Chapter 8

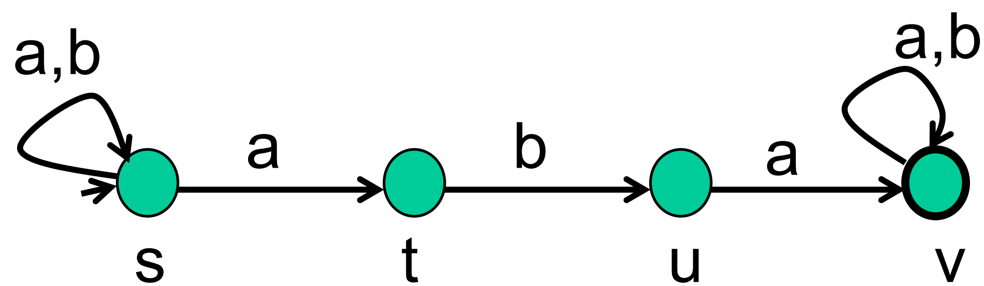
DFA state minimization

Motivations

Problems:

1. Given a DFA M with k states, is it possible to find an equivalent DFA M' (I.e., L(M) = L(M')) with state number fewer than k ?
2. Given a regular language A, how to find a machine with minimum number of states ?

Ex: A = L((a+b)*aba(a+b)*) can be accepted by the following NFA:



By applying the subset construction, we can construct a DFA M2 with 2^4=16 states,

of which only 6 are accessible from the initial state {s}.

Inaccessible states

- A state $p \in Q$ is said to be inaccessible (or unreachable) [from the initial state] if there exists no string x in Σ^* s.t. $\Delta(s,x) = p$ (I.e., $p \notin \{q \mid \exists x \in \Sigma^*, \Delta(s,x) = q \}$.)

Theorem: Removing inaccessible states from a machine M does not affect the language it accepts.

Pf: $M = \langle Q, \Sigma, \delta, s, F \rangle$: a DFA; p : an inaccessible state

Let $M' = \langle Q \setminus \{p\}, \Sigma, \delta', s, F \setminus \{p\} \rangle$ be the DFA M with p removed. Where $\delta' : (Q \setminus \{p\}) \times \Sigma \rightarrow Q \setminus \{p\}$ is defined by

$\delta'(q,a) = r$ if $\delta(q, a) = r$ and $q, r \in Q \setminus \{p\}$.

For M and M' it can be proved by induction on x that

for all x in Σ^*, $\Delta(s,x) = \Delta'(s,x)$.

Hence for all $x \in \Sigma^*$, $x \in L(M)$ iff $\Delta(s,x) = q \in F$

iff $\Delta'(s,x) = q \in F \setminus \{p\}$ iff $x \in L(M')$.

Inaccessible states are redundant

- M : any DFA with n inaccessible states $p_1,p_2,\ldots,p_n$.

Let $M_1,M_2,..,M_{n+1}$ are DFAs s.t. DFA M_{i+1} is constructed from M_i by removing p_i from M_{i}. I.e.,

M -rm(p_1)-> M_1 -rm(p_2)-> M_2 - … M_n -rm(p_n)-> M_n

By previous lemma: $L(M) = L(M_1) = \ldots = L(M_n)$ and

M_n has no inaccessible states.

- Conclusion: Removing all inaccessible sates simultaneously from a DFA will not affect the language it accepts.
- In fact the conclusion holds for all NFAs we well.

Pf: left as an exercise.

- Problem: Given a DFA (or NFA), how to find all inaccessible states ?

How to find all accessible states

- A state is said to be accessible if it is not inaccessible.

Note: the set of accessible states A(M) of a NFA M is

$$\{q \mid \exists\ x \in \Sigma^*,\ q \in \Delta(S,x)\}$$

and hence can be defined by induction.

- Let A_k be the set of states accessible from initial states of M by at most k steps of transitions.

I.e., $A_k = \{q \mid \exists\ x \in \Sigma^*$ with $|x| \le k$ and $q \in \Delta(S,x)\}$

- What is the relationship b/t A(M) and A_ks ?
  - sol: $A(M) = \bigcup_{k\ge 0} A_k$. Moreover $A_k \subseteq A_{k+1}$
- What is A_0 and the relationship b/t A_k and A_{k+1} ?

Formal definition: $M=<Q,\Sigma,\delta, S,F>$: any NFA.

- Basis: Every start state $q \in S$ is accessible.($A_0 \subseteq A(M)$)
- Induction: If q is accessible and p in $\delta(q,a)$ for some $a \in \Sigma$, then p is accessible.

  ($A_{k+1}=A_k \cup \{p \mid p \in \delta(q,a)$ for some $q \in A_k$ and $a \in \Sigma.\}$)

An algorithm to find all accessible states:

- **REACH(M) {** // $M = \langle Q, \Sigma, \delta, S, F \rangle$

1. A = S; // $A = A_0$
2. B = Δ (A) - A ; // $B = A_1 - A_0$
3. For k = 0 to |Q| do { // $A = A_k$; $B = A_{K+1} - A_k$
4. A = A U B ; // $A = A_{K+1}$

   B = Δ(B) - A; // $B = \Delta(B) - A = \Delta(A_{K+1} - A_k) - A_{K+1} = A_{K+2} - A_{k+1}$;

   if B = {} then break };
5. Return(A) }

Function Δ(S) { // = $\bigcup_{p \in S,\, a \in \Sigma,} q \in \delta(p,a)$

1. Δ = {};
2. For each q in S do

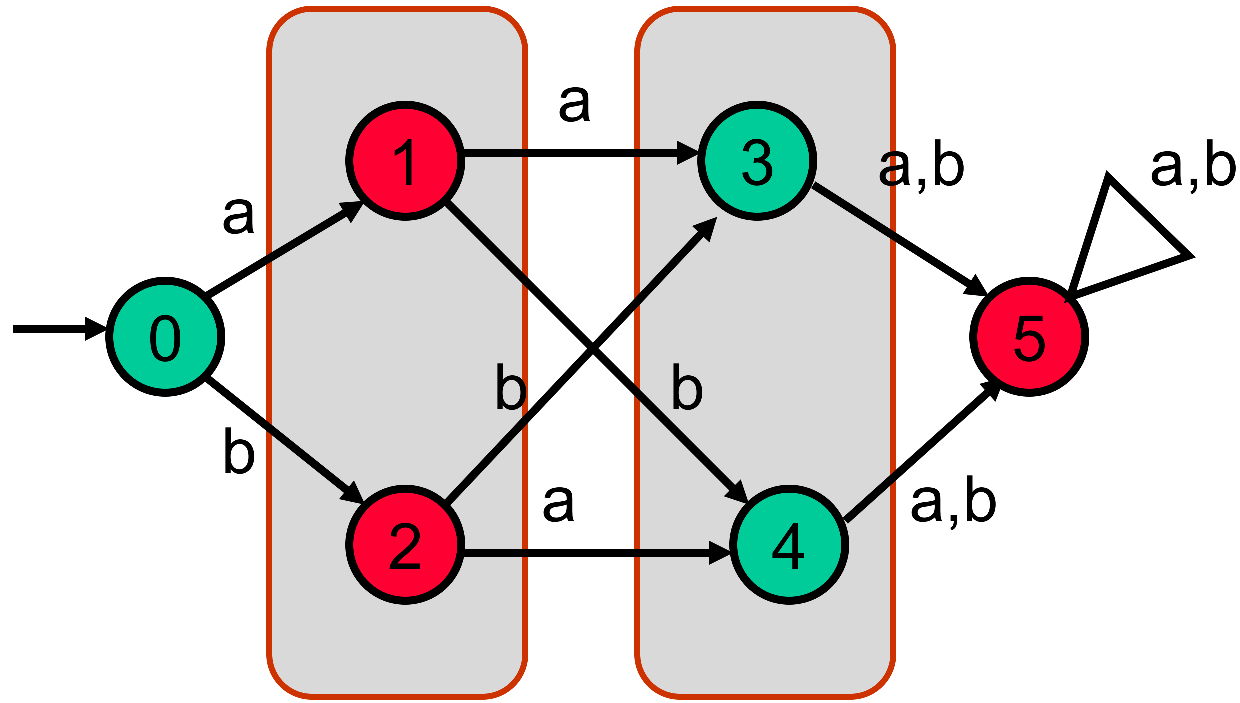
   for each a in Σ do

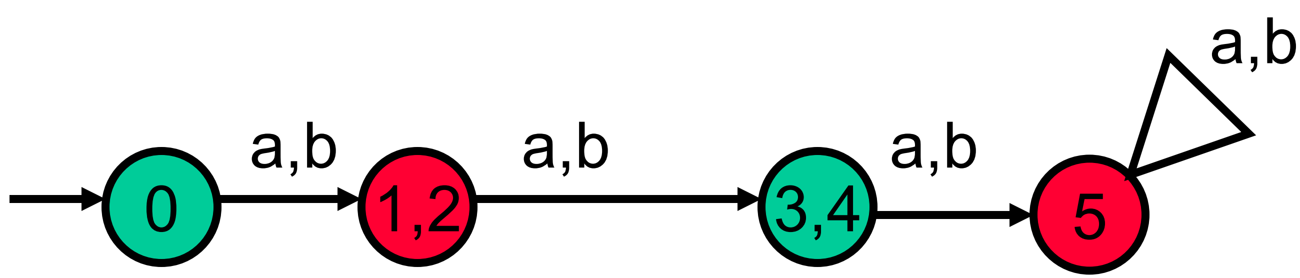
   Δ = Δ U δ (q,a);
3. Return(Δ) }

Minimization process

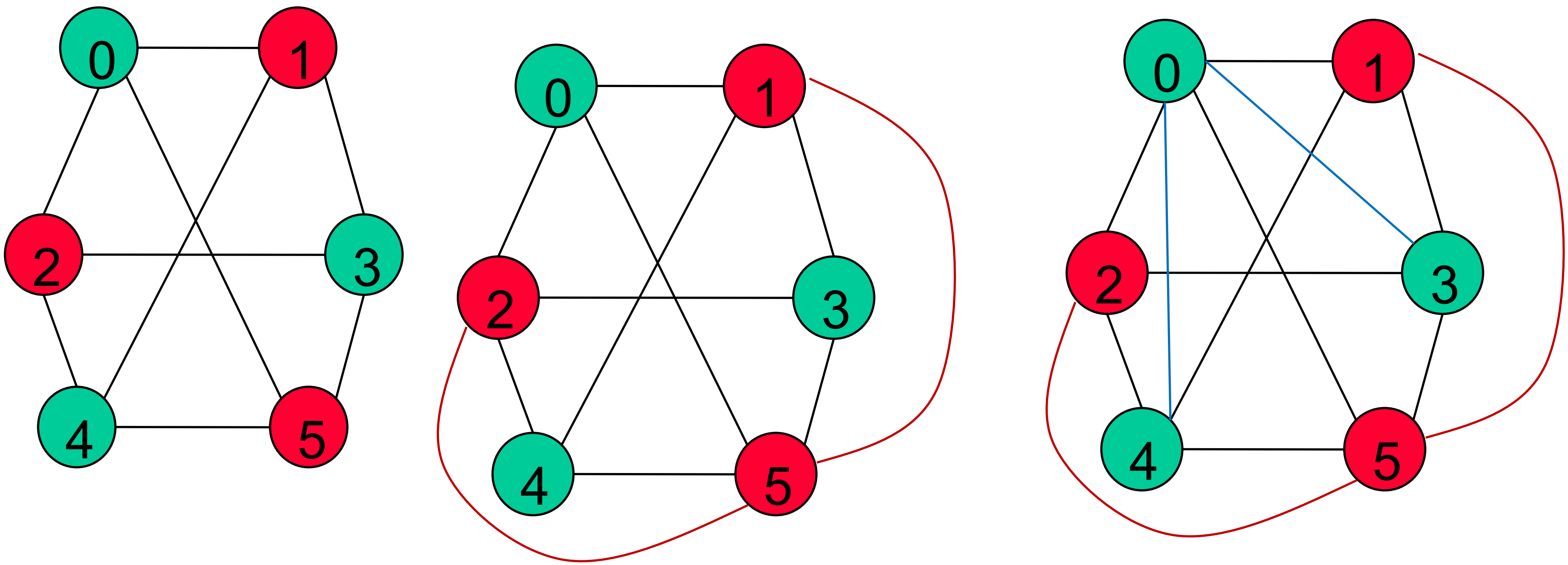
- Minimization process for a DFA:
  - 1. Remove all inaccessible states
  - 2. Merge all *equivalent* states
- What does it mean that two states are equivalent?
  - both *have the same observable behaviors* .i.e.,
  - there is no way to distinguish their difference.
- Definition: we say state p and q are *distinguishable* if there exists a string $x \in \Sigma^*$ s.t. $(\Delta(p,x) \in F \Leftrightarrow \Delta(q,x) \notin F)$.
  - If there is no such string, i.e. $\forall x \in \Sigma^* (\Delta(p,x) \in F \Leftrightarrow \Delta(q,x) \in F)$, we say p and q are equivalent (or indistinguishable).
- Example[13.2]: (next slide)
  - state 3 and 4 are equivalent.
  - States 1 and 2 are equivalent.
- Equivalents sates can be merged to form a simpler machine.

Example 13.2:





Example 13.2: Witness for states that are distinguishable



1. States b/t {0,3,4} and {1,2,5} can be distinguishsed by the empty string ε.
2. States b/t {1,2} and {5} can be distinguished by a or b.
3. States b/t {0} and {3,4} can be distinguished by aa,ab, ba or bb.
4. There is no way to distinguish b/t 1 and 2, and b/t 3 and 4.

Quotient Construction

- $M=(Q, \Sigma, \delta, s, F)$: a DFA.
- $\approx$: a relation on Q defined by:

  $p \approx q \iff \forall x \in \Sigma^* \quad \Delta(p,x) \in F \text{ iff } \Delta(q,x) \in F$
- Property: $\approx$ is an equivalence (i.e., reflexive, symmetric and transitive) relation.
- Hence it partitions Q into equivalence classes :
  - $[p] =_{def} \{q \in Q \mid p \approx q\}$ for $p \in Q$.
  - $Q/\approx \; =_{def} \{[p] \mid p \in Q\}$ is the quotient set.
  - Every $p \in Q$ belongs to exactly one class (which is [p])
  - $p \approx q$ iff $[p]=[q]$ //why? since $p \approx q$ implies ($p \approx r$ iff $q \approx r$).
- Ex: From Ex 13.2, we have $0,\ 1 \approx 2,\ 3 \approx 4,\ 5$.
  - => [0] = {0}, [1] = {1,2}, [2]={1,2}, [3]={3,4},[4]={3,4} and
  - [5] = {5}. As a result, [1] = [2] = {1,2}, [3]=[4]= {3,4} and
  - $Q/\approx$ = { {0},{1,2},{3,4},{5}} = { [0],[1],[2],[3],[4],[5] } = {[0],[1],[3],[5] }.

the function δ' is well-defined.

- Define a DFA called the quotient machine $M/\approx = \langle Q', \Sigma, \delta', s', F' \rangle$ where
  - $Q' = Q/\approx$; $s' = [s]$; $F' = \{[p] \mid p \in F\}$; and
  - $\delta'([p], a) = [\delta(p, a)]$ for all $p \in Q$ and $a \in \Sigma$. But well-defined?

Lem 13.5. if $p \approx q$ then $\delta(p,a) \approx \delta(q,a)$.

Hence $[p]=[q] \Rightarrow p \approx q \Rightarrow \delta(p,a) \approx \delta(q,a) \Rightarrow [\delta(p,a)] = [\delta(q,a)]$

Pf: By def. $[\delta(p,a)] = [\delta(q,a)]$ iff $\delta(p,a) \approx \delta(q,a)$

iff $\forall y \in \Sigma^* \; \Delta(\delta(p,a), y) \in F \Leftrightarrow \Delta(\delta(q,a), y) \in F$

iff $\forall y \in \Sigma^* \; \Delta(p, ay) \in F \Leftrightarrow \Delta(q, ay) \in F$

if $p \approx q$.

Lemma 13.6. $p \in F$ iff $[p] \in F'$.

pf: => : trival.

<=: need to show that if $q \approx p$ and $p \in F$, then $q \in F$.

But this is trivial since $p = \Delta(p, \varepsilon) \in F$ iff $\Delta(q, \varepsilon) = q \in F$

Properties of the quotient machine.

Lemma 13.7: $\forall x \in \Sigma^*$, $\Delta'([p],x) = [\Delta(p,x)]$.

Pf: By induction on $|x|$.

Basis $x = \varepsilon$: $\Delta'([p], \varepsilon] = [p] = [\Delta(p, \varepsilon)]$.

Ind. step: Assume $\Delta'([p],x) = [\Delta(p,x)]$ and let $a \in \Sigma$.

$\Delta'([p],xa) = \delta'(\Delta'(p,x),a) = \delta'([\Delta(p,x)],a)$ --- ind. hyp.

$=[\delta(\Delta(p,x),a)]$ -- def. of δ'

$= [\Delta(p,xa)]$. -- def. of Δ.

Theorem 13.8: $L(M/\approx) = L(M)$.

Pf: $\forall x \in \Sigma^*$,

$x \in L(M/\approx)$ iff $\Delta'(s',x) \in F'$

iff $\Delta'([s],x) \in F'$ iff $[\Delta(s,x)] \in F'$ --- lem 13.7

iff $\Delta(s,x) \in F$ --- lem 13.6

iff $x \in L(M)$.

M/≈ need not be merged further

- **Theorem: $((M/\approx) / \approx) = M/\approx$**

Pf: Denote the second ≈ by ~. I.e.

$[p] \sim [q]$ iff $\forall x \in \Sigma^*$, $\Delta'([p],x) \in F' \Leftrightarrow \Delta'([q],x) \in F'$

Now

$[p] \sim [q]$

iff $\forall x \in \Sigma^*$, $\Delta'([p],x) \in F' \Leftrightarrow \Delta'([q],x) \in F'$ -- def.of~

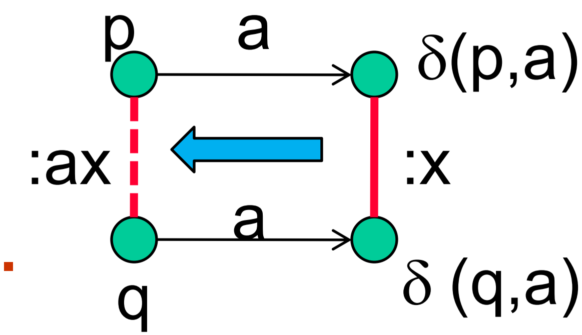
iff $\forall x \in \Sigma^*$, $[\Delta(p,x)] \in F' \Leftrightarrow [\Delta(q,x)] \in F'$ -- lem 13.7

iff $\forall x \in \Sigma^*$, $\Delta(p,x) \in F \Leftrightarrow \Delta(q,x) \in F$ -- lem 13.6

iff $p \approx q$ -- def of ≈

iff $[p] = [q]$ -- property of equivalence ≈

A minimization algorithm



1. Write down a table of all pairs {p,q}, initially unmarked.
2. mark {p,q} if $p \in F$ and $q \notin F$ or vice versa.
3. Repeat until no additional pairs marked:
   3.1 if ∃ unmarked pair {p,q} s.t. {δ(p,a), δ(q,a) } is marked for some $a \in \Sigma$, then mark {p,q}.
4. When done, $p \approx q$ iff {p,q} is not marked.

Let M_k ($k \geq 0$) be the set of pairs marked after the k-th iteration of step 3. [and M_0 is the set of pairs before step 3.]

Notes: (1) $M = \cup_{k \geq 0} M_k$ is the final set of pairs marked by the alg. (2) The algorithm must terminate since there are totally only C(n,2) pairs and each iteration of step 3 must mark at least one pair for it to not terminate..

An Example:

- **The DFA: (Ex 13.2)**

	a	b
>0	1	2
1F	3	4
2F	4	3
3	5	5
4	5	5
5F	5	5

Initial Table

1	-				
2	-	-			
3	-	-	-		
4	-	-	-	-	
5	-	-	-	-	-
	0	1	2	3	4

After step 2 (M_0)

1	M				
2	M	-			
3	-	M	M		
4	-	M	M	-	
5	M	-	-	M	M
	0	1	2	3	4

After first pass of step 3 (M_1)

1	M				
2	M	-			
3	-	M	M		
4	-	M	M	-	
5	M	M	M	M	M
	0	1	2	3	4

2nd pass of step 3. (M_2 & M_3)

- The result : 1 ≈ 2 and 3 ≈ 4.

1	M				
2	M	-			
3	M2	M	M		
4	M2	M	M	-	
5	M	M1	M1	M	M
	0	1	2	3	4

Correctness of the minimization algorithm

Let M_k ($k \geq 0$) be the set of pairs marked after the k-th itration of step 3. [and M_0 is the set of pairs befer step 3.]

Lemma: $\{p,q\} \in M_k$ iff $\exists x \in \Sigma^*$ of length $\leq k$ s.t. $\Delta(p,x) \in F$ and $\Delta(q,x) \notin F$ or vice versa,

Pf: By ind. on k. **Basis** k = 0. trivial.

Ind. step: $\exists\, x \in \Sigma^*$ of length $\leq k+1$ s.t. $\Delta(p,x) \in F \Leftrightarrow \Delta(q,x) \notin F$,

iff $\exists\, y \in \Sigma^*$ of length $\leq k$ s.t. $\Delta(p,y) \in F \Leftrightarrow \Delta(q,y) \notin F$, or

$\exists\, ay \in \Sigma^*$ of length $\leq k+1$ s.t. $\Delta(\delta(p,a),y) \in F \Leftrightarrow \Delta(\delta(q,a),y) \notin F$,

iff $\{p, q\} \in M_k$ or $\{\delta(p,a), \delta(q,a)\} \in M_k$ for some $a \in \Sigma$.

iff $\{p,q\} \in M_{k+1}$.

Theorem 14.3: The pair {p,q} is marked by the algorithm iff not($p \approx q$) (i.e., $\exists\, x \in \Sigma^*$ s.t. $\Delta(p,x) \in F \Leftrightarrow \Delta(q,x) \notin F$)

Pf: not($p \approx q$) iff $\exists\, x \in \Sigma^*$ s.t. $\Delta(p,x) \in F \Leftrightarrow \Delta(q,x) \notin F$

iff $\{p,q\} \in M_k$ for some $k \geq 0$

iff $\{p,q\} \in M = \bigcup_{k \geq 0} M_k$.