Convolution Theorem

Convolution theorem

Theorem

$$\mathcal{F}(f*g)=\mathcal{F}(f)\mathcal{F}(g)$$

$$\mathcal{F}(fg)=\mathcal{F}(f)*\mathcal{F}(g)$$

$$\mathcal{F}^{-1}(F*G)=\mathcal{F}^{-1}(F)\mathcal{F}^{-1}(G)$$

$$\mathcal{F}^{-1}(FG)=\mathcal{F}^{-1}(F)*\mathcal{F}^{-1}(G)$$

Proof (1)

$$\mathcal{F}(f*g)=\int_{-\infty}^{\infty}f(t')g(t-t')\int_{-\infty}^{\infty}e^{-i\omega t}dt\,dt'$$

$$=\int_{-\infty}^{\infty}f(t')e^{-i\omega t'}dt'\int_{-\infty}^{\infty}g(t-t')e^{-i\omega(t-t')}dt$$

$$=\int_{-\infty}^{\infty}f(t')e^{-i\omega t'}dt'\int_{-\infty}^{\infty}g(t'')e^{-i\omega t''}dt''$$

$$=\mathcal{F}(f)\mathcal{F}(g)$$

Convolution Theorem
Example

$$f(x) = x \quad g(x) = e^{-x^2}$$

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - \xi) e^{-\xi^2} d\xi$$

$$\int_{-\infty}^{\infty} \xi e^{-\xi^2} d\xi = 0 \text{ by symmetry}$$

$$\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi}$$

$$\text{Hence} \quad f * g = \frac{x\sqrt{\pi}}{\sqrt{2\pi}} = \frac{x}{\sqrt{2}}$$