Fourier Integrals

From Fourier Series to Fourier Integral

- Consider any periodic function $f_L(x)$ of period $2L$ that is represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \qquad w_n = \frac{n\pi}{L}$$

  - what happens if we let $L \to \infty$?

- We should expect an integral (instead of a series) involving $\cos wx$ and $\sin wx$ with w no longer restricted to integer multiples $w = w_n = n\pi/L$ of π/L but taking *all* values.

- If we insert a_n and b_n , and denote the variable of integration by v, the Fourier series of $f_L(x)$ becomes

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v\,dv + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v\,dv\right].$$

We now set

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$

- Then $1/L = \Delta w/\pi$, and we may write the Fourier series in the form

(1)
$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[(\cos w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\cos w_n v\,dv + (\sin w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\sin w_n v\,dv\Big].$$

- Let $L \to \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L\to\infty} f_L(x)$$

is **absolutely integrable** on the x-axis; that is, the following limits exist:

$$\lim_{a\to-\infty}\int_a^0 |f(x)|\,dx + \lim_{b\to\infty}\int_0^b |f(x)|\,dx \quad \left(\text{written } \int_{-\infty}^{\infty} |f(x)|\,dx\right).$$

$1/L \to 0$, and the value of the first term on the right side of (1) $\to$ zero. Also $\Delta w = \pi/L \to dw$. The infinite series in (1) becomes an integral from 0 to ∞, which represents $f(x)$, namely,

(3) $$f(x) = \frac{1}{\pi}\int_0^\infty \left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\, dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv\, dv\right] dw.$$

If we introduce the notations

(4) $$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv$$

Fourier integral

we can write this in the form

(5) $$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\, dw.$$

This is called a representation of $f(x)$ by a **Fourier integral.**

Fourier Integral

THEOREM 1

Fourier Integral

If $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and a left-hand derivative at every point and if the integral exists, then $f(x)$ can be represented by a Fourier integral with A and B given by (4). At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point.

Sine Integral

- The case $x = 0$ is of particular interest. If $x = 0$, then (7) gives

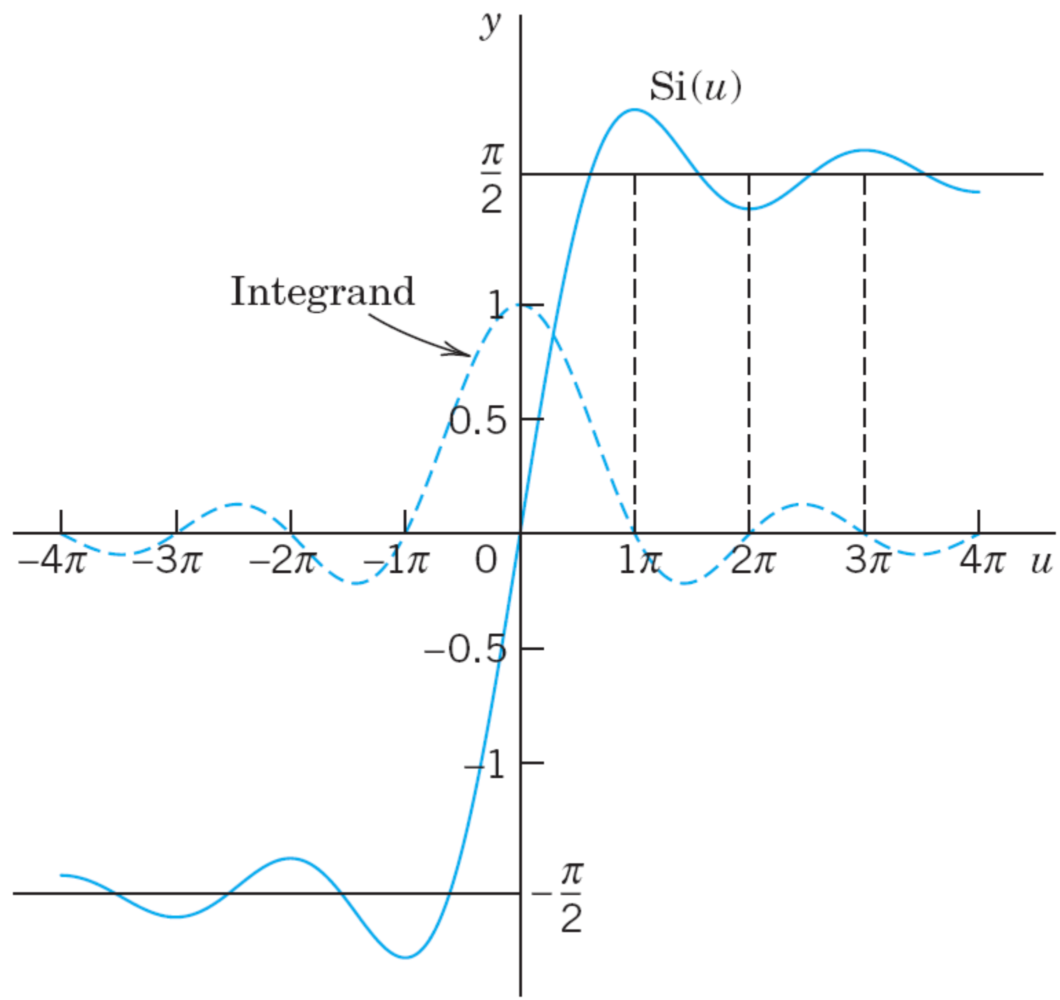
(8*) $$\int_0^\infty \frac{\sin w}{w}\, dw = \frac{\pi}{2}.$$

We see that this integral is the limit of the so-called **sine integral**

(8) $$\mathrm{Si}(u) = \int_0^u \frac{\sin w}{w}\, dw$$

as $u \to \infty$. The graphs of Si(u) and of the integrand are shown in Fig. 279.

Fig. 279. Sine integral Si(*u*) and integrand



- In the case of the Fourier integral, approximations are obtained by replacing ∞ by numbers a. Hence the integral

$$(9)\qquad \frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\, dw$$

which approximates $f(x)$.

Fourier Cosine Integral and Fourier Sine Integral

- If $f(x)$ is an ***even*** function, then $B(w) = 0$ and

(10) $$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\, dv.$$

The Fourier integral (5) then reduces to the **Fourier cosine integral**

$$f(x) = \int_0^\infty A(w)\cos wx\, dw \qquad (f \text{ even}).$$

Fourier Cosine Integral and Fourier Sine Integral

- If $f(x)$ is an ***odd*** function, then $A(w) = 0$ and

(12) $$B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv.$$

The Fourier integral (5) then reduces to the **Fourier cosine integral**

$$f(x) = \int_0^\infty B(w) \sin wx \, dw \qquad (f \text{ odd}).$$