Algorithms

Introduction to heapsort

Review: The Master Theorem

- Given: a *divide and conquer* algorithm
  - An algorithm that divides the problem of size n into a subproblems, each of size n/b
  - Let the cost of each stage (i.e., the work to divide the problem + combine solved subproblems) be described by the function $f(\mathrm{n})$
- Then, the Master Theorem gives us a cookbook for the algorithm's running time:

Review: The Master Theorem

- if T(n) = aT(n/b) + f(n) then

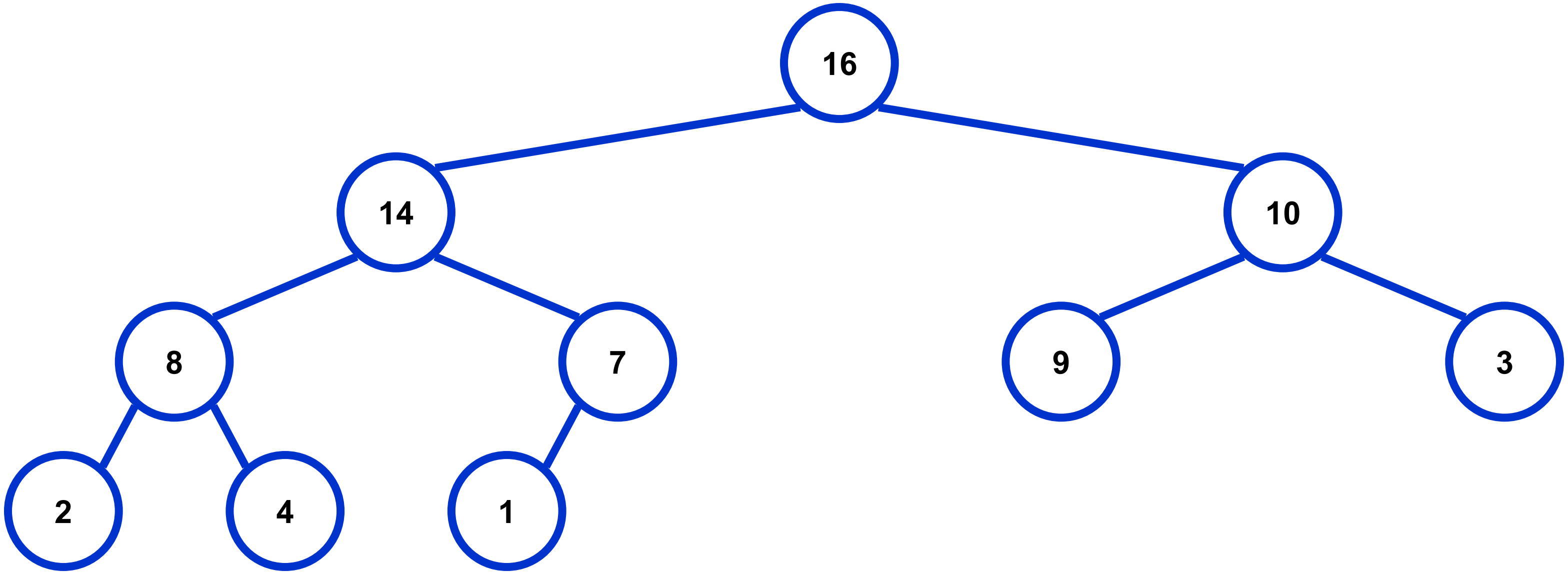
$$T(n) = \begin{cases} \Theta\left(n^{\log_b a}\right) & f(n) = O\left(n^{\log_b a - \varepsilon}\right) \\ \Theta\left(n^{\log_b a} \log n\right) & f(n) = \Theta\left(n^{\log_b a}\right) \\ \Theta\left(f(n)\right) & f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right) \text{ AND} \\ & af(n/b) < cf(n) \text{ for large } n \end{cases} \Bigg\} \begin{matrix} \varepsilon > 0 \\ c < 1 \end{matrix}$$

Sorting Revisited

- So far we've talked about two algorithms to sort an array of numbers
  - What is the advantage of merge sort?
    - Answer: O(n lg n) worst-case running time
  - What is the advantage of insertion sort?
    - Answer: sorts in place
    - Also: When array "nearly sorted", runs fast in practice
- Next on the agenda: *Heapsort*
  - Combines advantages of both previous algorithms

Heaps

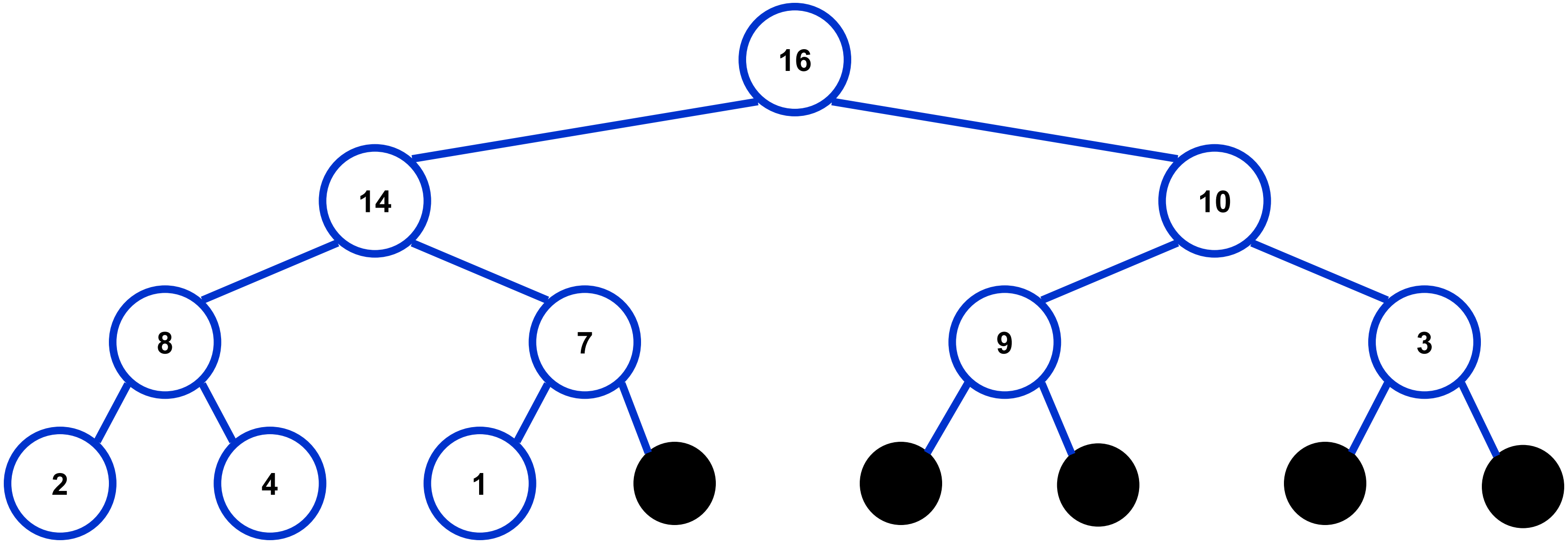
- A *heap* can be seen as a complete binary tree:



  - *What makes a binary tree complete?*
  - *Is the example above complete?*

Heaps

- A *heap* can be seen as a complete binary tree:



  - The book calls them "nearly complete" binary trees; can think of unfilled slots as null pointers

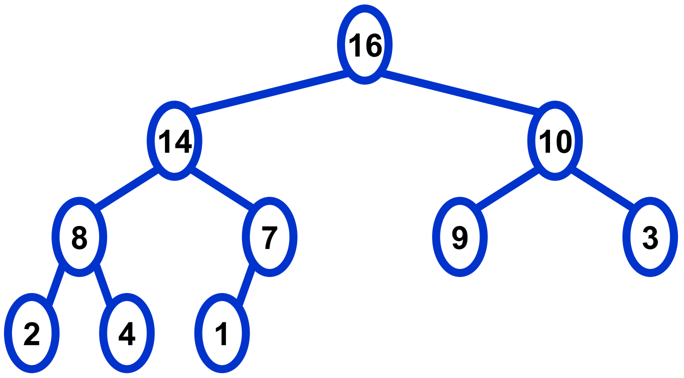
Heaps

- In practice, heaps are usually implemented as arrays:

A =

16	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

=



Heaps

- To represent a complete binary tree as an array:
  - The root node is A[1]
  - Node i is $A[i]$
  - The parent of node i is $A[i/2]$ (note: integer divide)
  - The left child of node i is $A[2i]$
  - The right child of node i is $A[2i + 1]$

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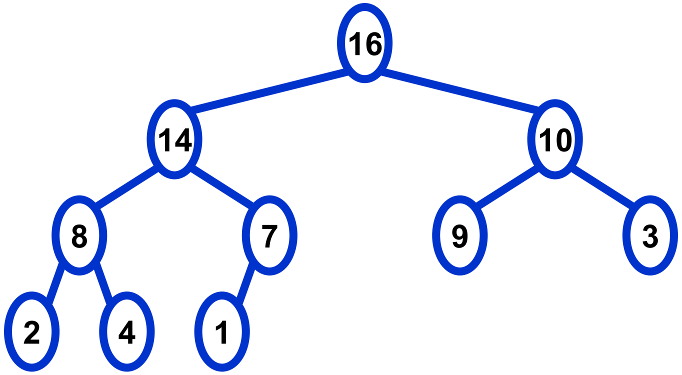


Referencing Heap Elements

- So…

```
Parent(i) { return ⌊i/2⌋; }
Left(i) { return 2*i; }
right(i) { return 2*i + 1; }
```

- An aside: *How would you implement this most efficiently?*
  - Trick question, I was looking for "`i << 1`", etc.
  - But, any modern compiler is smart enough to do this for you (and it makes the code hard to follow)

The Heap Property

- Heaps also satisfy the *heap property*:

  A[*Parent*(i)] $\geq$ A[i] for all nodes $i > 1$

  - In other words, the value of a node is at most the value of its parent
  - *Where is the largest element in a heap stored?*

Heap Height

- Definitions:
  - The *height* of a node in the tree = the number of edges on the longest downward path to a leaf
  - The height of a tree = the height of its root
- *What is the height of an n-element heap? Why?*
- This is nice: basic heap operations take at most time proportional to the height of the heap

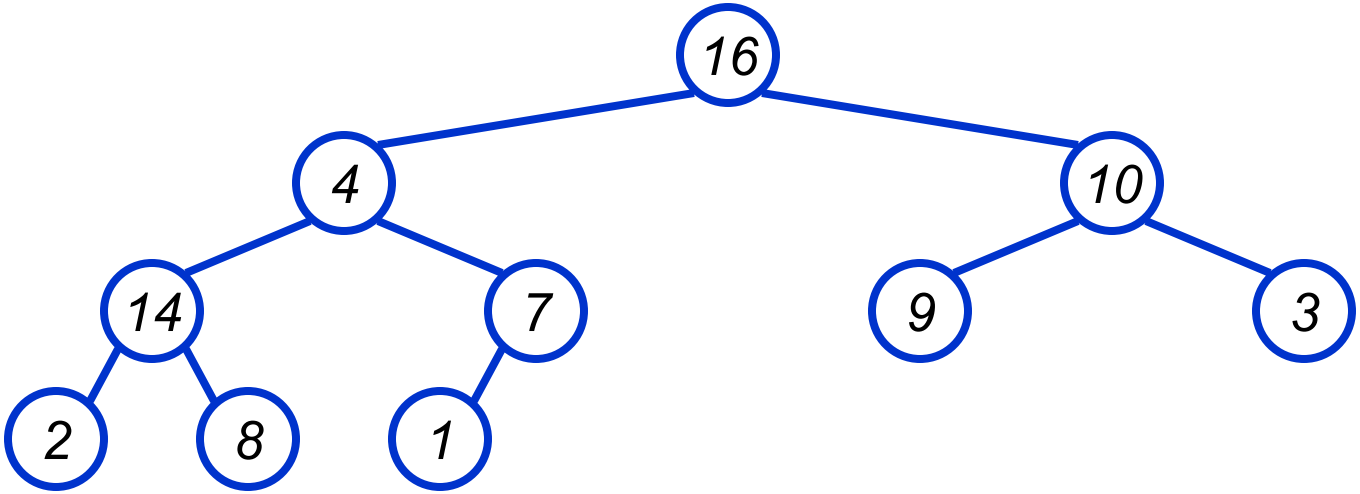
Heap Operations: Heapify()

- **`Heapify()`**: maintain the heap property
  - Given: a node *i* in the heap with children *l* and *r*
  - Given: two subtrees rooted at *l* and *r*, assumed to be heaps
  - Problem: The subtree rooted at *i* may violate the heap property (*How?*)
  - Action: let the value of the parent node "float down" so subtree at *i* satisfies the heap property
    - *What do you suppose will be the basic operation between i, l, and r?*

Heap Operations: Heapify()

```
Heapify(A, i)
{
  l = Left(i); r = Right(i);
  if (l <= heap_size(A) && A[l] > A[i])
      largest = l;
  else
      largest = i;
  if (r <= heap_size(A) && A[r] > A[largest])
      largest = r;
  if (largest != i)
      Swap(A, i, largest);
      Heapify(A, largest);
}
```

Heapify() Example



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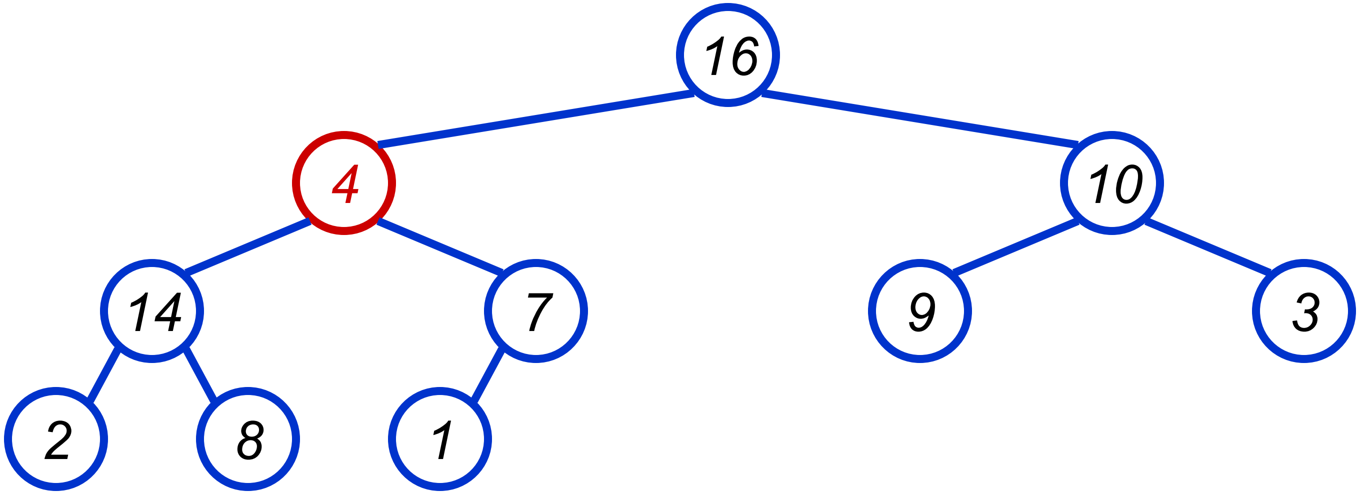

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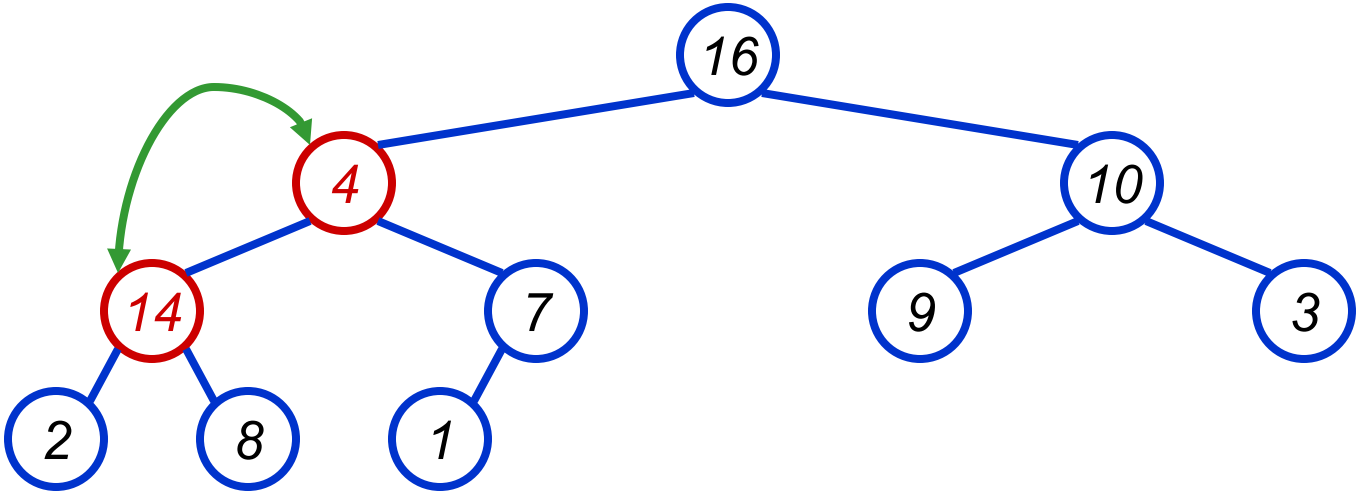

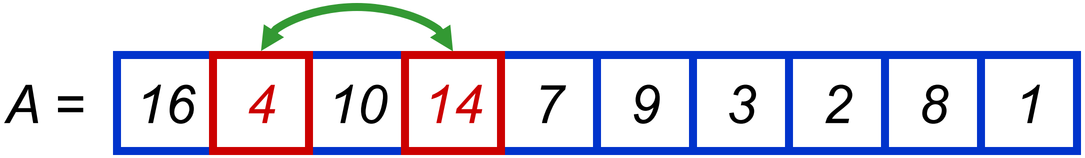

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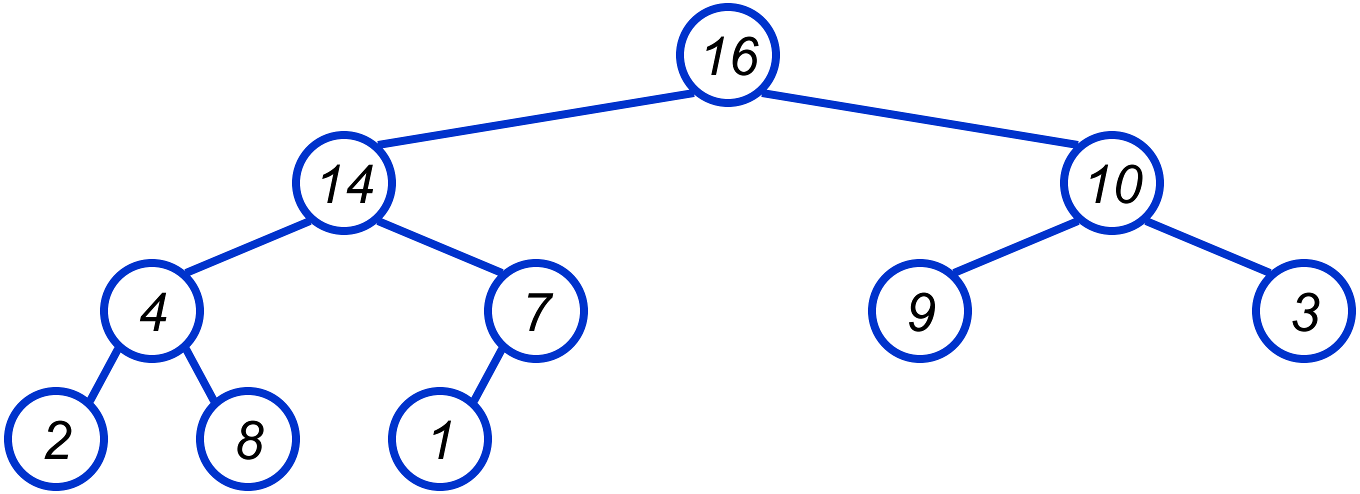

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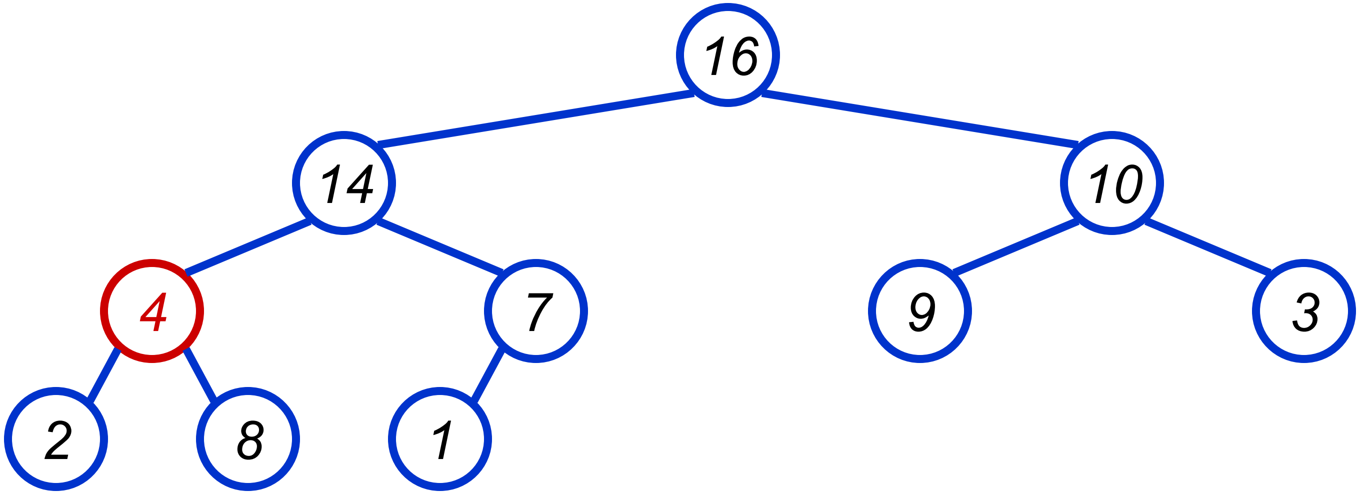

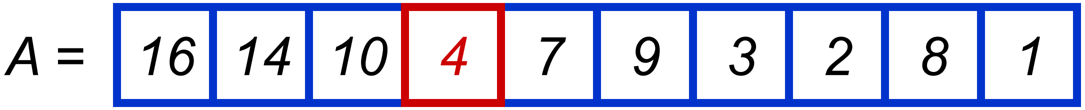

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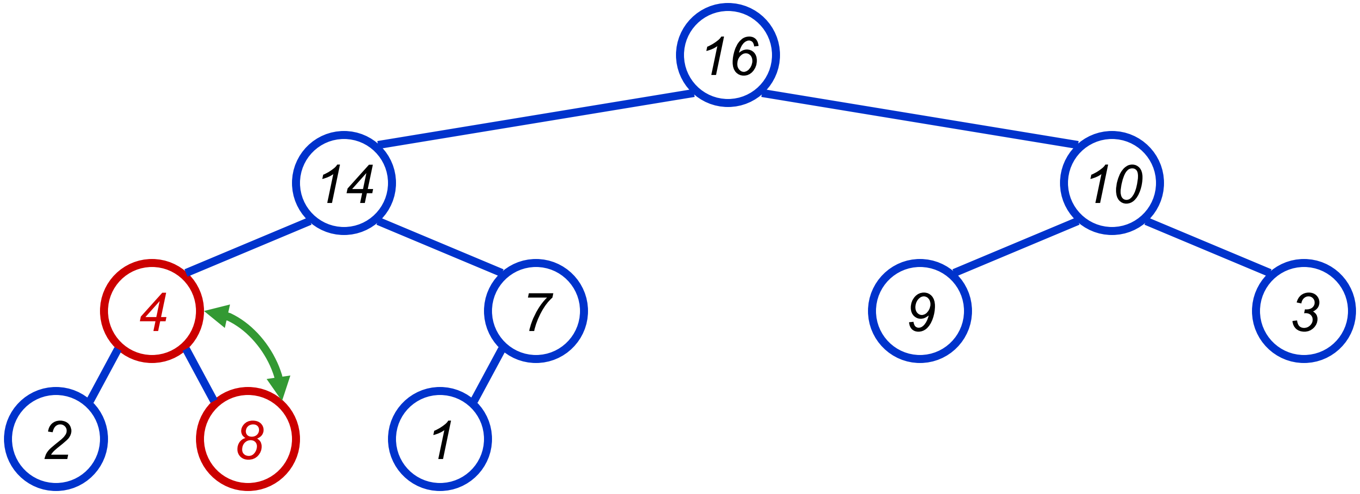

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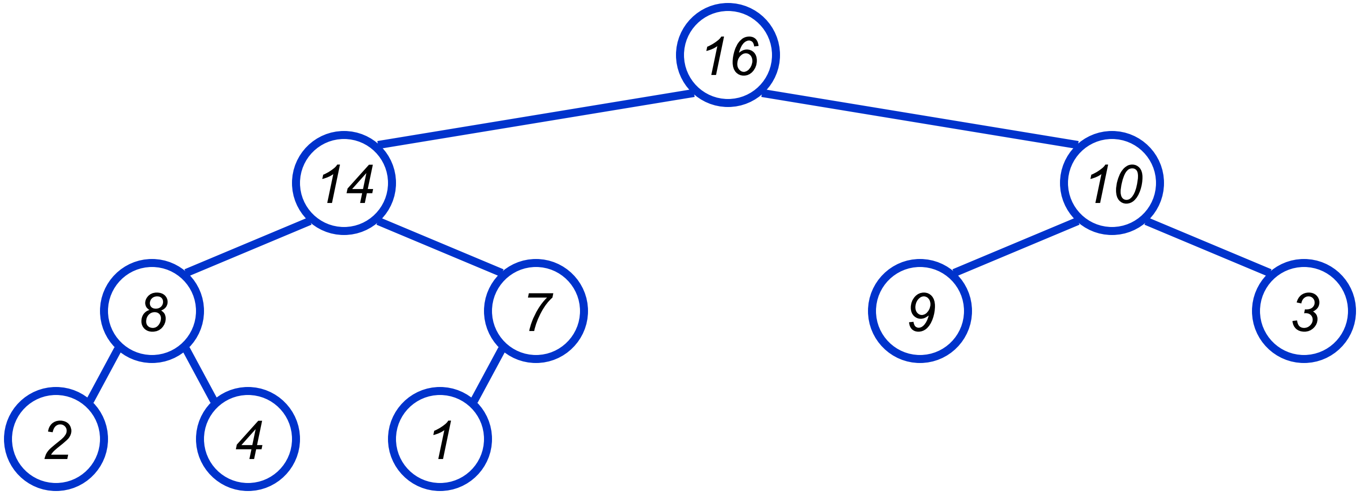

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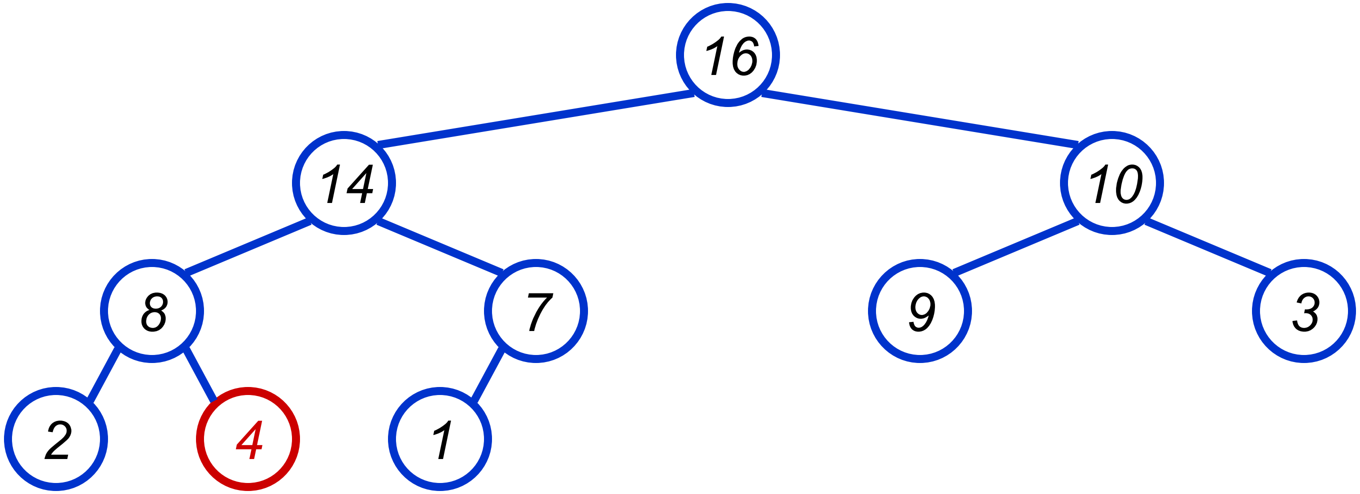

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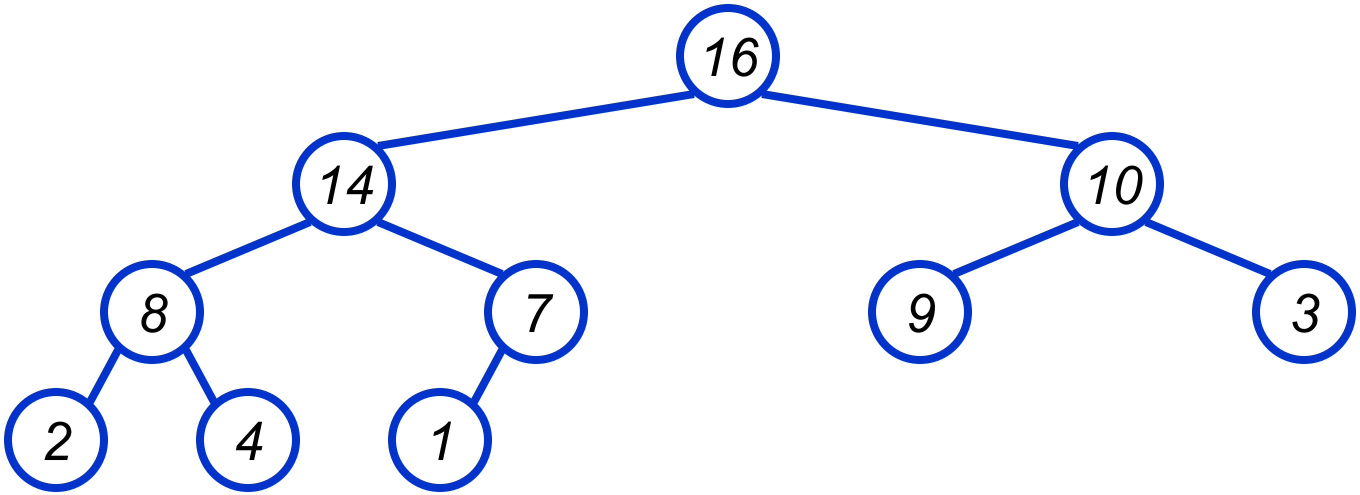

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Analyzing Heapify(): Informal

- *Aside from the recursive call, what is the running time of* **`Heapify()`***?*
- *How many times can* **`Heapify()`** *recursively call itself?*
- *What is the worst-case running time of* **`Heapify()`** *on a heap of size n?*

Analyzing Heapify(): Formal

- Fixing up relationships between i, l, and r takes Θ(1) time
- *If the heap at i has n elements, how many elements can the subtrees at l or r have?*
    - Draw it
- Answer: $2n/3$ (worst case: bottom row 1/2 full)
- So time taken by **`Heapify()`** is given by

$$T(n) \leq T(2n/3) + \Theta(1)$$

Analyzing Heapify(): Formal

- So we have

$$T(n) \le T(2n/3) + \Theta(1)$$

- By case 2 of the Master Theorem,

$$T(n) = O(\lg n)$$

- Thus, **`Heapify()`** takes logarithmic time

Heap Operations: BuildHeap()

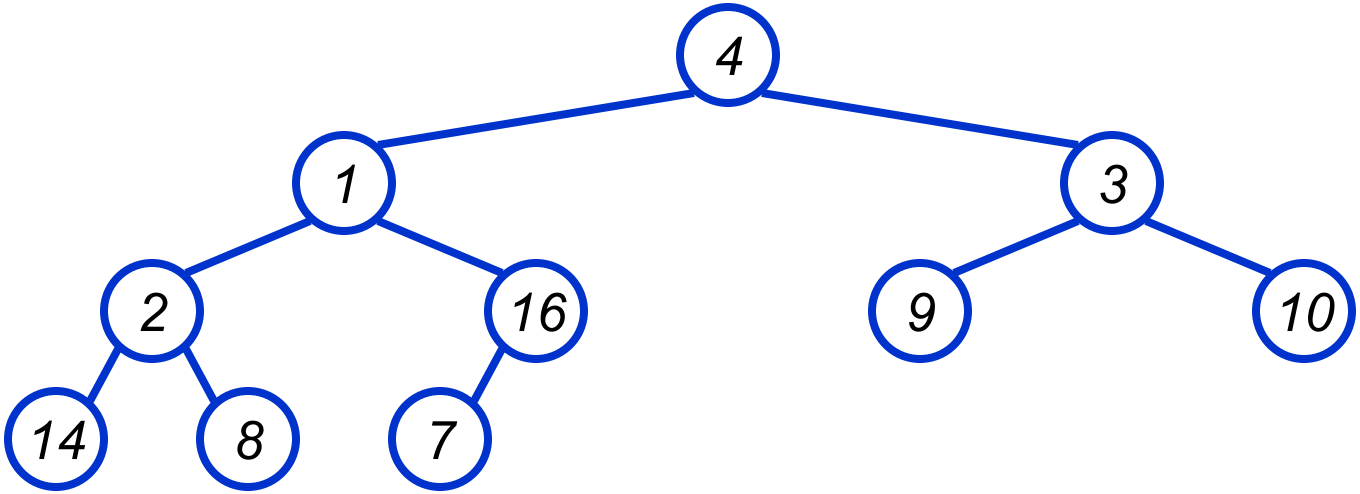
- We can build a heap in a bottom-up manner by running **`Heapify()`** on successive subarrays
  - Fact: for array of length *n*, all elements in range A[$\lfloor n/2 \rfloor$ + 1 .. n] are heaps (*Why?*)
  - So:
    - Walk backwards through the array from n/2 to 1, calling **`Heapify()`** on each node.
    - Order of processing guarantees that the children of node *i* are heaps when *i* is processed

BuildHeap()

```
// given an unsorted array A, make A a heap
BuildHeap(A)
{
  heap_size(A) = length(A);
  for (i = $\lfloor length[A]/2 \rfloor$ downto 1)
      Heapify(A, i);
}
```

BuildHeap() Example

- Work through example
  A = {4, 1, 3, 2, 16, 9, 10, 14, 8, 7}



Analyzing BuildHeap()

- Each call to **`Heapify()`** takes O(lg n) time
- There are O(n) such calls (specifically, $\lfloor n/2 \rfloor$)
- Thus the running time is O(n lg n)
    - *Is this a correct asymptotic upper bound?*
    - *Is this an asymptotically tight bound?*
- A tighter bound is O(n)
    - *How can this be? Is there a flaw in the above reasoning?*

Analyzing BuildHeap(): Tight

- To **`Heapify()`** a subtree takes O(h) time where h is the height of the subtree
  - $h = O(\lg m)$, m = # nodes in subtree
  - The height of most subtrees is small
- Fact: an n-element heap has at most $\lceil n/2^{h+1} \rceil$ nodes of height h
- CLR 7.3 uses this fact to prove that **`BuildHeap()`** takes O(n) time

Heapsort

- Given **`BuildHeap()`**, an in-place sorting algorithm is easily constructed:
  - Maximum element is at A[1]
  - Discard by swapping with element at A[n]
    - Decrement heap_size[A]
    - A[n] now contains correct value
  - Restore heap property at A[1] by calling **`Heapify()`**
  - Repeat, always swapping A[1] for A[heap_size(A)]

Heapsort

```
Heapsort(A)
{
      BuildHeap(A);
      for (i = length(A) downto 2)
      {
            Swap(A[1], A[i]);
            heap_size(A) -= 1;
            Heapify(A, 1);
      }
}
```

Analyzing Heapsort

- The call to **`BuildHeap()`** takes $O(n)$ time
- Each of the $n - 1$ calls to **`Heapify()`** takes $O(\lg n)$ time
- Thus the total time taken by **`HeapSort()`**
  $= O(n) + (n - 1)\, O(\lg n)$
  $= O(n) + O(n \lg n)$
  $= O(n \lg n)$

Priority Queues

- Heapsort is a nice algorithm, but in practice Quicksort (coming up) usually wins
- But the heap data structure is incredibly useful for implementing *priority queues*
  - A data structure for maintaining a set *S* of elements, each with an associated value or *key*
  - Supports the operations **`Insert()`**, **`Maximum()`**, and **`ExtractMax()`**
  - ***What might a priority queue be useful for?***

Priority Queue Operations

- **Insert(S, x)** inserts the element x into set S
- **Maximum(S)** returns the element of S with the maximum key
- **ExtractMax(S)** removes and returns the element of S with the maximum key
- *How could we implement these operations using a heap?*