

# *Trapezoidal rule*

# Trapezoid Rule

- Simplest way to approximate the area under a curve – using **first order polynomial** (a straight line)
- Using Newton's form of the interpolating polynomial:

$$p_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

- Now, solve for the integral:

$$I = \int_a^b f(x)dx \approx \int_a^b p_1(x)dx$$

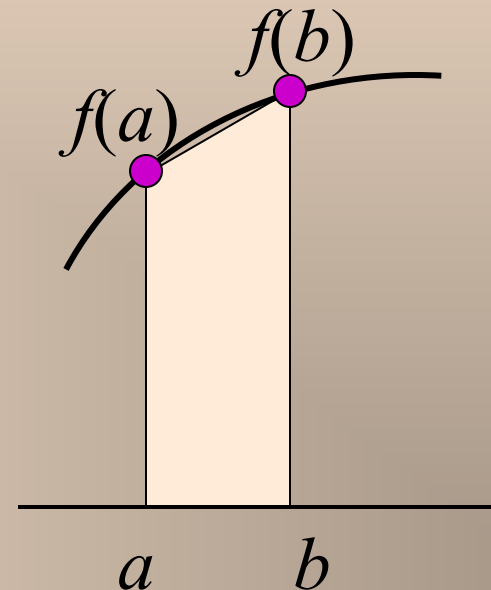
# Trapezoid Rule

$$I \approx \int_a^b \left[ f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right] dx$$

Trapezoid Rule

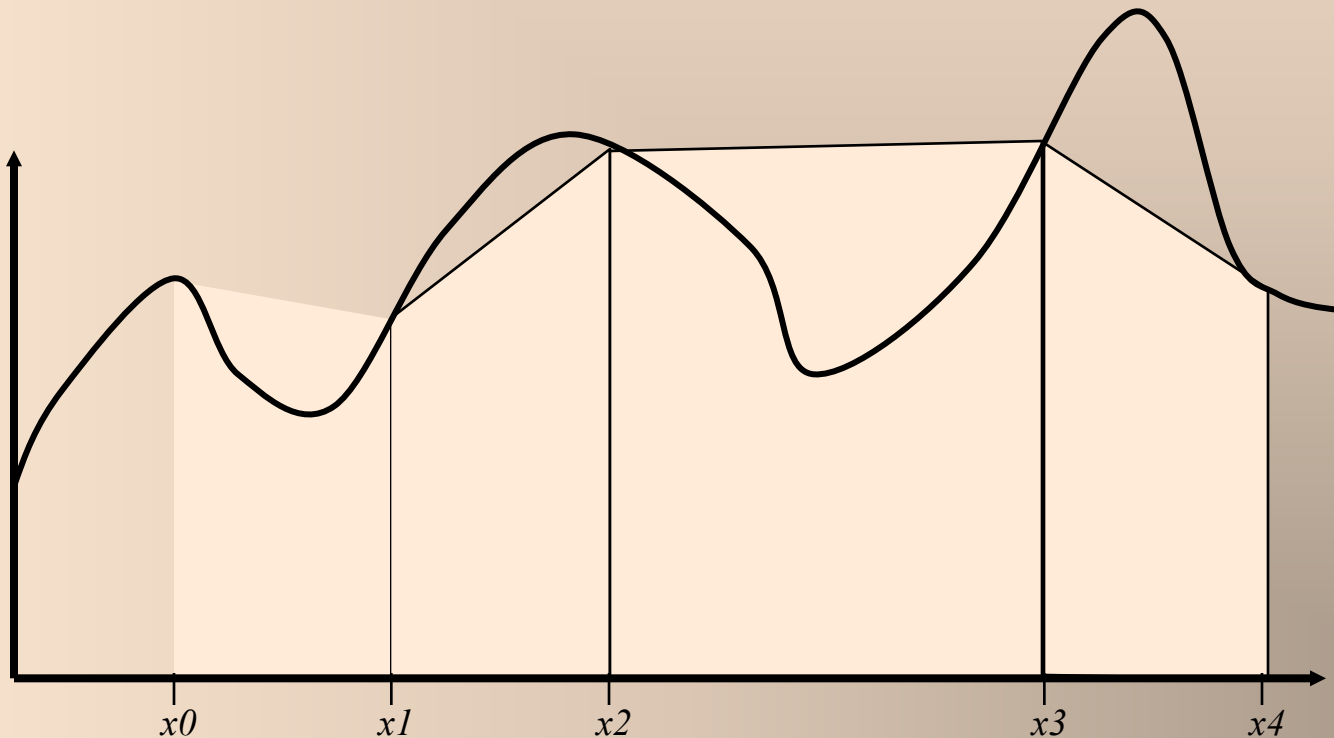
$$I \approx \frac{(b - a)}{2} [f(a) + f(b)]$$

$I \approx \text{width} \times \text{average height}$



# *Trapezoid Rule*

- Improvement?



# Trapezoid Rule Error

- The integration error is:

$$E_t = -\frac{1}{12} f''(\xi)h^3 = -\frac{(b-a)}{12} f''(\xi)h^2 \quad O(h^3)$$

- Where  $h = b - a$  and  $\xi$  is an unknown point where  $a < \xi < b$  (intermediate value theorem)
- You get exact integration if the function,  $f$ , is linear ( $f'' = 0$ )

# Example

Integrate from  $f(x) = e^{-x^2}$   $a = 0$  to  $b = 2$

Use trapezoidal rule:

$$\begin{aligned} I &= \int_0^2 e^{-x^2} dx \\ &\approx \frac{(b-a)}{2} [f(a) + f(b)] = \frac{(2-0)}{2} [f(2) + f(0)] \\ &= 1 \times (e^{-4} + e^0) = 1.0183 \end{aligned}$$

# Example

Estimate error:  $E_t = -\frac{1}{12} f''(\xi)h^3$

Where  $h = b - a$  and  $a < \xi < b$

Don't know  $\xi$  - use average value

$$f''(x) = (-2 + 4x^2)e^{-x^2}$$

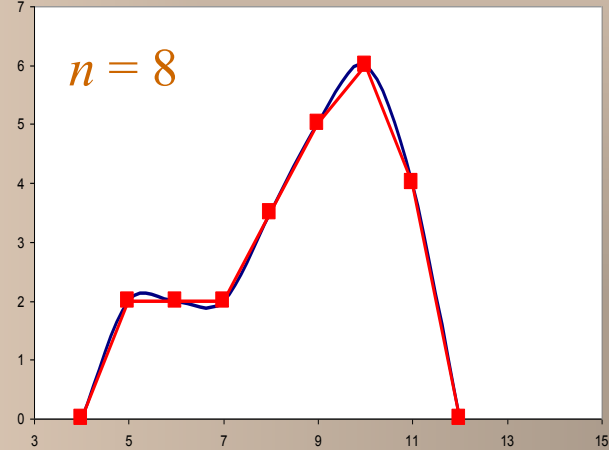
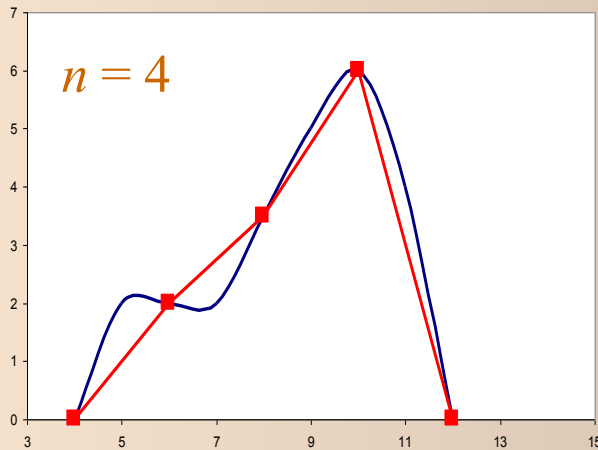
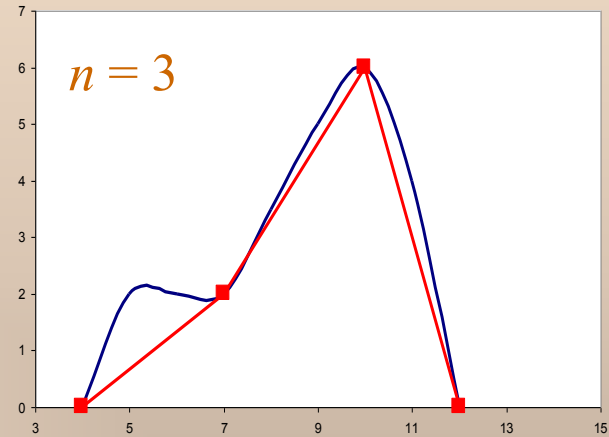
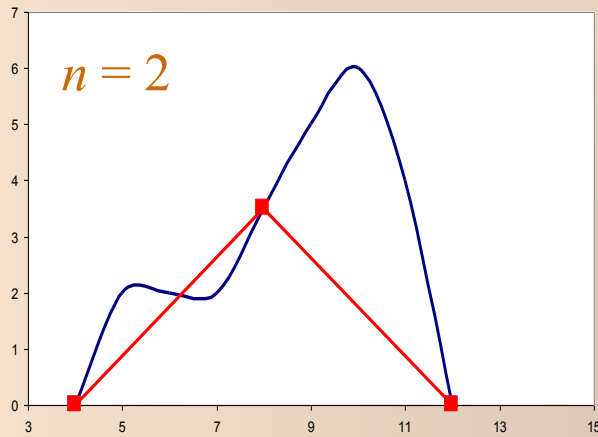
$$f''(0) = -2$$

$$h = 2 - 0 = 2$$

$$f''(2) = 0.2564$$

$$E_t \approx E_a = -\frac{2^3}{12} \frac{[f''(0) + f''(2)]}{2} = 0.58$$

More intervals, better result [error  $\sim O(h^2)$ ]





# Composite Trapezoid Rule

- If we do multiple intervals, we can avoid duplicate function evaluations and operations:
- Use  $n+1$  equally spaced points.
- Each interval has:  $h = \frac{b-a}{n}$
- Break up the limits of integration and expand.

$$I = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \dots + \int_{b-h}^b f(x)dx$$

# Composite Trapezoid Rule

- Substituting the trapezoid rule for each integral.

$$\begin{aligned} I &= \int_a^{a+h} f(x) dx + \int_{a+h}^{a+2h} f(x) dx + \dots + \int_{b-h}^b f(x) dx \\ &= \frac{(a+h-a)}{2} [f(a) + f(a+h)] + \frac{(a+2h-a-h)}{2} [f(a+h) + f(a+2h)] \\ &\quad + \dots + \frac{(b-b+h)}{2} [f(b-h) + f(b)] \end{aligned}$$

- Results in the Composite Trapezoid Formula:

$$I = \frac{h}{2} \left[ f(a) + 2 \sum_{i=1}^{n-1} f(a + ih) + f(b) \right]$$

# Composite Trapezoid Rule

- Think of this as the *width* times the average *height*.

$$I = \frac{h}{2} \left[ f(a) + 2 \sum_{i=1}^{n-1} f(a + ih) + f(b) \right]$$
$$= \underbrace{(b - a)}_{\text{width}} \underbrace{\frac{f(a) + 2 \sum_{i=1}^{n-1} f(a + ih) + f(b)}{2n}}_{\text{Average height}}$$

# Error

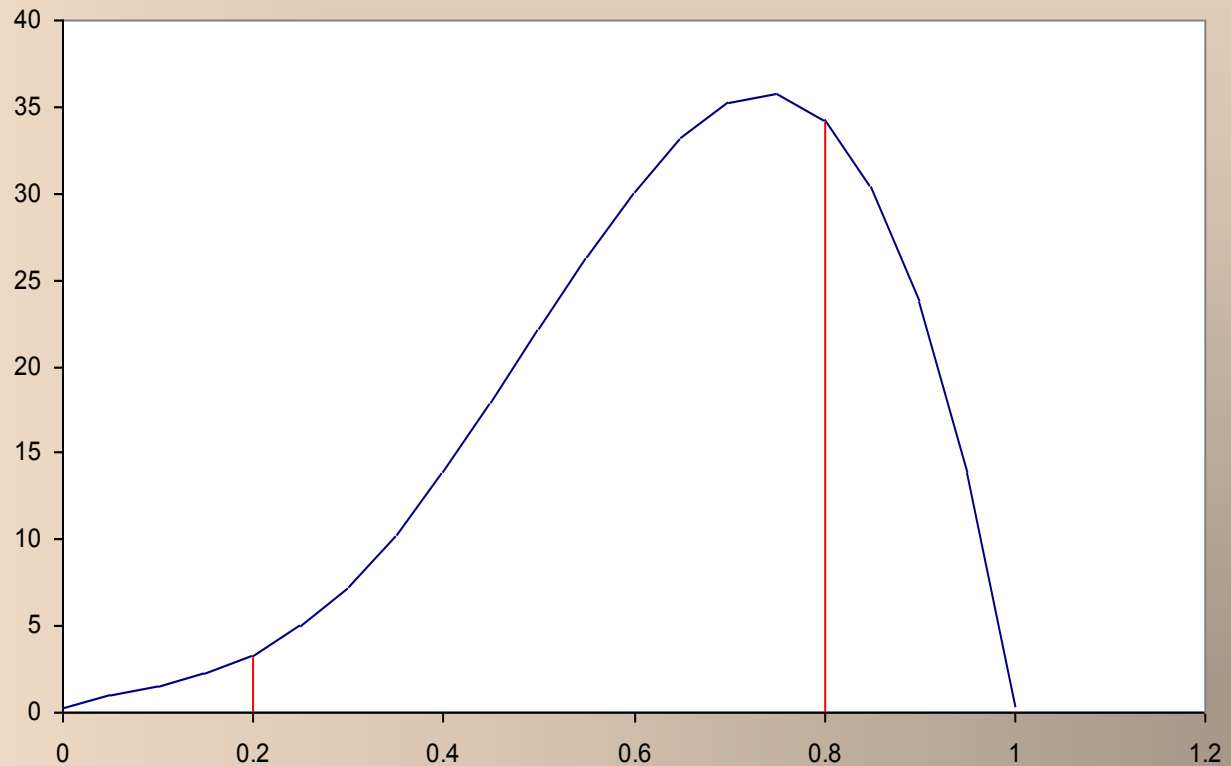
- The error can be estimated as:

$$E_a = \frac{(b-a)h^2}{12} \bar{f}'' = \frac{(b-a)^3}{12n^2} \bar{f}'' \quad O(h^2)$$

- Where,  $\bar{f}''$  is the average second derivative.
- If  $n$  is doubled,  $h \rightarrow h/2$  and  $E_a \rightarrow E_a/4$
- Note, that the error is dependent upon the *width* of the area being integrated.

# Example

- Integrate:  $f(x) = 0.3 + 20x - 140x^2 + 730x^3 - 810x^4 + 200x^5$
- from  $a=0.2$   
to  $b=0.8$



## Example

- A single application of the Trapezoid rule.

$$\begin{aligned} I &= (b - a) \frac{f(a) + f(b)}{2} \\ &= (0.8 - 0.2) \frac{34.22 + 3.81}{2} \\ &= 11.26 \end{aligned}$$

- Error: 
$$E_t = -\frac{1}{12} f''(\xi)(b - a)^3$$

# Example

- We don't know  $\xi$  so approximate with average  $f''$

$$f'(x) = 20 - 280x + 2190x^2 - 3240x^3 + 1000x^4$$

$$f''(x) = -280 + 4380x - 9720x^2 + 4000x^3$$

$$\begin{aligned}\bar{f}''(x) &= \frac{\int_{0.2}^{0.8} f'' dx}{0.8 - 0.2} \\ &= \frac{f'(0.8) - f'(0.2)}{0.8 - 0.2} = -131.6\end{aligned}$$

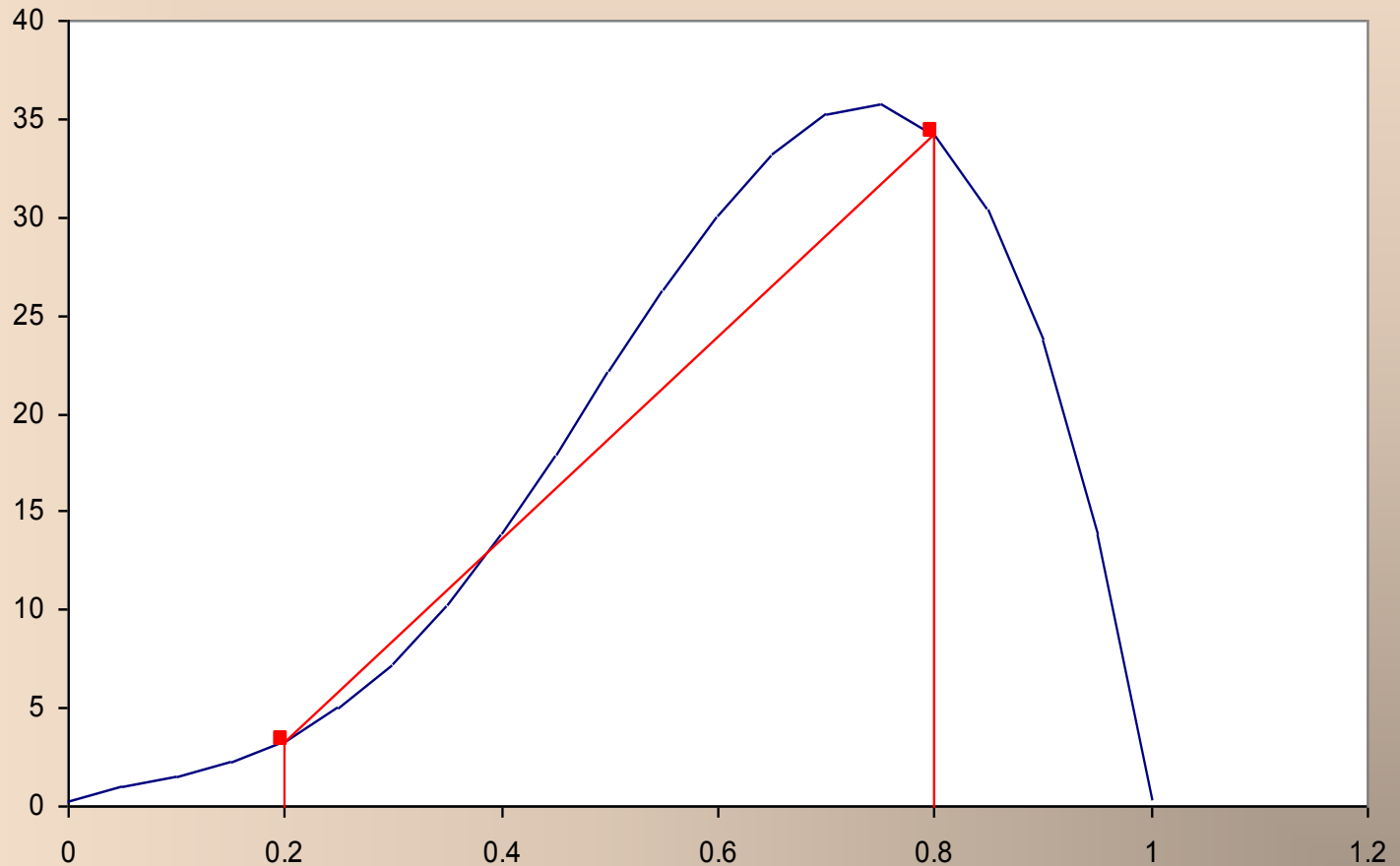
# Example

- The error can thus be estimated as:

$$\begin{aligned} E_t &= \frac{(b-a)h^2}{12} \bar{f}'' = \frac{(b-a)^3}{12n^2} \bar{f}'' \\ &= -\frac{1}{12}(-131.6)(0.8-0.2)^3 = 2.37 \end{aligned}$$



True value of integral is 12.82. Trapezoid rule is 11.26 - within approx error -  $E_t$  is 12%



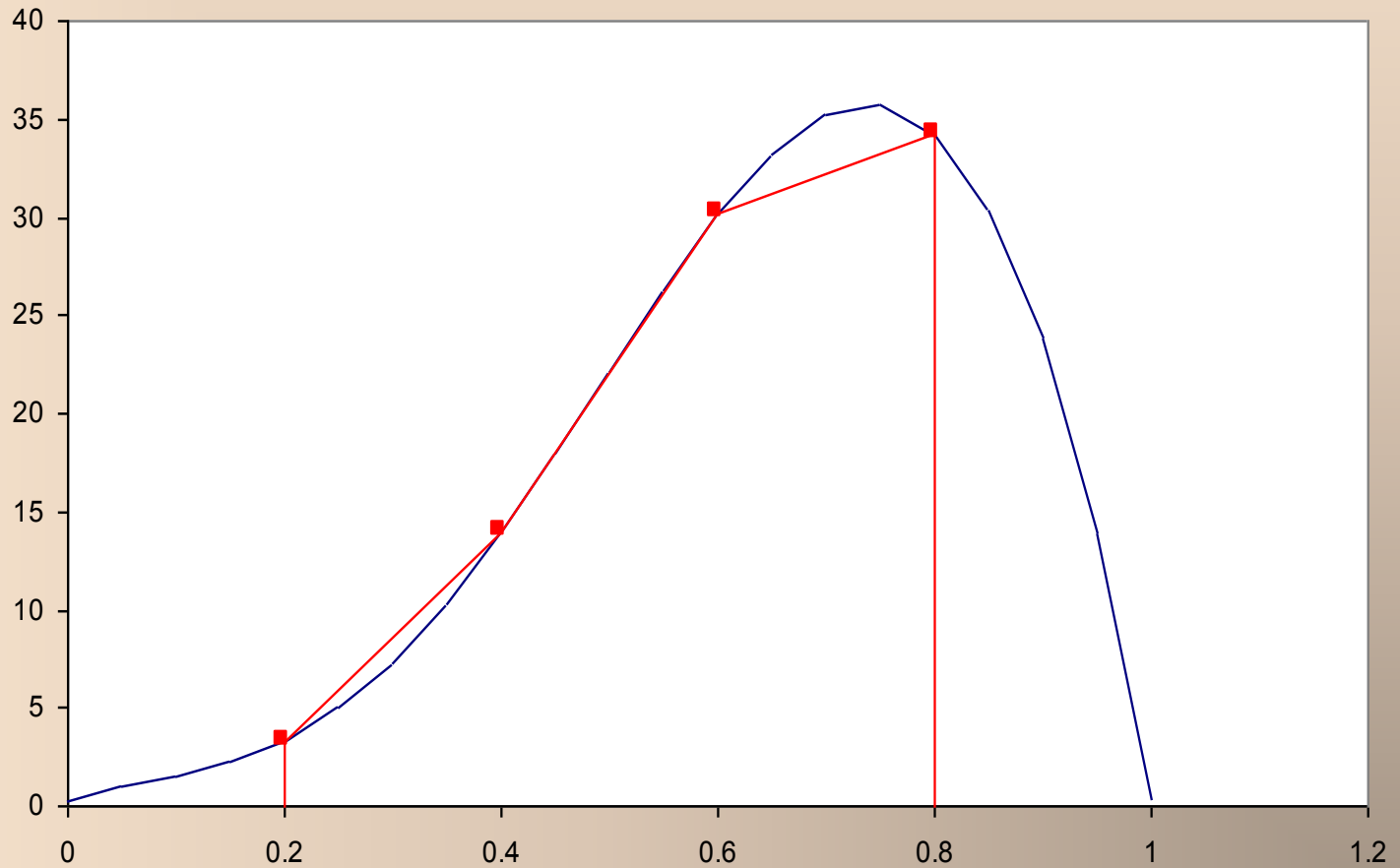
# Using Three Intervals

- Use intervals (0.2,0.4),(0.4,0.6),(0.6,0.8):
  - ( $n = 3, h = 0.2$ )

$$\begin{aligned} I &= (b - a) \frac{f(a) + 2 \sum_{i=1}^{n-1} f(a + ih) + f(b)}{2n} \\ &= (0.8 - 0.2) \frac{f(0.2) + 2[f(0.4) + f(0.6)] + f(0.8)}{(2)(3)} \\ &= 0.6 \frac{3.31 + 2(13.93 + 30.16) + 34.22}{6} \\ &= 12.57 \end{aligned}$$

True value of integral is 12.82

$E_t$  is now 2%



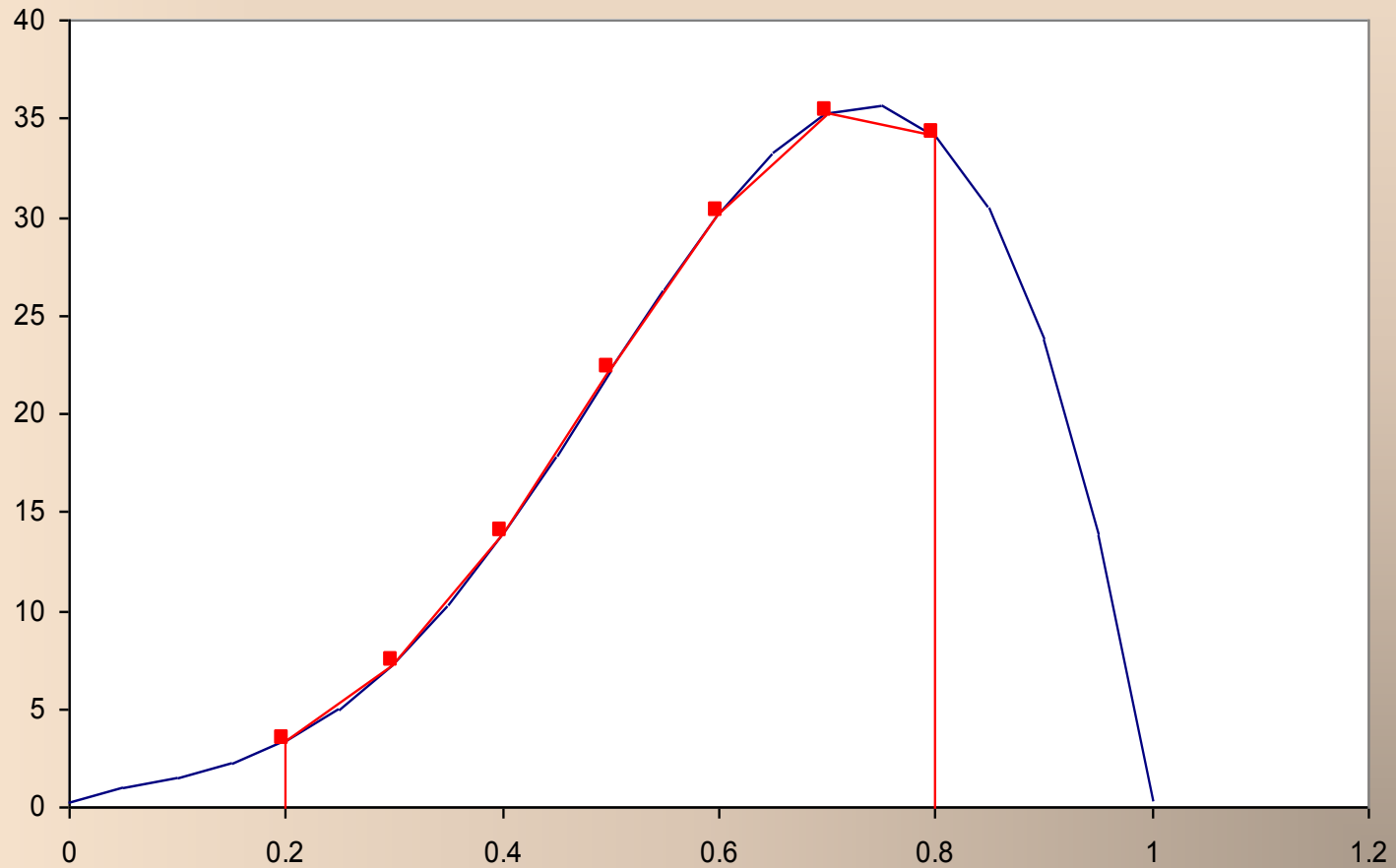
## Using Six Intervals

- Use intervals  $(0.2,0.3), (0.3,0.4)$ , etc.
  - $(n = 6, h = 0.1)$

$$\begin{aligned} I &= (0.8 - 0.2) \frac{f(0.2) + 2[f(0.3) + f(0.4) + f(0.5) + f(0.6) + f(0.7)] + f(0.8)}{(2)(6)} \\ &= 0.6 \frac{3.31 + 2(7.34 + 13.93 + 22.18 + 30.16 + 35.22) + 34.22}{12} \\ &= 12.76 \end{aligned}$$

True value of integral is 12.82

$E_t$  is now 0.5%



# Example

- Trapezoid Rules:

				$k = 0$	$k = 1$
		intervals	$h$	Integral	
$j$	$k \rightarrow$				
	$j = 0$	1	0.8	0.1728	}
	$j = 1$	2	0.4	1.0688	
$j = 2$	4	0.2	1.4848		

$$I = \frac{4}{3}(1.0688) - \frac{1}{3}(0.1728) = 1.3674667 \quad (j=1, k=1)$$

*Exact integral is 1.64053334*

# Example

			$k = 0$	$k = 1$	
$j$	$k$	segments	$O(h^2)$	$O(h^4)$	
		1	0.8	0.1728	
		2	0.4	1.0688	1.3674667
		4	0.2	1.4848	

$$I = \frac{4}{3}(1.4848) - \frac{1}{3}(1.0688) = 1.62346667 \quad (j=2, k=1)$$

*Exact integral is 1.64053334*

# Example

				$k = 1$	$k = 2$
$j$	$k$	segments	$h$	$O(h^2)$	$O(h^4)$
		1	0.8	0.1728	
2	0.4	1.0688	1.3674667		
4	0.2	1.4848	1.62346667		

$(j=2, k=2)$  
$$I = \frac{16}{15}(1.62346667) - \frac{1}{15}(1.3674667) = 1.64053334$$

*Exact integral is 1.64053334*



# Example

		$k$			
		$k = 1$	$k = 2$	$k = 3$	
$j$	segments	$h$	$O(h^2)$	$O(h^4)$	$O(h^6)$
	1	0.8	0.1728		
	2	0.4	1.0688	1.3674667	
	4	0.2	1.4848	1.62346667	1.64053334

# Example

- Better and better results can be obtained by continuing this

		$k = 3$					
$j$	$k$	segments	$h$	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
			1	0.8	0.1728		
		2	0.4	1.0688	1.3674667		
		4	0.2	1.4848	1.62346667	1.64053334	
		8	0.1	??	??	??	??

$$I = \frac{64}{63} (??) - \frac{1}{63} (1.64053334) = ??$$

$(j=3, k=3)$

# Higher-Order Polynomials

- Recall:

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_0+m_1} p_{m_1}(x)dx + \int_{x_0+m_1}^{x_0+m_1+m_2} p_{m_2}(x)dx + \dots + \int_{x_{n-m_n}}^{x_n} p_{m_n}(x)dx$$

$m$	Polynomial	Formula	Error
1	linear	Trapezoid	$O(h^2)$
2	quadratic	Simpson's 1/3	$O(h^4)$
3	cubic	Simpson's 3/8	$O(h^4)$