \ldots$$

Dividing throughout by $(z-a)^n$

$$f(z) = \frac{F(a)}{(z-a)^n} + \frac{F'(a)}{(z-a)^{n-1}} + \ldots + \frac{F^{n-1}(a)}{(n-1)!(z-a)} + \ldots$$

The residue at z = a is the coefficient of $(z-a)^{-1}$

The residue at a pole of order *n* situated at *z* = *a* is:

$$B_1 = \left.\frac{F^{n-1}(a)}{(n-1)!}\right|_{z=a} = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]_{z=a}$$

EXAMPLE:

Evaluate $\int_C \frac{\cos 2z}{(z-a)^3} dz$ around a circle of radius |z| > |a|.

$\frac{\cos 2z}{(z-a)^3}$ has a pole of order 3 at z = a, and the residue is:

$$B_1 = \frac{F^{n-1}(a)}{(n-1)!}\Big|_{z=a} = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]_{z=a}$$

$$= \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-a)^3 \frac{\cos 2z}{(z-a)^3}\right]_{z=a} = -2\cos 2a$$

$$\therefore \int_C \frac{\cos 2z}{(z-a)^3} dz = 2\pi i(-2\cos 2a)$$