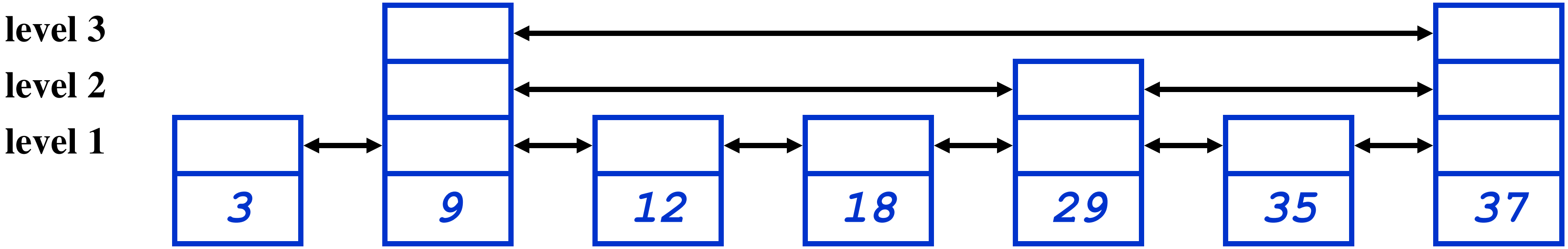
Algorithms

Hash Tables

Review: Skip Lists

- The basic idea:



- Keep a doubly-linked list of elements
  - Min, max, successor, predecessor: O(1) time
  - Delete is O(1) time, Insert is O(1)+Search time
- During insert, add each level-i element to level i+1 with probability p (e.g., p = 1/2 or p = 1/4)

Summary: Skip Lists

- O(1) expected time for most operations
- O(lg n) expected time for insert
- O(n^2) time worst case
  - But random, so no particular order of insertion evokes worst-case behavior
- O(n) expected storage requirements
- Easy to code

Review: Hash Tables

- Hash table:
  - Given a table T and a record x, with key (= symbol) and satellite data, we need to support:
    - Insert (T, x)
    - Delete (T, x)
    - Search(T, x)
  - We want these to be fast, but don't care about sorting the records
  - In this discussion we consider all keys to be (possibly large) natural numbers

Review: Direct Addressing

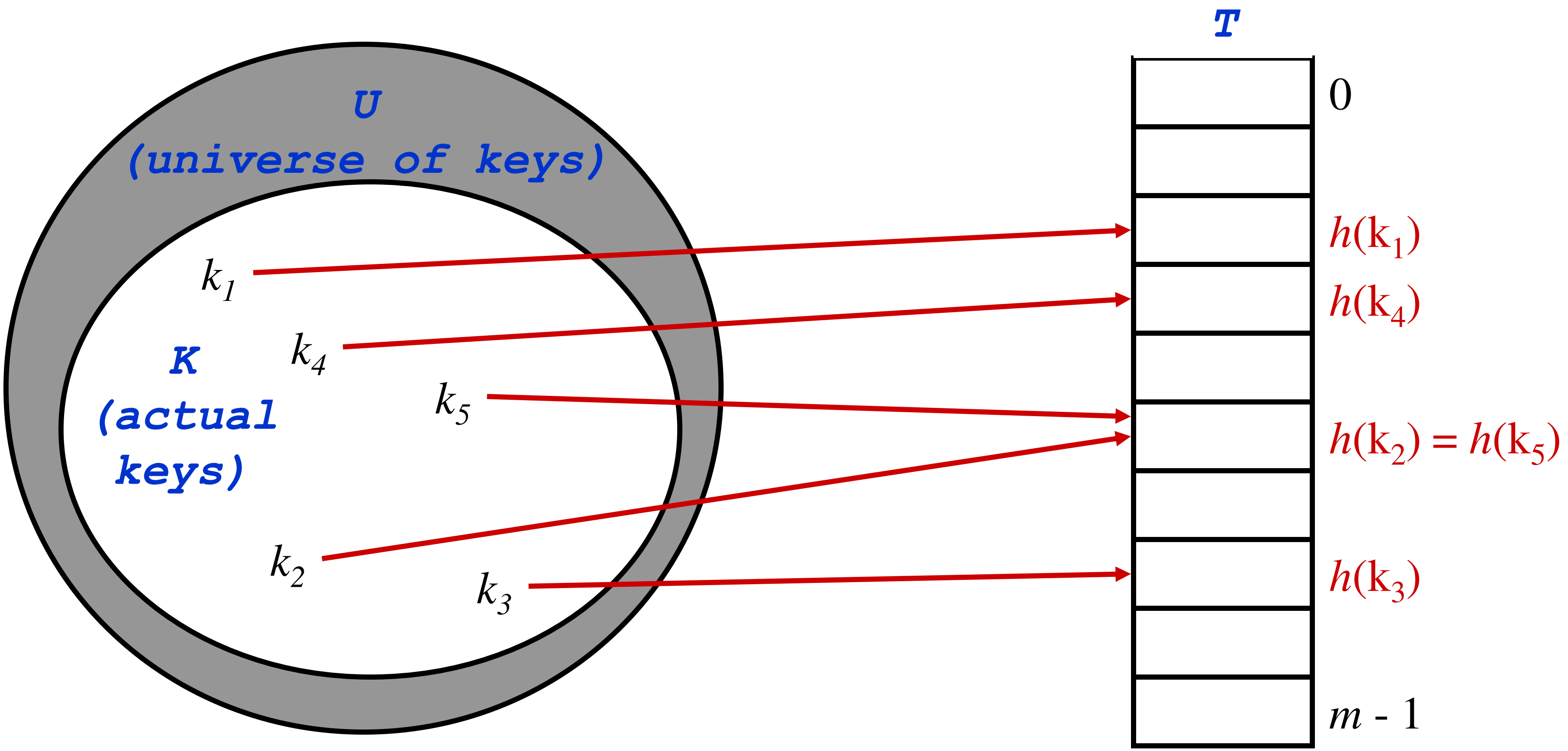
- Suppose:
  - The range of keys is 0..m-1
  - Keys are distinct
- The idea:
  - Set up an array T[0..m-1] in which
    - T[i] = x if $x \in T$ and key[x] = i
    - T[i] = NULL otherwise
  - This is called a *direct-address table*
    - Operations take O(1) time!

Review: The Problem With Direct Addressing

- Direct addressing works well when the range *m* of keys is relatively small
- But what if the keys are 32-bit integers?
  - Problem 1: direct-address table will have 2^{32} entries, more than 4 billion
  - Problem 2: even if memory is not an issue, the time to initialize the elements to NULL may be
- Solution: map keys to smaller range 0..*m*-1
- This mapping is called a *hash function*

Hash Functions

- Next problem: *collision*



Resolving Collisions

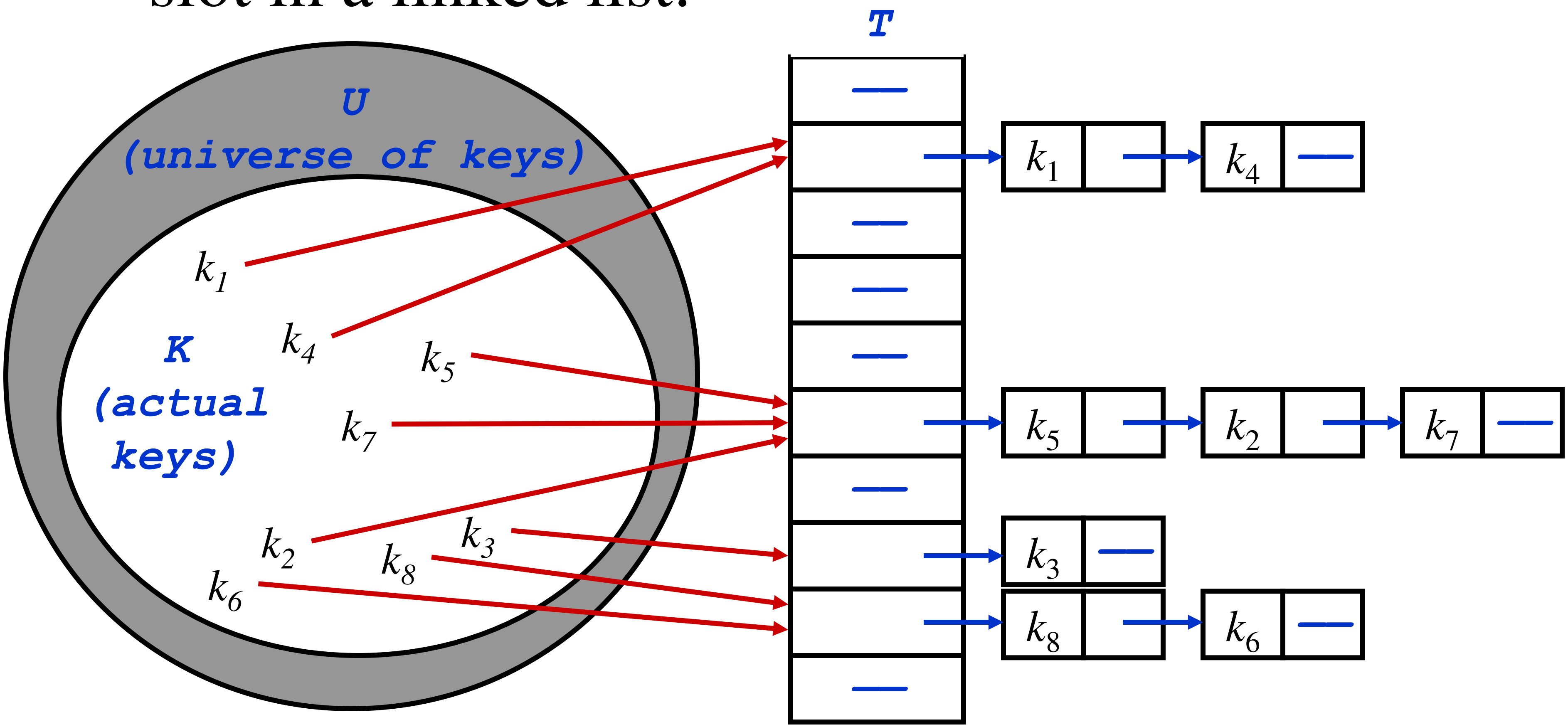
- *How can we solve the problem of collisions?*
- Solution 1: *chaining*
- Solution 2: *open addressing*

Open Addressing

- Basic idea (details in Section 12.4):
  - To insert: if slot is full, try another slot, …, until an open slot is found (*probing*)
  - To search, follow same sequence of probes as would be used when inserting the element
    - If reach element with correct key, return it
    - If reach a NULL pointer, element is not in table
- Good for fixed sets (adding but no deletion)
  - Example: spell checking
- Table needn't be much bigger than n

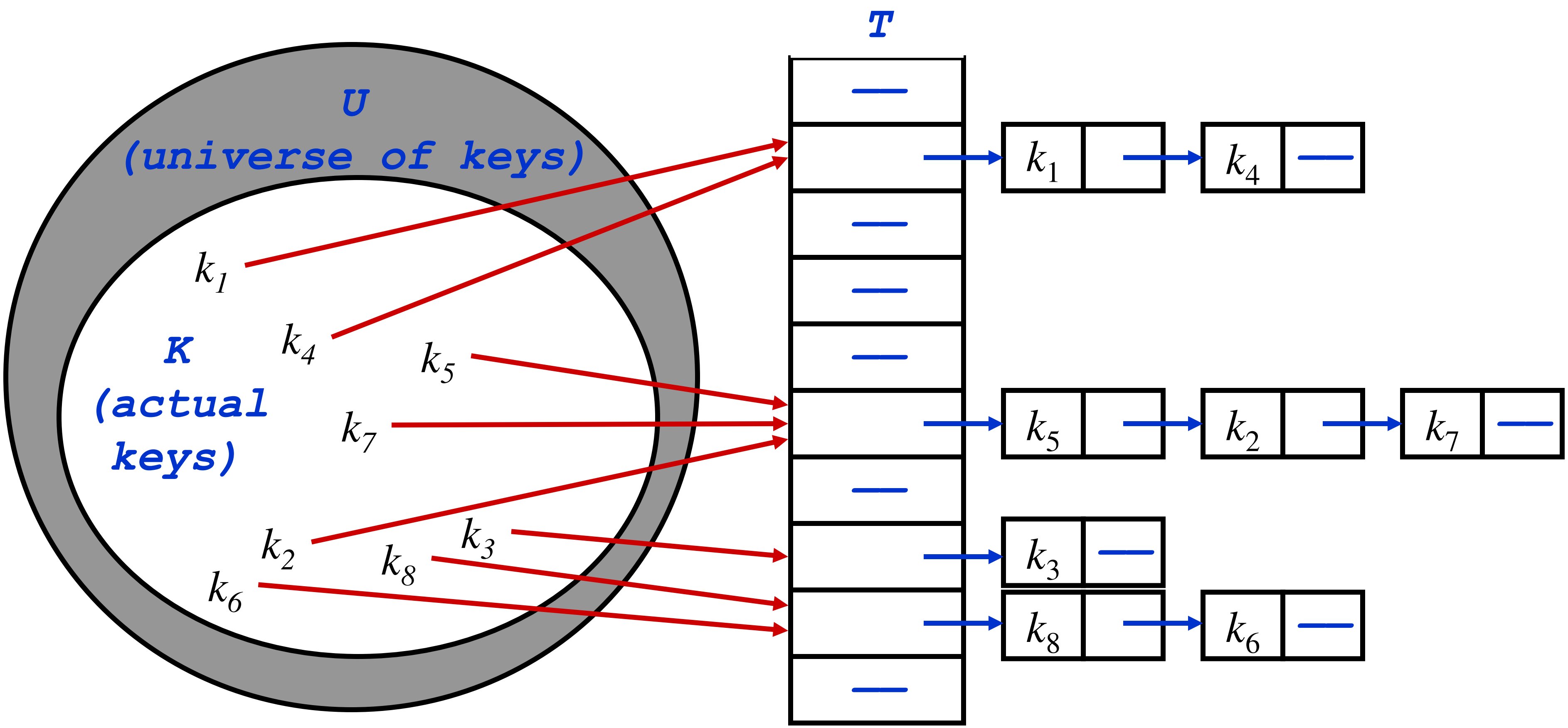
Chaining

- Chaining puts elements that hash to the same slot in a linked list:



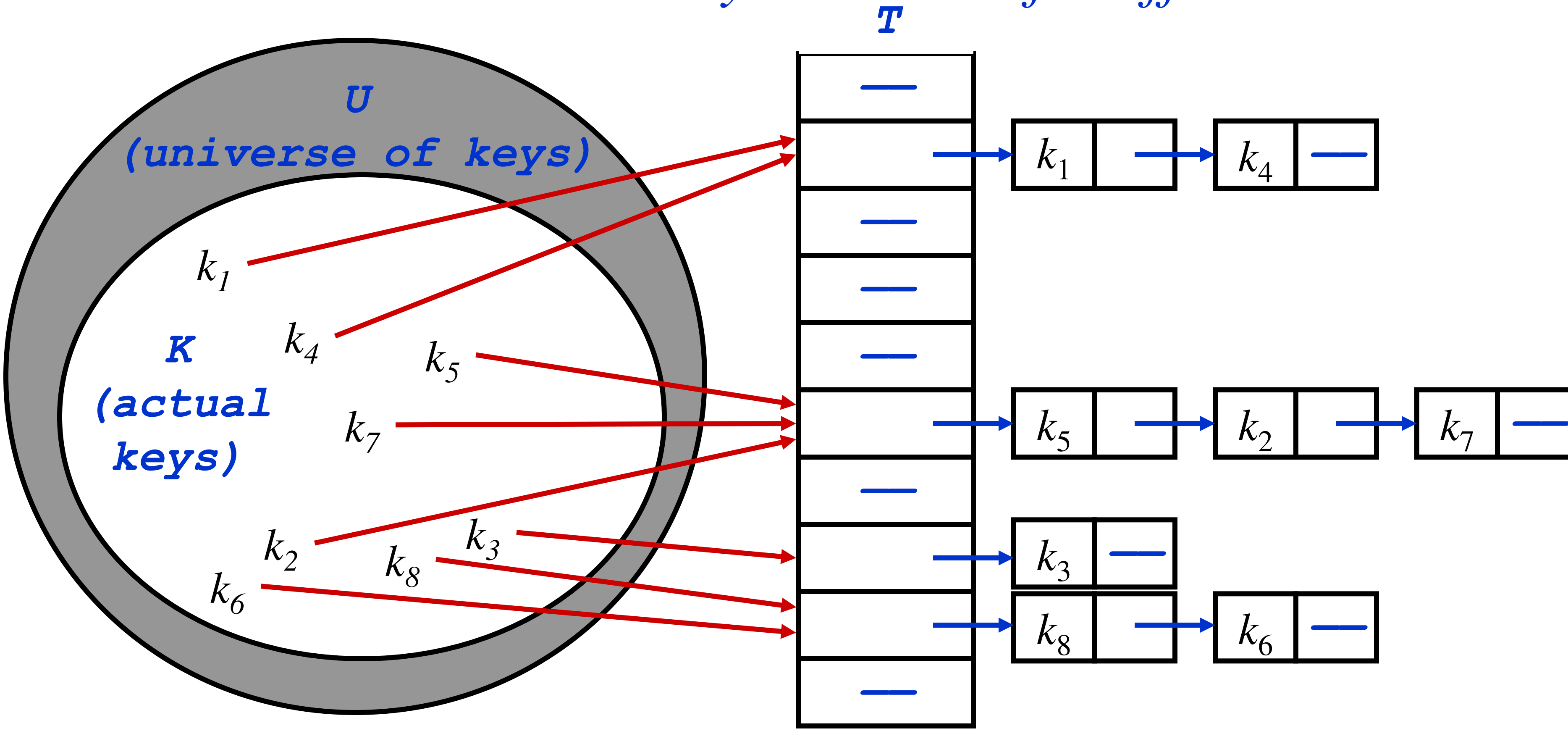
Chaining

- *How do we insert an element?*



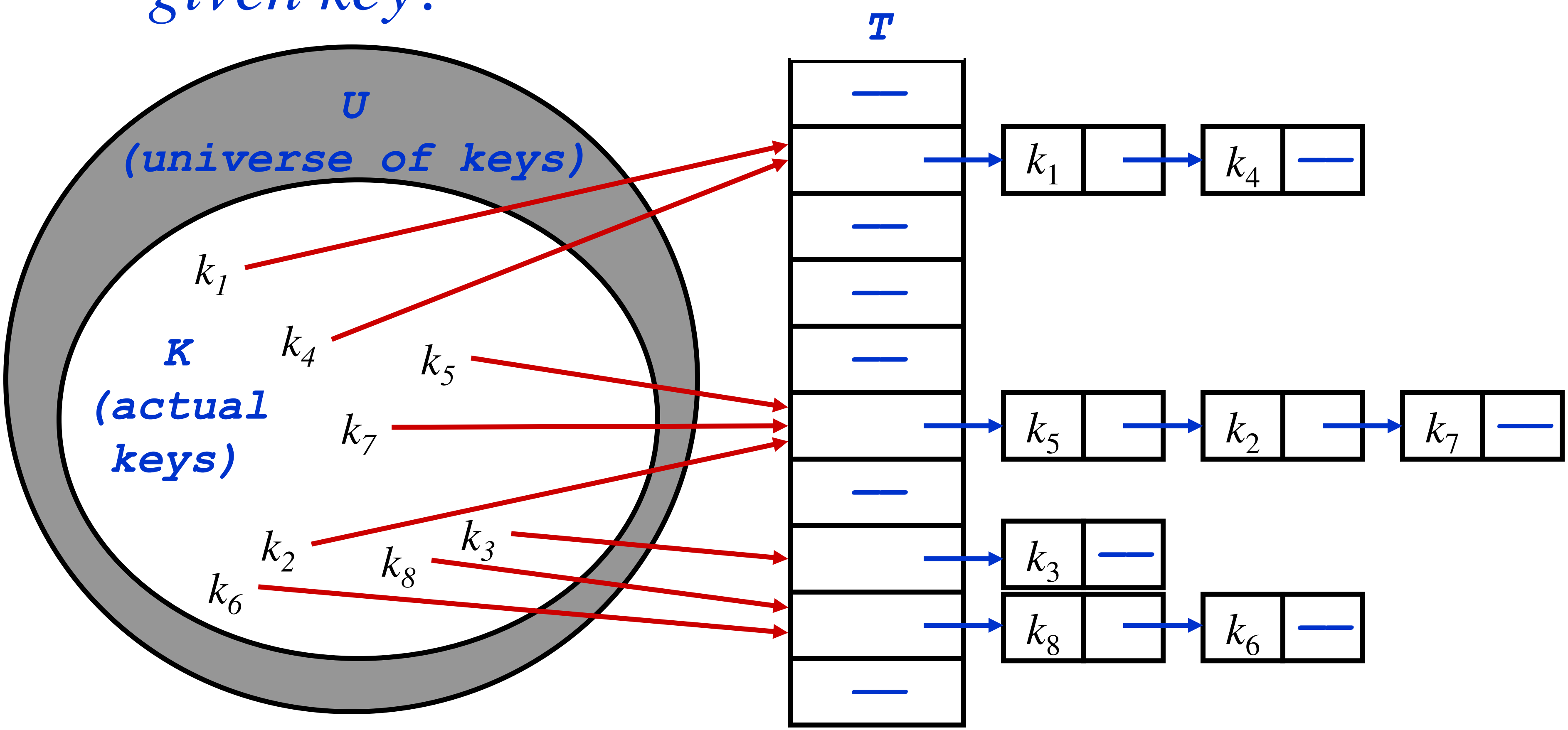
Chaining

- *How do we delete an element?*
  - *Do we need a doubly-linked list for efficient delete?*



Chaining

- *How do we search for a element with a given key?*



Analysis of Chaining

- Assume *simple uniform hashing*: each key in table is equally likely to be hashed to any slot
- Given n keys and m slots in the table: the *load factor* $\alpha = n/m$ = average # keys per slot
- *What will be the average cost of an unsuccessful search for a key?*

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- *What will be the average cost of a successful search?* A: $O(1 + \alpha/2) = O(1 + \alpha)$

Analysis of Chaining Continued

- So the cost of searching = O(1 + α)
- *If the number of keys n is proportional to the number of slots in the table, what is α?*
- A: α = O(1)
  - In other words, we can make the expected cost of searching constant if we make α constant

Choosing A Hash Function

- Clearly choosing the hash function well is crucial
  - *What will a worst-case hash function do?*
  - *What will be the time to search in this case?*
- *What are desirable features of the hash function?*
  - Should distribute keys uniformly into slots
  - Should not depend on patterns in the data

Hash Functions: The Division Method

- $h(k) = k \bmod m$
  - In words: hash k into a table with m slots using the slot given by the remainder of k divided by m
- *What happens to elements with adjacent values of k?*
- *What happens if m is a power of 2 (say 2^P)?*
- *What if m is a power of 10?*
- Upshot: pick table size m = prime number not too close to a power of 2 (or 10)

Hash Functions: The Multiplication Method

- For a constant A, $0 < A < 1$:
- $h(k) = \left\lfloor m \underbrace{(kA - \lfloor kA \rfloor)}_{\textit{What does this term represent?}} \right\rfloor$

Hash Functions: The Multiplication Method

- For a constant A, $0 < A < 1$:
- $h(k) = \lfloor m \underbrace{(kA - \lfloor kA \rfloor)}_{\textit{Fractional part of } kA} \rfloor$

- Choose $m = 2^P$
- Choose A not too close to 0 or 1
- Knuth: Good choice for $A = (\sqrt{5} - 1)/2$

Hash Functions: Worst Case Scenario

- Scenario:
  - You are given an assignment to implement hashing
  - You will self-grade in pairs, testing and grading your partner's implementation
  - In a blatant violation of the honor code, your partner:
    - Analyzes your hash function
    - Picks a sequence of "worst-case" keys, causing your implementation to take $O(n)$ time to search
- *What's an honest CS student to do?*

Hash Functions: Universal Hashing

- As before, when attempting to foil an malicious adversary: randomize the algorithm
- *Universal hashing*: pick a hash function randomly in a way that is independent of the keys that are actually going to be stored
  - Guarantees good performance on average, no matter what keys adversary chooses