# PART III

## Turing Machines and Effective Computability

# PART III Chapter 1

## Turing Machines

## Turing machines

- the most powerful automata (> FAs and PDAs )
- invented by Turing in 1936
- can compute any function normally considered computable
- Turing-Church Thesis:
  - Anything (function, problem, set etc.) that is (though to be) computable is computable by a Turing machine (i.e., Turing-computable).
- Other equivalent formalisms:
  - post systems (string rewriting system)
  - Formal Grammars (Chomsky Hierarchy): on strings
  - $\mu$-recursive function : on numbers
  - $\lambda$-calculus, combinatory logic: on $\lambda$-term
  - C, BASIC, PASCAL, JAVA languages,… : on strings

## Informal description of a Turing machine

1. Finite automata (DFAs, NFAs, etc.):
   - limited input tape: one-way, read-only
   - no working-memory
   - finite-control store (program)
2. PDAs:
   - limited input tape: one-way, read-only
   - one additional stack as working memory
   - finite-control store (program)
3. Turing machines (TMs):
   - a semi-infinite tape storing input and supplying additional working storage.
   - finite control store (program)
   - can read/write and two-way(move left and right) depending on the program state and input symbol scanned.

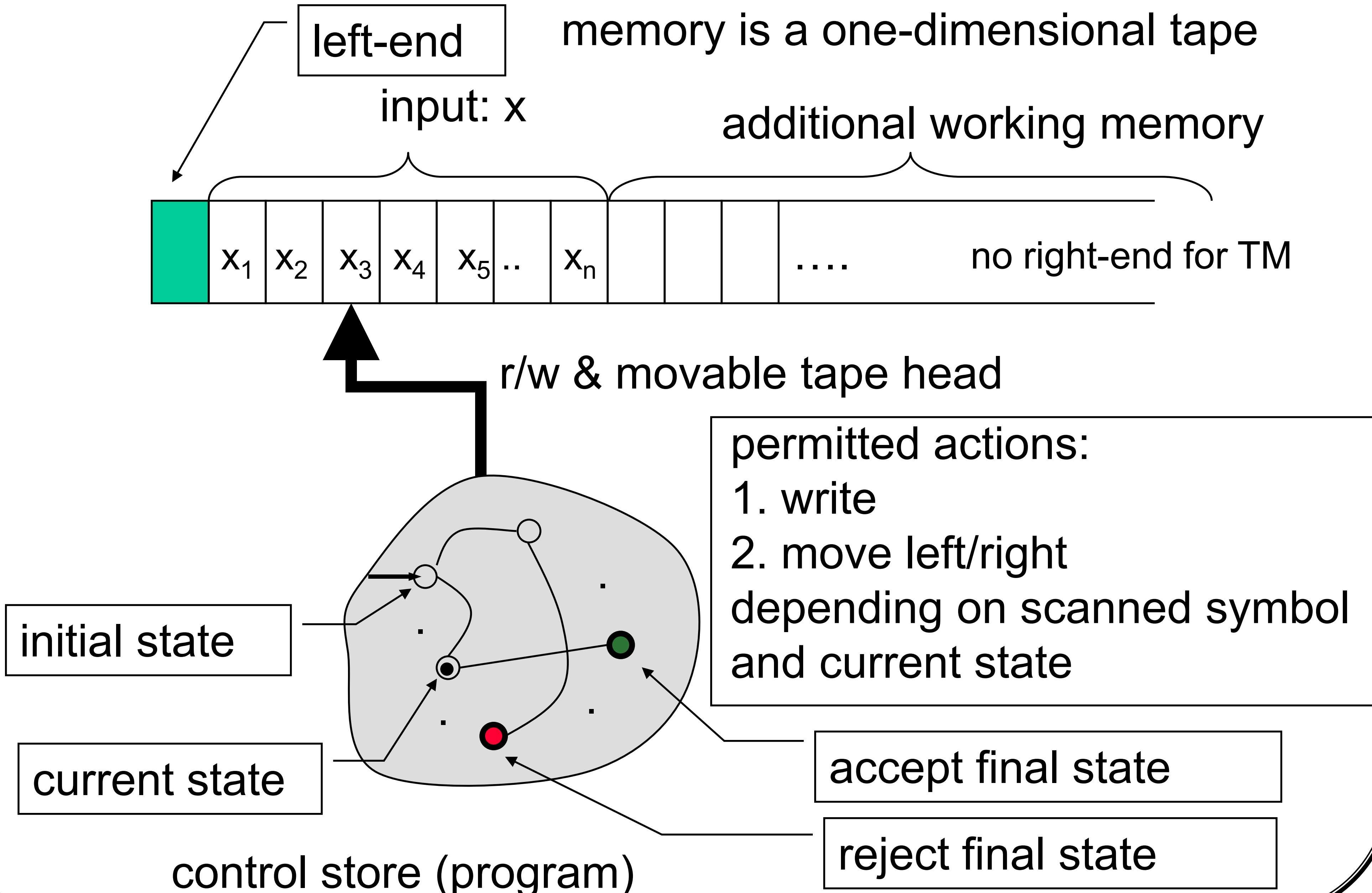
## Turing machines and LBAs

4. Linear bounded automata (LBA): special TMs
   -  the input tape is of the same size as the input length
   (i.e., no additional memory supplied except those used to store the input)
   -  can read/write and move left/right depending on the program state and input symbol scanned.

- Primitive instructions of a TM (like +,-,*, etc in C or BASIC):
  1. L, R // moving the tape head left or right
  2. $a \in \Gamma$, // write the symbol $a \in \Gamma$ on the current scanned position

  depending on the precondition:
  1. current state and
  2. current scanned symbol of the tape head

# The model of a Turing machine



## The structure of a TM instruction:

- **An instruction of a TM is a tuple:**

$$(q, \quad a, \quad p, \quad d) \in Q \times \Gamma \times Q \times (\Gamma \cup \{L,R\})$$

where

- q is the current state
- a is the symbol scanned by the tape head
- (q,a) defines a precondition that the machine may encounter
- (p,d) specify the actions to be done by the TM once the machine is in a condition matching the precondition (i.e., the symbol scanned by the tape head is 'a' and the machine is at state q )
- p is the next state that the TM will enter
- d is the action to be performed:
  - ✪ $d = b \in \Gamma$ means "write the symbol b to the tape cell currently scanned by the tape head".
  - ✪ d = R (or L) means "move the tape head one tape cell in the right (or left, respectively) direction.

- **A Deterministic TM program $\delta$ is simply a set of TM instructions (or more formally a function: $\delta$: $Q \times \Gamma$ --> $Q \times (\Gamma \cup \{L,R\})$)**

## Formal Definition of a standard TM (STM)

- **A deterministic 1-tape Turing machine (STM) is a 9-tuple**

  **M = (Q,Σ,Γ, [, □, δ, s, t,r ) where**

  - **Q : is a finite set of (program) states with a role like labels in traditional programs**
  - **Γ : tape alphabet**
  - **$\Sigma \subset \Gamma$ : input alphabet**
  - **[ $\in \Gamma - \Sigma$ : The left end-of-tape mark**
  - **□ $\in \Gamma - \Sigma$ is the blank tape symbol**
  - **s $\in$ Q : initial state**
  - **t $\in$ Q : the accept state**
  - **r $\neq$ t $\in$ Q: the reject state and**
  - **δ: (Q - {t,r})x Γ --> Qx(Γ U {L,R}) is a *total* transition function with the restriction:** if δ(p, [ ) =(q, d) then d = R. i.e., the STM cannot write any symbol at left-end and never move off the tape to the left.

## Configurations and acceptances

- Issue: h/w to define configurations like those defined in FAs and PDAs ?
- At any time $t_0$ the TM M's tape contains a semi-infinite string of the form

  $$\text{Tape}(t_0) = [\, y_1 y_2 \ldots y_m \,\square\,\square\,\square\,\square \ldots\ldots \quad (y_m \neq \square)$$

- Let $\square^\omega$ denotes the semi-infinite string:

  $$\square\,\square\,\square\,\square\,\square \ldots\ldots$$

Note: Although the tape is an infinite string, it has a finite canonical representation: y, where $y = [\, y_1 \ldots y_m$ (with $y_m \neq \square$)

A configuration of the TM M is a global state giving a snapshot of all relevant info about M's computation at some instance in time.

## Formal definition of a configuration

Def: a cfg of a STM M is an element of

$$CF_M =_{def} Q \times \{ [\, y \mid y \in (\Gamma - \{[\})^* \} \times N \qquad // \; N = \{0,1,2,\ldots\} \; //$$

When the machine M is at cfg (p, z, n) , it means M is

1. at state p
2. Tape head is pointing to position n and
3. the input tape content is z.

Obviously cfg gives us sufficient information to continue the execution of the machine.

Def: 1. [Initial configuration:] Given an input x and a STM M, the initial configuration of M on input x is the triple:

$$(s, [x, 0)$$

2. If cfg1 = (p, y, n), then cfg1 is an accept configuration if p = t (the accept configuration), and cfg1 is an reject cfg if p = r ( the reject cfg). cfg1 is a halting cfg if it is an accept or reject cfg.

## One-step and multi-step TM computations

- one-step Turing computation ( $|\!-\!-_M$) is defined as follows:
- $|\!-\!-_M \subseteq CF_M^2$ is the least binary relation over $CF_M$ s.t.

0. $(p,z,n)\ |\!-\!-_M\ (q, s^n_b(z), n)$ if $\delta(p,z_n) = (q, b)$ where $b \in \Gamma$
1. $(p,z,n)\ |\!-\!-_M\ (q,z, n-1)$ if $\delta(p,z_n) = (q, L)$
2. $(p,z,n)\ |\!-\!-_M\ (q,z, n+1)$ if $\delta(p,z_n) = (q, R)$

   - where $s^n_b(z)$ is the resulting string with the n-th symbol of z replaced by 'b'.
   - ex: $s^4_b($ [baa<u>a</u>cabc ) = [baa<u>b</u>cabc
   - $s^6_b($ [baa ) = [baa□□b

- $|\!-\!-_M$ is defined to be the set of all pairs of configurations each satisfying one of the above three rules.

**Notes:** 1. if $C=(p,z,n)\ |\!-\!-_M\ (q,y,m)$ then $n \geq 0$ and $m \geq 0$ (why?)

2. $|\!-\!-_M$ is a function [from nonhalting cfgs to cfgs] (i.e., if $C\ |\!-\!-_M\ D$ & $C\ |\!-\!-_M\ E$ then D=E).

3. define $|\!-\!-^n_M$ and $|\!-\!-^*_M$ (ref. and tran. closure of $|\!-\!-_M$) as usual.

## Accepting and rejecting of TM on inputs

- $x \in \Sigma$ is said to be accepted by a STM M if

  $icfg_M(x) =_{def} (s, [x, 0) \mid\text{--}^*_M (t,y,n)$ for some y and n

  - I.e, there is a finite computation

    $(s, [x, 0) = C_0 \mid\text{--}_M C_1 \mid\text{--}_M \dots \mid\text{--}_M C_k = (t,\_,\_)$

  starting from the initial configuration and ending at an accept configuration.

- x is said to be rejected by a STM M if

  $(s, [x, 0) \mid\text{--}^*_M (r,y,n)$ for some y and n

  - I.e, there is a finite computation
  - $(s, [x, 0) = C_0 \mid\text{--}_M C_1 \mid\text{--}_M \dots \mid\text{--}_M C_k = (t,\_,\_)$
  - starting from the initial configuration and ending at a reject configuration.

Notes: 1. It is impossible that x is both accepted and rejected by a STM. (why ?)

2. It is possible that x is neither accepted nor rejected. (why ?)

## Languages accepted by a STM

**Def:**

1. M is said to *halt* on input x if either M accepts x or rejects x.
2. M is said to *loop* on x if it does not halt on x.
3. A TM is said to be *total* if it halts on all inputs.
4. The language accepted by a TM M,

   $L(M) =_{def} \{x \text{ in } \Sigma^* \mid x \text{ is accepted by } M, \text{ i.e., } (s, [x\Box^{\omega}, 0) \vdash^*_M (t, -,-)\}$

5. If L = L(M) for some STM M

   ==> L is said to be *recursively enumerable (r.e.)*

6. If L = L(M) for some total *STM M*

   ==> L is said to be *recursive*

7. If $\sim L =_{def} \Sigma^* - L = L(M)$ for some STM M (or total STM M)

   ==> L is said to be *Co-r.e. (or Co-recursive, respectively)*

## Some examples

**Ex1: Find a STM to accept $L_1$ = { w # w | w ∈ {a,b}* }**

note: $L_1$ is not a CFL.

The STM has tape alphabet Γ = {a, b,#, -, □, [} and behaves as follows:

on input z : (Hopefully of the form: w # w ∈ {a,b,#}* )

1. if z is not of the form {a,b}* # {a,b}* => goto reject
2. move left until '[' is encountered and in that case move right
3. while I/P (i.e., symbol scanned by input head) = '-' move right;
4. if I/P = 'a' then
   - 4.1 write '-'; move right until # is encountered; Move right;
   - 4.2 while I/P = '-' move right
   - 4.3 case (I/P) of { 'a' : (write '-'; goto 2); o/w: goto reject }
5. if I/p = 'b' then … // like 4.1~ 4.3
6. If I/P = '#' then // All symbols left to # have been compared
   - 6.1 move right
   - 6.2 while I/P = '-" move right
   - 6.3 case (I/P) of {'□' : goto Accept; o/w: go to Reject }

## More detail of the STM

Step 1 can be accomplished as follows:

1.1 while I/P matches (~# /\ ~ □) R; // i.e, I/P ≠ # and I/P ≠ □

//or equivalently, while I/P matches (a \/ b\/ [ \/ - ) R

if □ => reject // no # found on the input

if # => R;

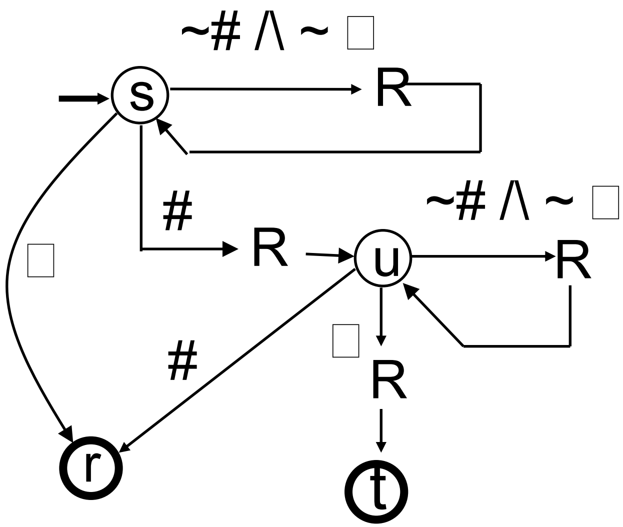
1.2 While ( ~# /\ ~ □ ) R;

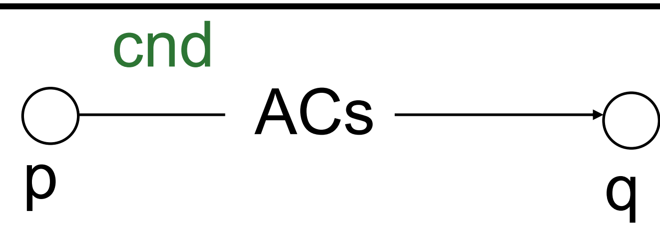
if □ => goto accept [or goto 2 if regarded as a subroutine]

if # => goto Reject; // more than one #s found

Step 1 requires only two states:

# Graphical representation of a TM




means:

if (state = p) /\ (cnd true for I/P)

then 1. perform ACs and

2. go to q

ACs can be primitive ones: R, L, a,…

or another subroutine TM $M_1$.

Ex: the arc from s to s in the left graph implies the existence of 4 instructions:

(s, a, s, R), (s,b,s,R),

(s, [,s,R), and (s,-, s,R)

## Tabular form of a STM

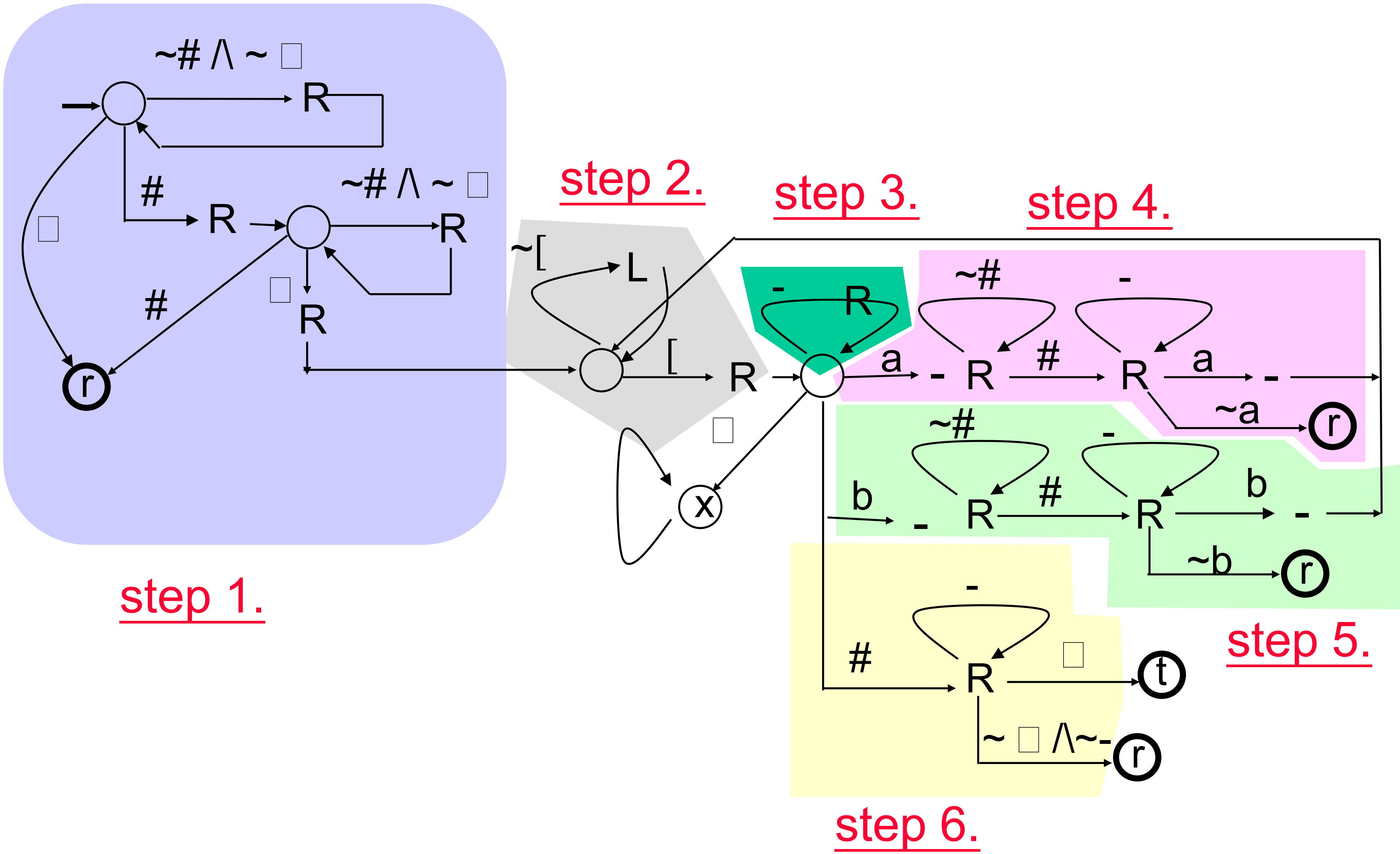
- **Translation of the graphical form to tabular form of a STM**

| δ \ Γ, Q | [ | a | b | # | - | □ |
|---|---|---|---|---|---|---|
| >s | s,R | s,R | s,R | u,R | x | r,x |
| u | x | u,R | u,R | r,x | x | t, □ |
| tF | halt | halt | halt | halt | halt | halt |
| rF | halt | halt | halt | halt | halt | halt |

X means don't care

**The rows for t & r indeed need not be listed!!**

# The complete STM accepting $L_1$



## R.e. and recursive languages

Recall the following definitions:

1. M is said to *halt* on input x if either M accepts x or rejects x.
2. M is said to *loop* on x if it does not halt on x.
3. A TM is said to be *total* if it halts on all inputs.
4. The language accepted by a TM M,

   $L(M) =_{def} \{x \in \Sigma^* \mid$ x is accepted by M, i.e., $(s, [x \square^{\omega}, 0) \mid\text{--}^*_M (t, -, -) \}$
5. If L = L(M) for some STM M

   ==> L is said to be *recursively enumerable (r.e.)*
6. If L = L(M) for some total *STM M*

   ==> L is said to be *recursive*
7. If $\sim L =_{def} \Sigma^* - L = L(M)$ for some STM M (or total STM M)

   ==> L is said to be *Co-r.e. (or Co-recursive, respectively)*

# Recursive languages are closed under complement

**Theorem 1:** Recursive languages are closed under complement. (i.e., If L is recursive, then $\sim L = \Sigma^* - L$ is recursive.)

pf: Suppose L is recursive. Then $L = L(M)$ for some total TM M.

Now let M* be the machine M with accept and reject states switched (i.e., the accepting state t* of M* is r of M, while rejecting state r* of M* is t of M ).

Now for any input x,

- $x \notin \sim L \Rightarrow x \in L(M) \Rightarrow icfg_M(x) \vdash_M^* (t,-,-) \Rightarrow$
- $icfg_{M^*}(x) \vdash_{M^*}^* (r^*,-,-) \Rightarrow x \notin L(M^*)$.
- $x \in \sim L \Rightarrow x \notin L(M) \Rightarrow icfg_M(x) \vdash_M^* (r,-,-) \Rightarrow$
- $icfg_{M^*}(x) \vdash_{M^*}^* (t^*,-,-) \Rightarrow x \in L(M^*)$.

Hence $\sim L = L(M^*)$ and is recursive.

**Note.** The same argument cannot be applied to r.e. languages. (why?)

**Exercise:** Are recursive sets closed under union, intersection, concatenation and/or Kleene's operation ?

## Some more termonology

**Set : Recursive and recursively enumerable(r.e.)**

**predicate: Decidability and semidecidability**

**Problem: Solvability and semisolvabilty**

- P : a statement about strings ( or a property of strings)
- A: a set of strings
- Q : a (decision) Problem.

We say that

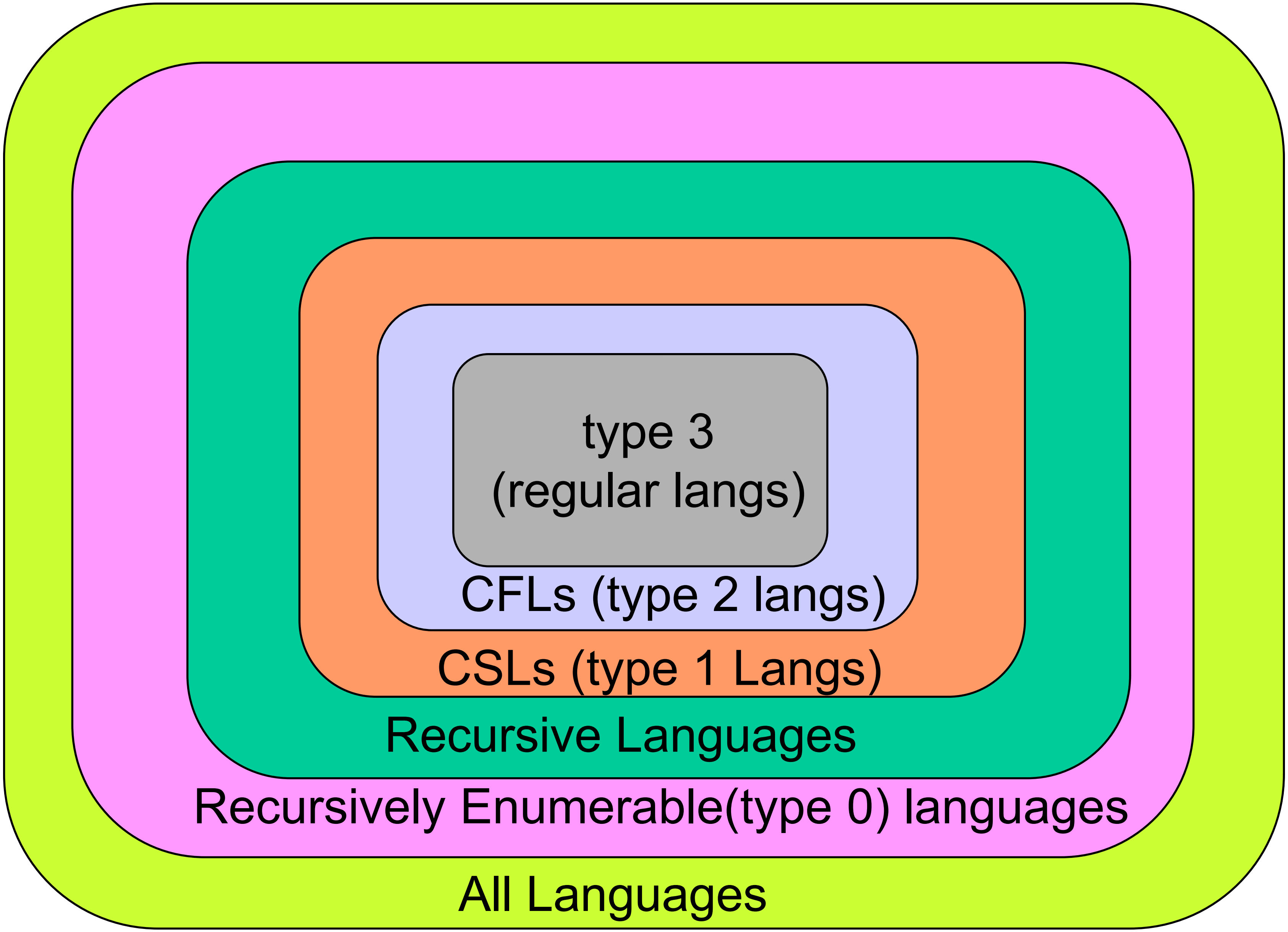
1. P is decidable <==> { x | P(x) is true } is recursive
2. A is recursive <==> "x $\in$ A" is decidable.
3. P is semidecidable <==> { x | P(x) is true } is r.e.
4. A is r.e. <==> "x $\in$ A" is semidecidable.
5. Q is solvable <=> Rep(Q) $=_{def}$ {"P" | P is a positive instance of Q } is recursive.
6. Q is semisolvale <==> Rep(Q) is r.e..

## The Chomsky Hierarchy

- **Relationship of Languages, Grammars and machines**

| Language | recognition model | generation model |
|---|---|---|
| Regular languages;<br>type 3 languages | Finite automata<br>(DFA, NFA) | regular expressions<br>type 3(right linear, regular) grammar |
| context-free language (CFL) ;<br>type 2 languages | Pushdown automata | Context free grammar (CFG) ;<br>type 2 grammar |
| context-sensitive language (CFL) ;<br>type 1 languages | LBA (Linear Bounded Automata) | Context sensitive grammar(CSG) ;<br>type 1 Grammar |
| Recursive Languages | Total Turing machines | - |
| R.E. (Recursively enumerative ) language;<br>type 0 language | Turing machines | type 0 grammar ;<br>unrestricted grammar |

# The Chomsky Hierarchy



## Phrase-structure (unrestricted) grammar

**Def.: A unrestricted grammar G is a tuple G=(N, Σ, S, P)**

**where**

- N, Σ, and S are the same as for CFG, and
- P, a finite subset of (NUΣ)* N (NUΣ)* x (NUΣ)* , is a set of production rules of the form:
- $\alpha \rightarrow \beta$ where
- $\alpha \in$ (NUΣ)* N (NUΣ)* is a string over (NUΣ)* containing at least on nonterminal.
- $\beta \in$ (NUΣ)* is a string over (NUΣ)*.

**Def: G is of type**

- 1 (context-sensitive) if $S \rightarrow \varepsilon$ or $|\alpha| \leq |\beta|$.
- 2 (context-free) if $\alpha \in N$ and $\beta \neq \varepsilon$ or $S \rightarrow \varepsilon$.
- 3 (right linear) if every rule is one of the forms:
  - $A \rightarrow a\,B$ or $A \rightarrow a$ ($a \neq \varepsilon$) or $S \rightarrow \varepsilon$.

## Derivations

- Derivation $\rightarrow_G \subseteq (N \cup \Sigma)^* \times (N \cup \Sigma)^*$ is the least set of pairs such that :

$\forall x,y \in (\Sigma \cup N)^*, \alpha \rightarrow \beta \in P, \quad x\alpha y \rightarrow_G x\beta y.$

- Let $\rightarrow^*_G$ be the ref. and tran. closure of $\rightarrow_G$.
- L(G) : the languages generated by grammar G is the set:

$$L(G) =_{def} \{x \in \Sigma^* \mid S \rightarrow^*_G x \}$$

## Example

- Design CSG to generate the language $L=\{0^n1^n2^n \mid n \geq 0\}$, which is known to be not context free.

Sol: Consider the CSG $G_1$ with the following productions:

$S \to \varepsilon$ , $S \to 0SA2$ $2A \to A2$,

$0A \to 01$ $1A \to 11$

For $G_1$ we have

$S \to 0SA2 \to \ldots \to 0^k(A2)^k \to^* 0^kA^k2^k \to 0^k1^k2^k \therefore L \subseteq L(G1)$.

Also note that

- if $S \to^* \alpha$ then $\#0(\alpha) = \#(A|1)(\alpha) = \#(2)(\alpha)$.
- if $S \to^* \alpha \in \{0,1,2\}^*$ then
  - $\alpha_k = 0$ implies $\alpha_j = 0$ for all $j < k$.
  - $\alpha_k = 1$ implies $\alpha_j = 1$ or $0$ for all $j < k$.

  where $\alpha_k$ is the k-th symbol in string $\alpha_k$ .
- Hence $\alpha$ must be of the form $0^*1^*2^* \Rightarrow \alpha \in L$. QED