Algorithms

Topological Sort

Minimum Spanning Trees

Review: Breadth-First Search

```
BFS(G, s) {
    initialize vertices;
    Q = {s};          // Q is a queue (duh); initialize to s
    while (Q not empty) {
        u = RemoveTop(Q);
        for each v ∈ u->adj {
            if (v->color == WHITE)
                v->color = GREY;
                v->d = u->d + 1;
                v->p = u;
                Enqueue(Q, v);
        }
        u->color = BLACK;
    }
}
```

`v->d` *represents depth in tree*

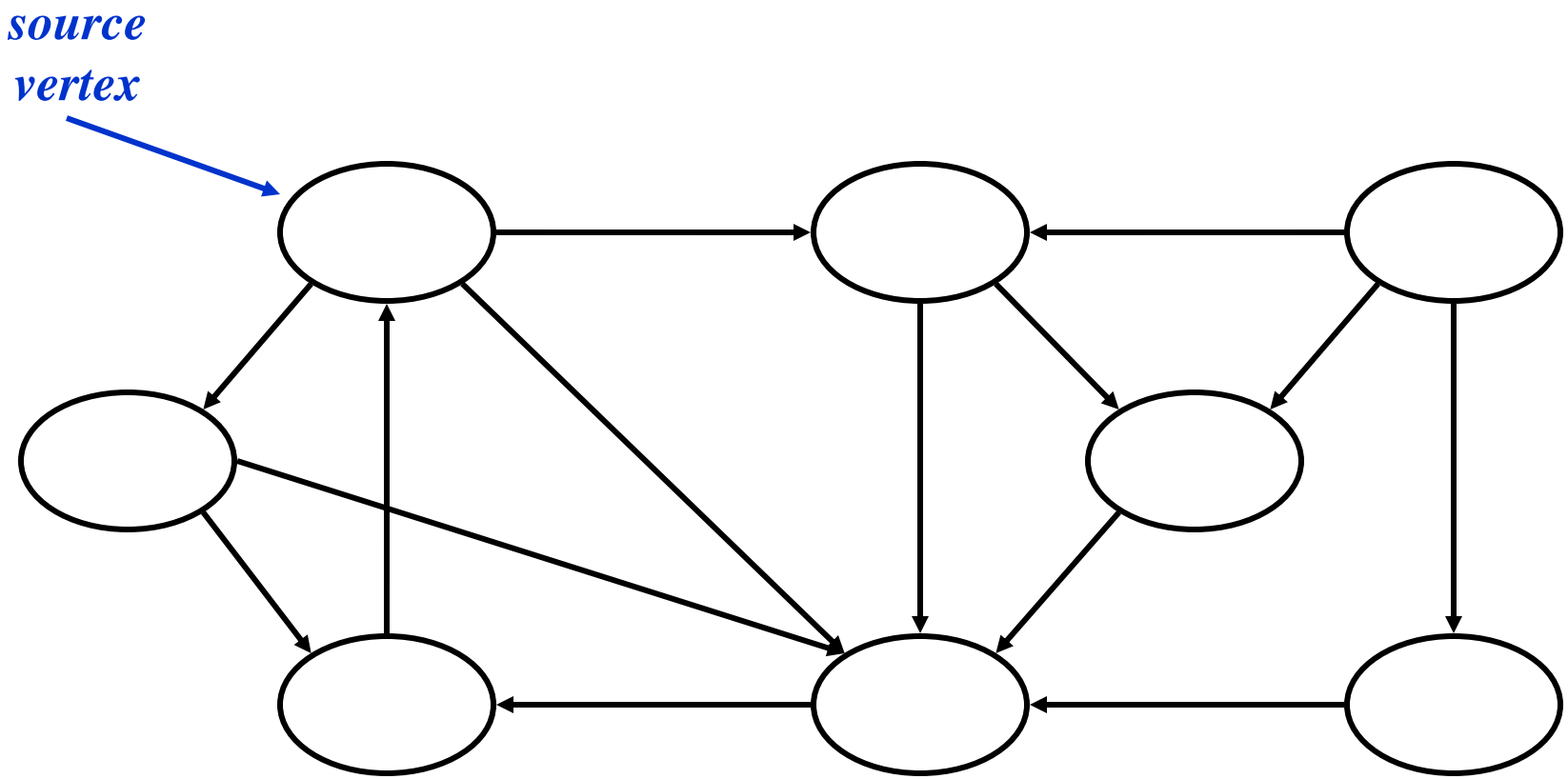
`v->p` *represents parent in tree*

Review: DFS Code

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

```
DFS_Visit(u)
{
    u->color = YELLOW;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

Review: DFS Example



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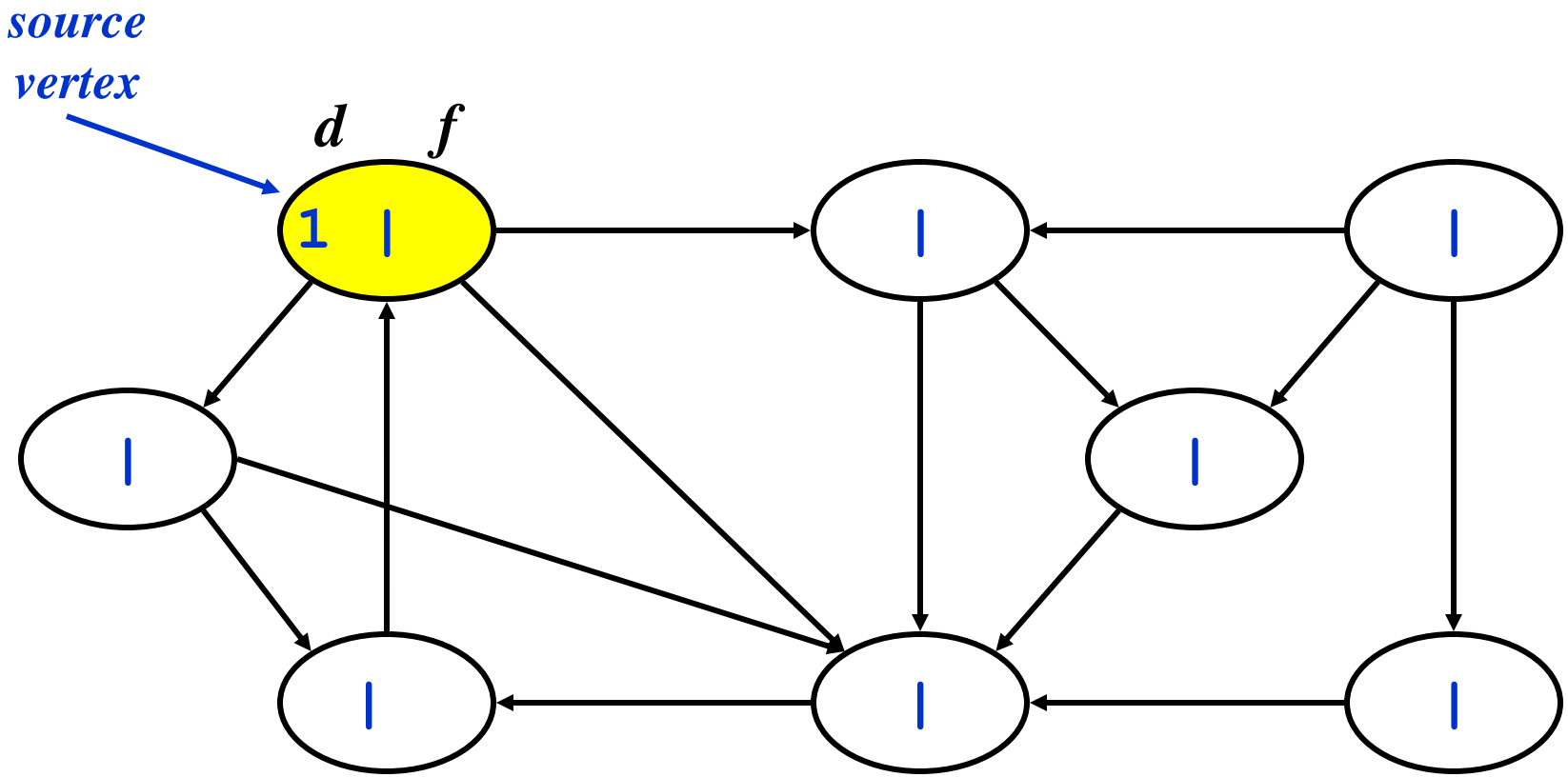

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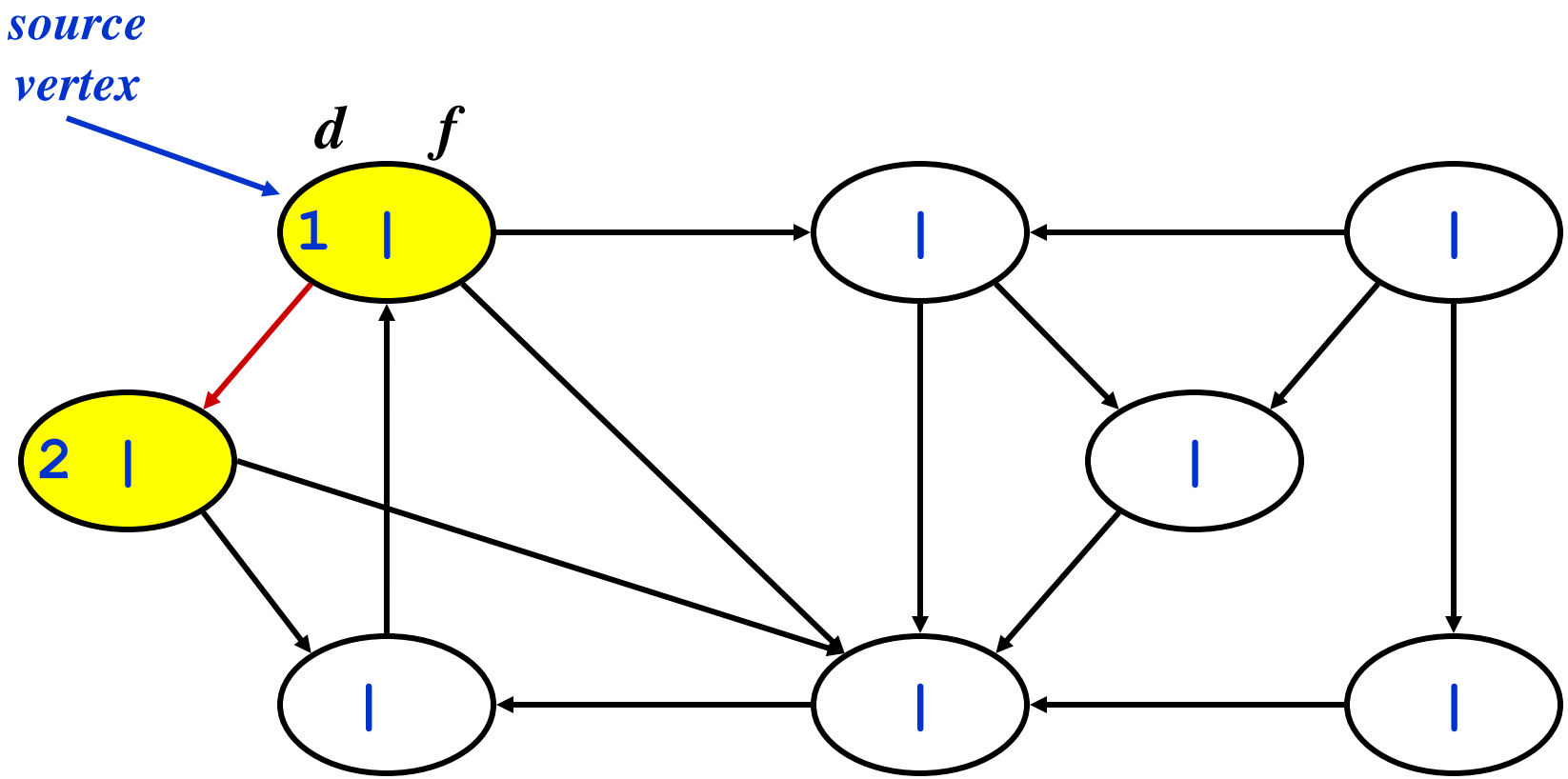

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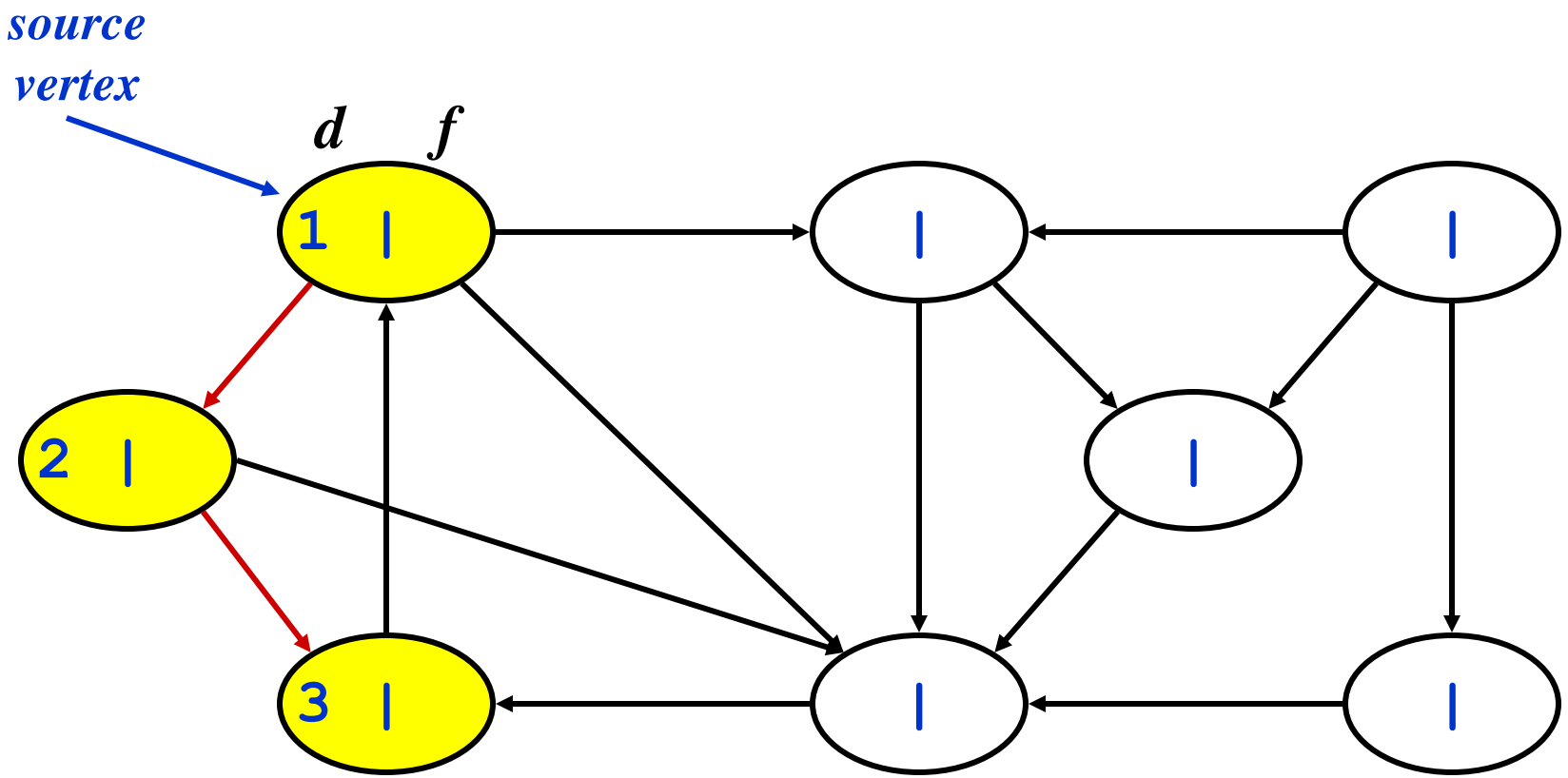

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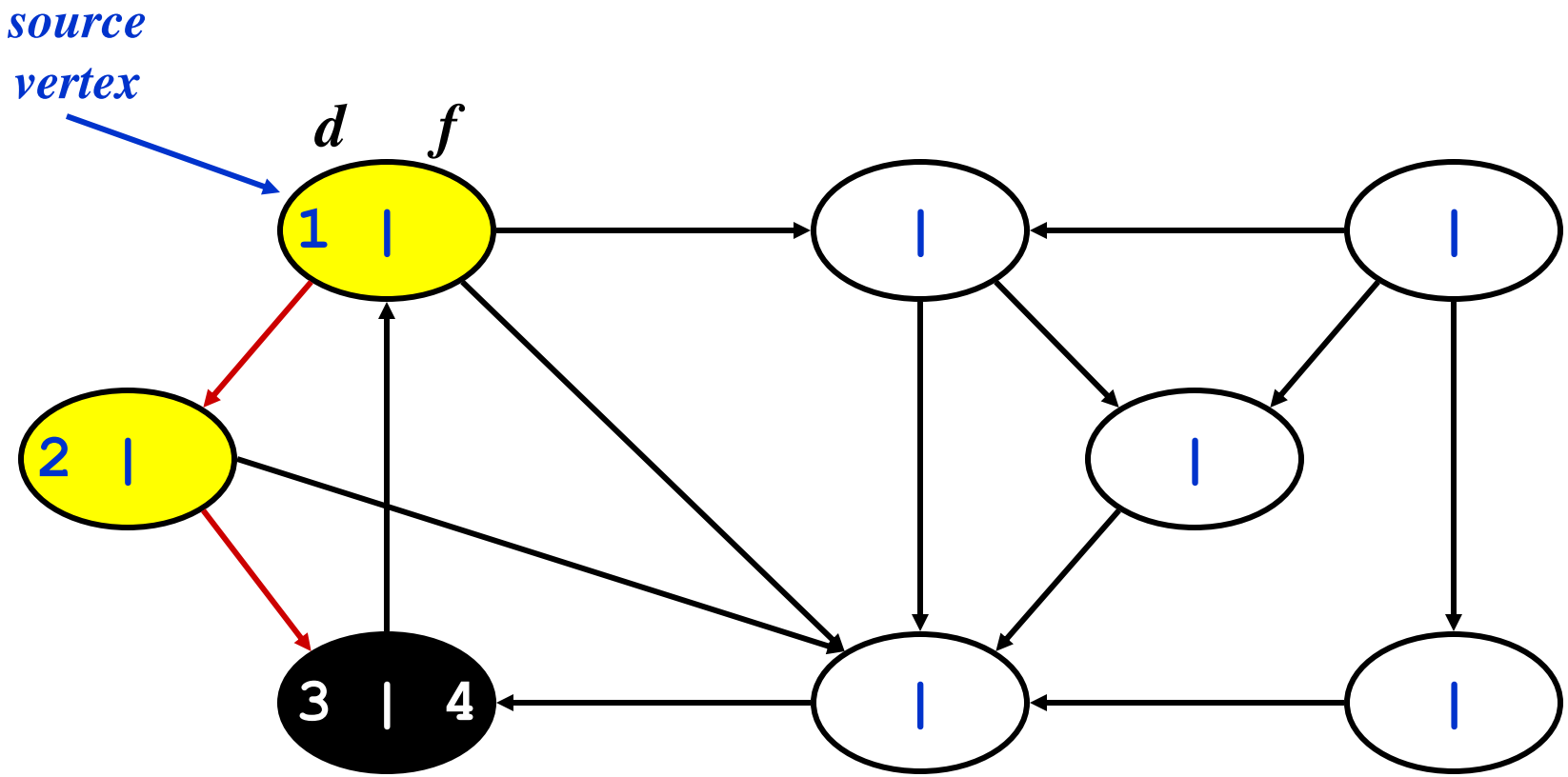

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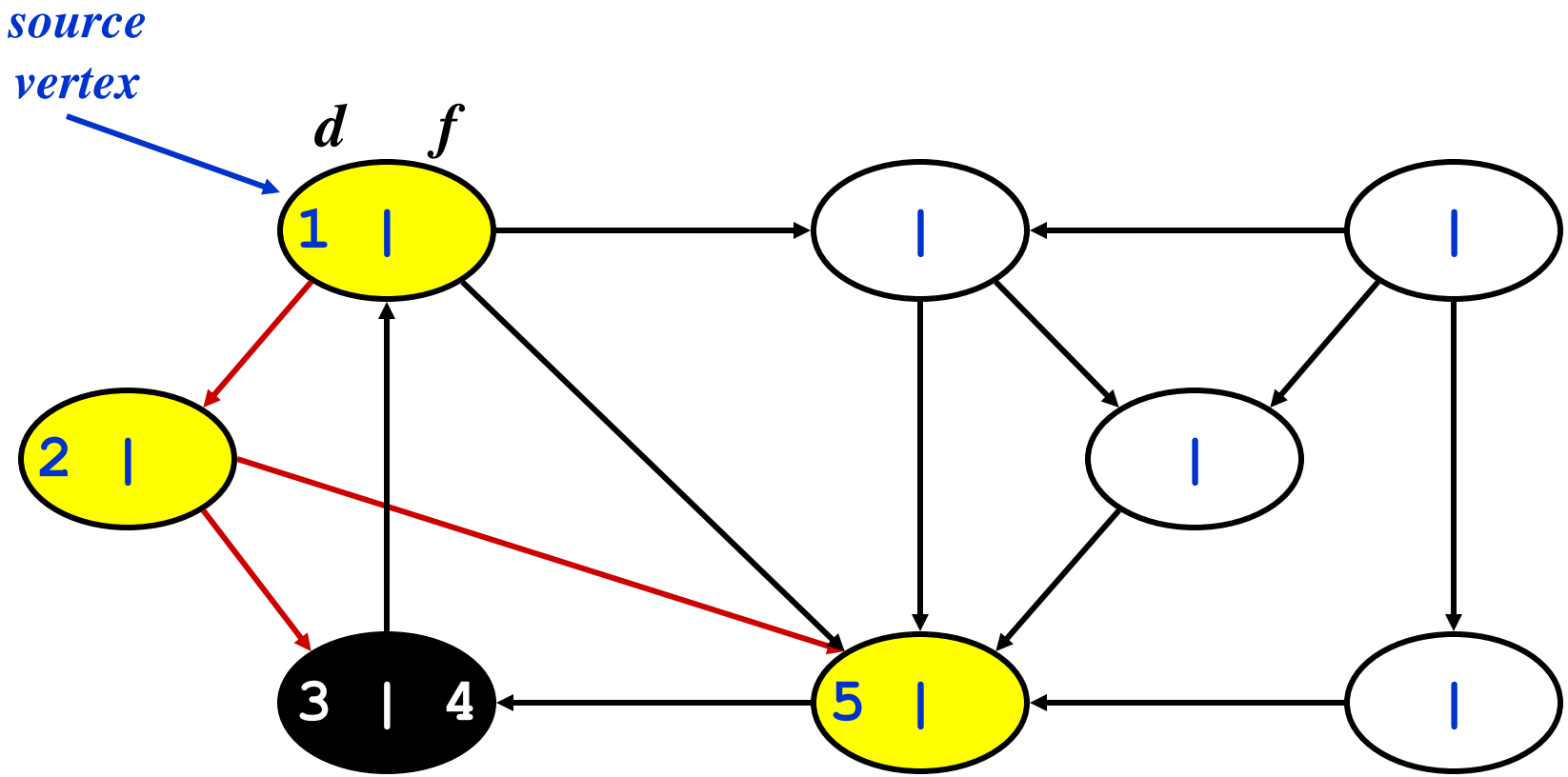

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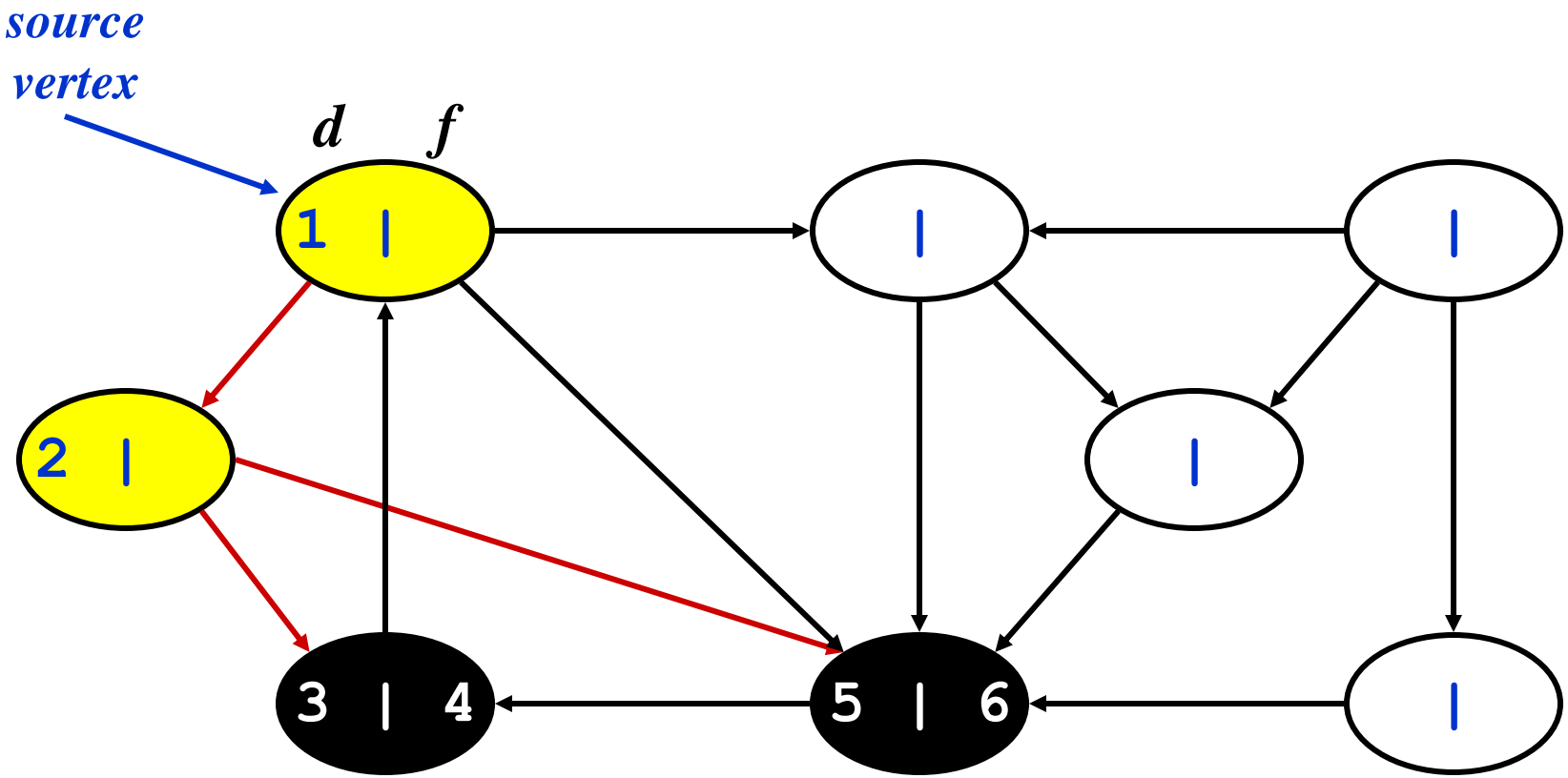

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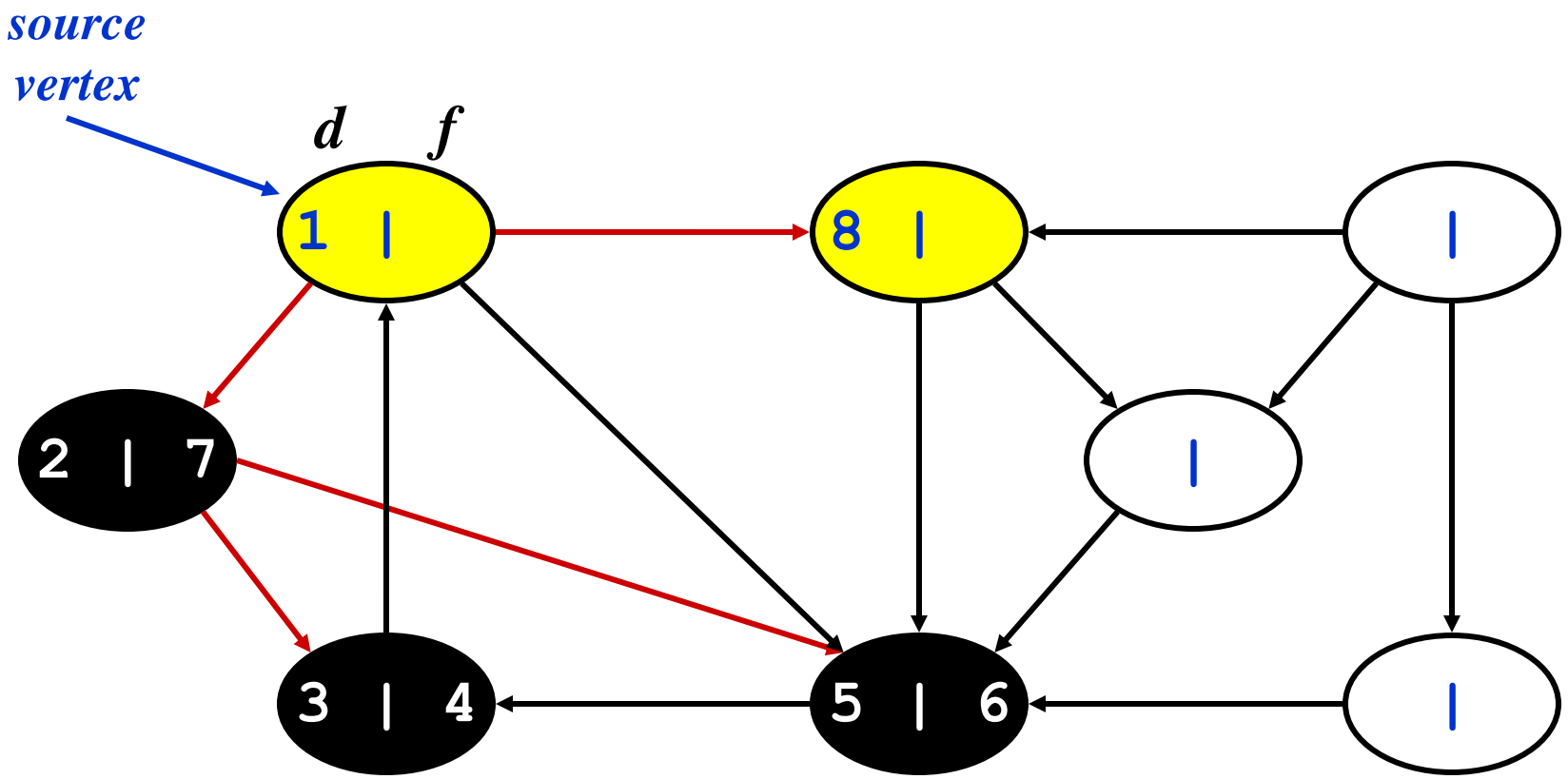

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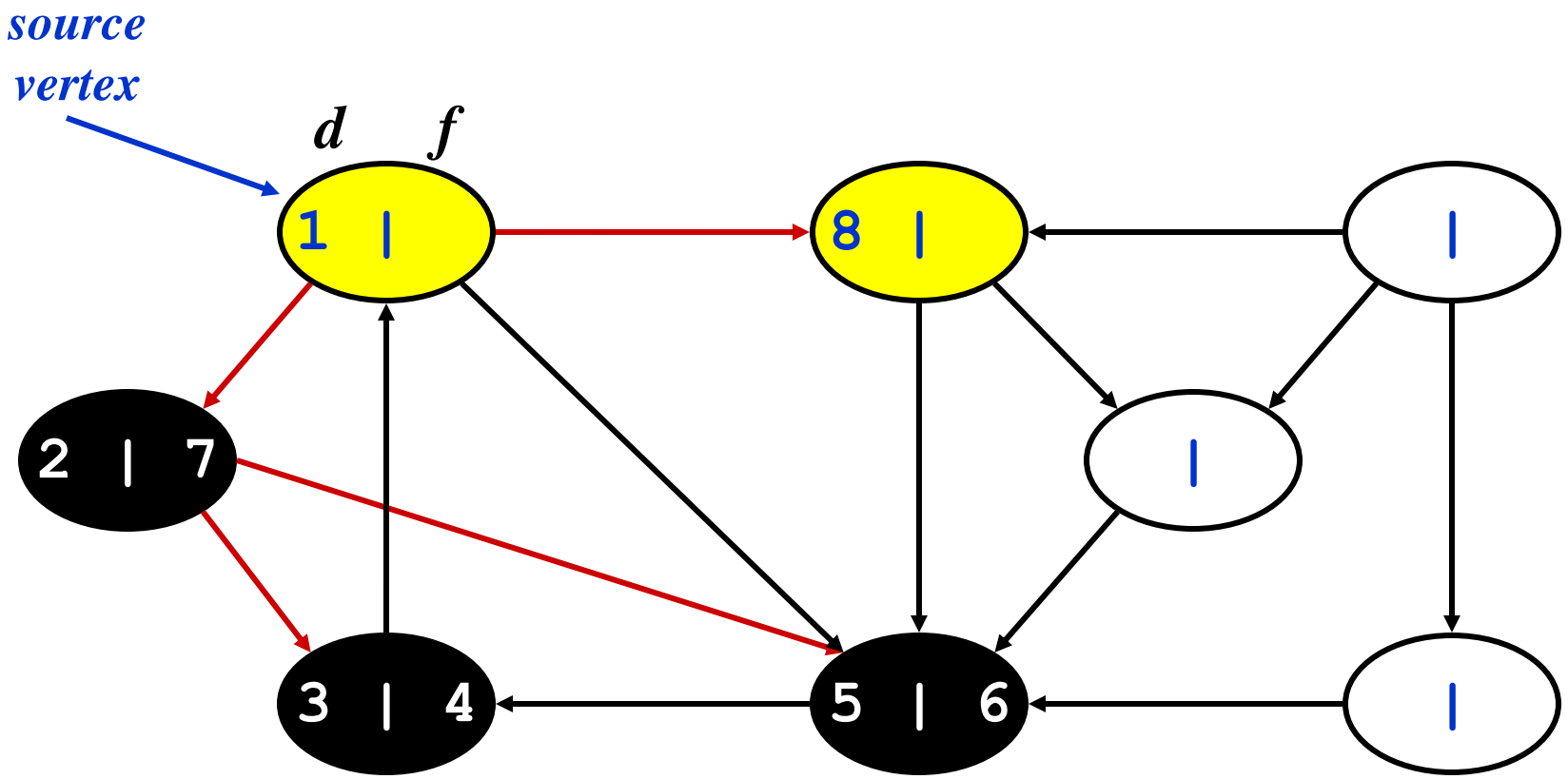

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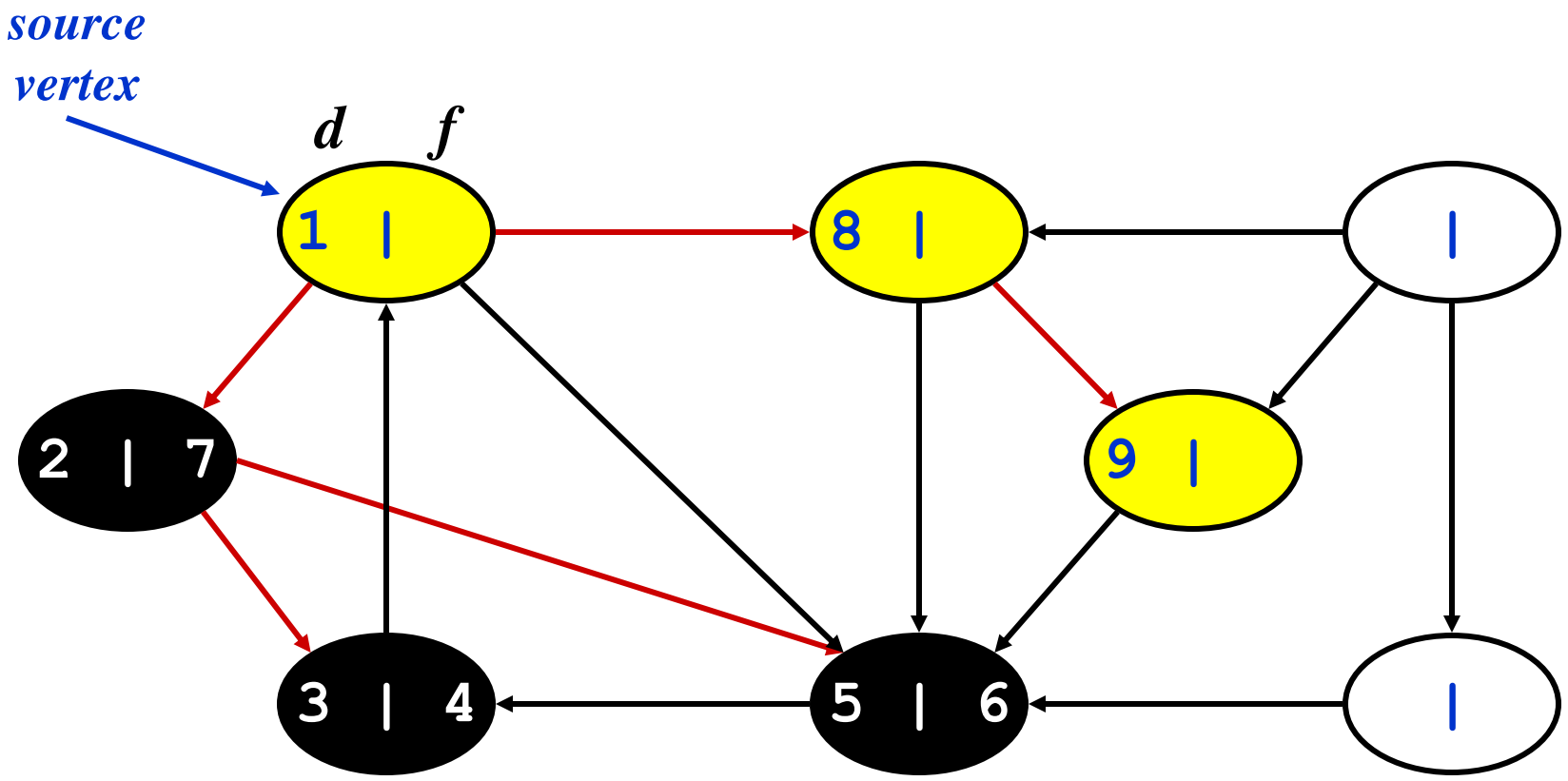

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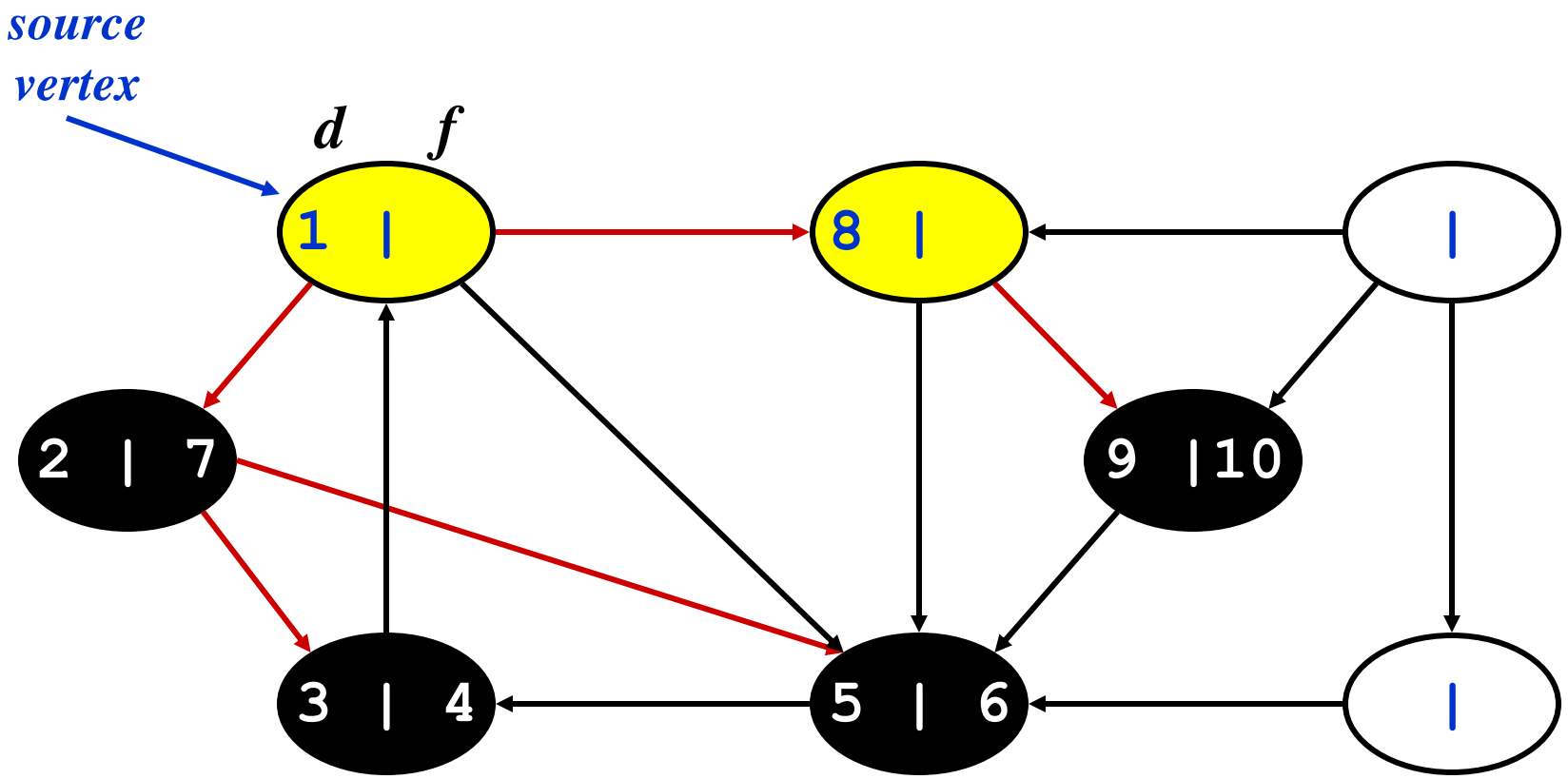

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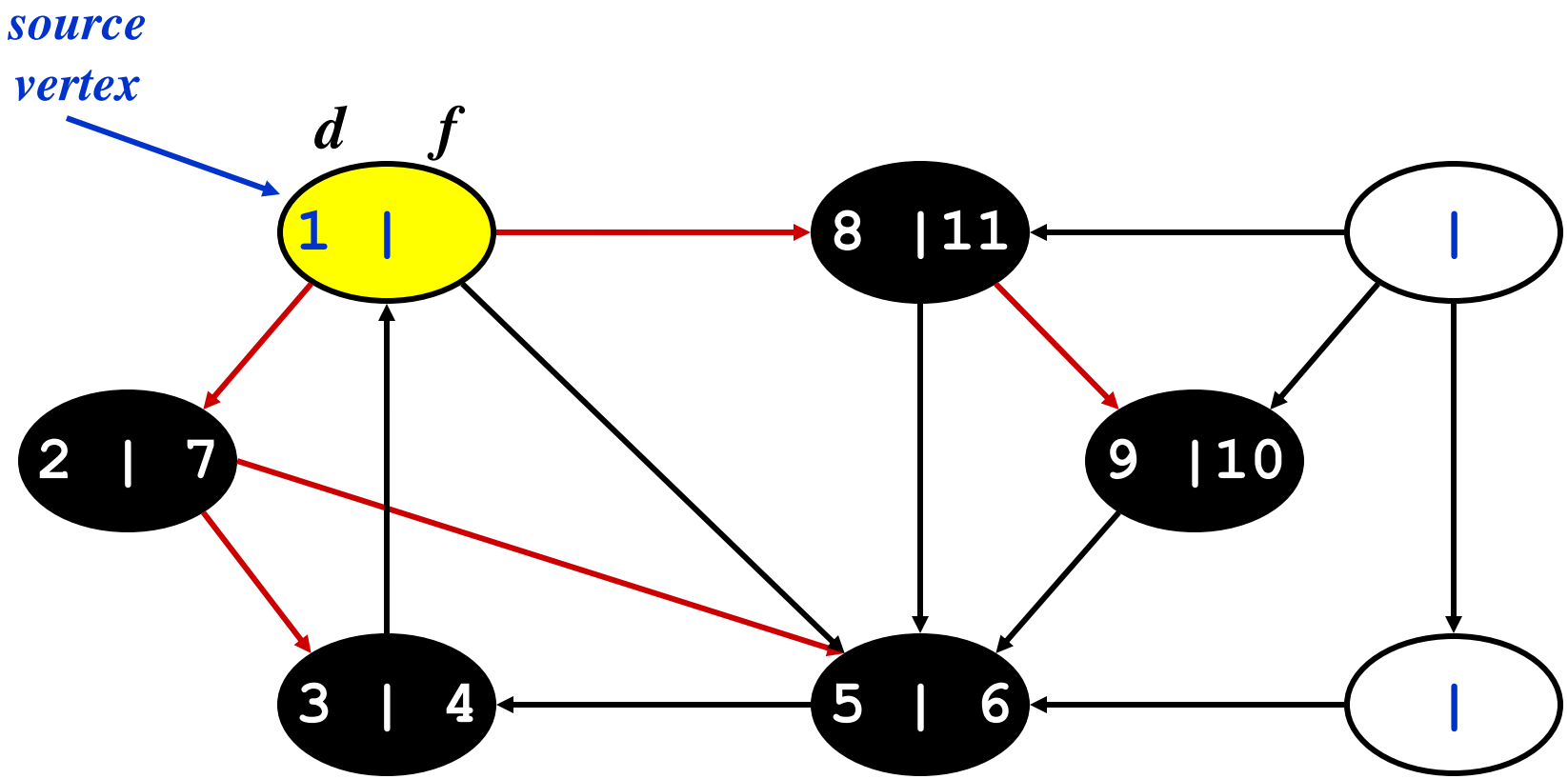

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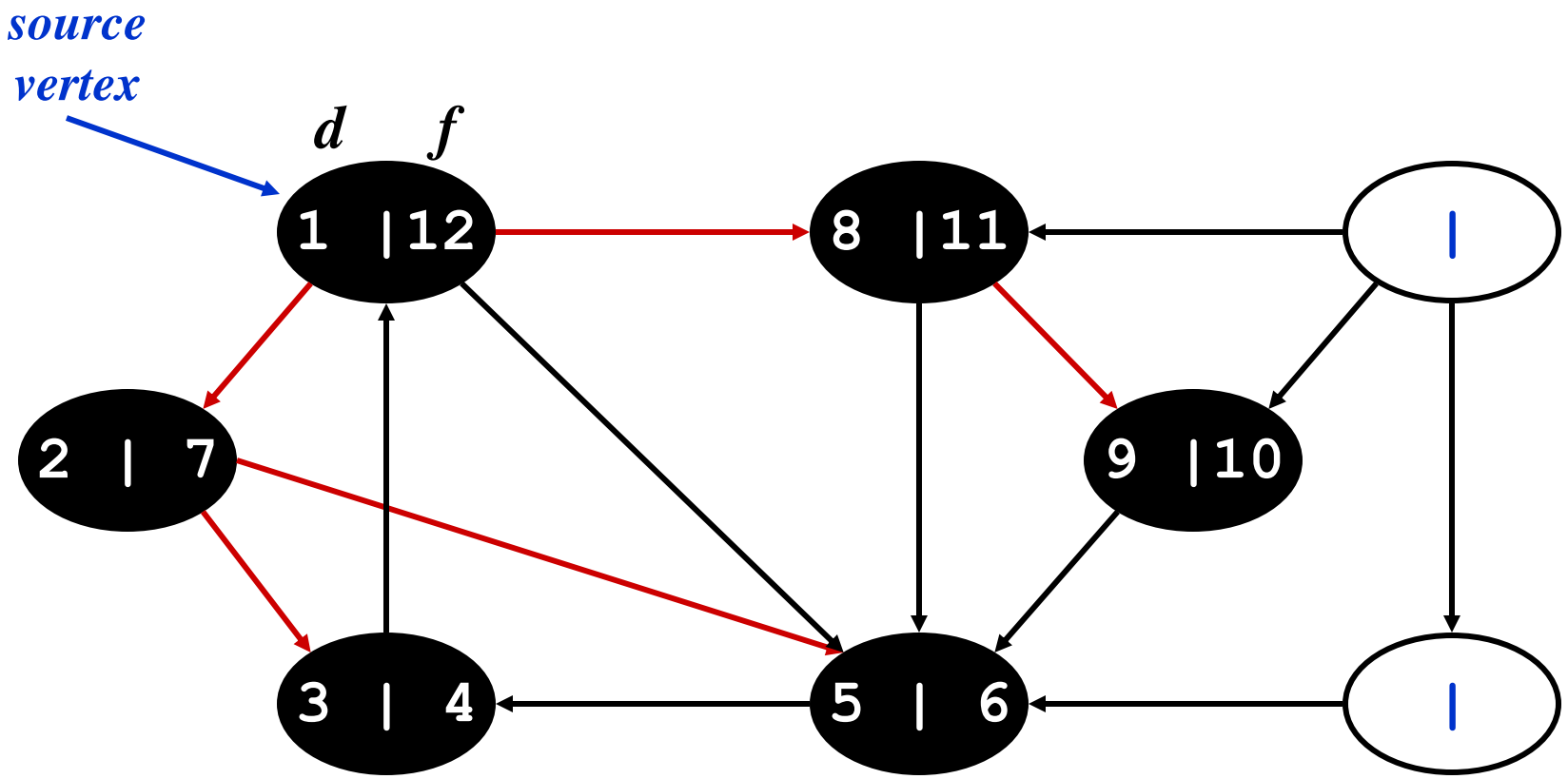

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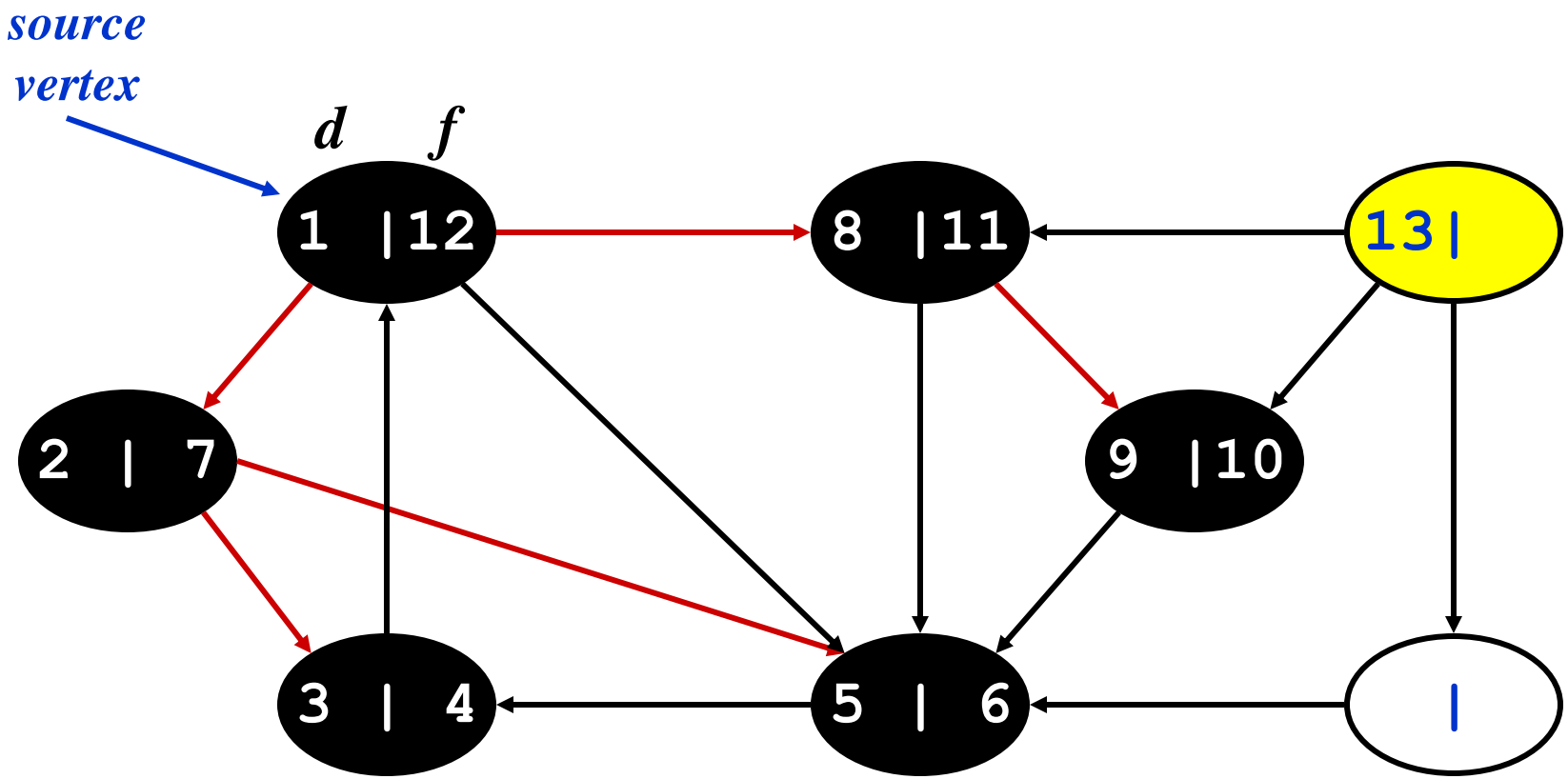

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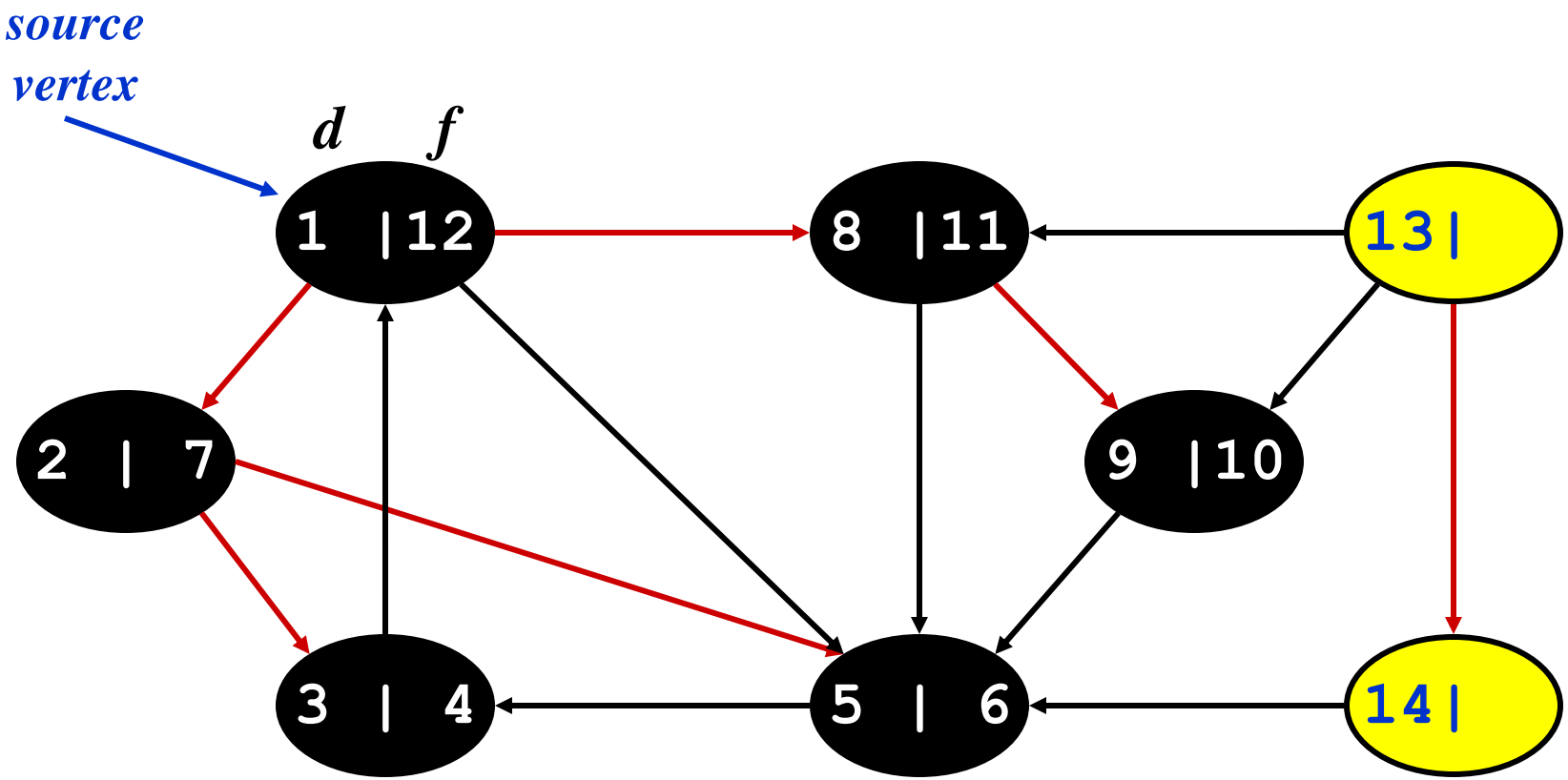

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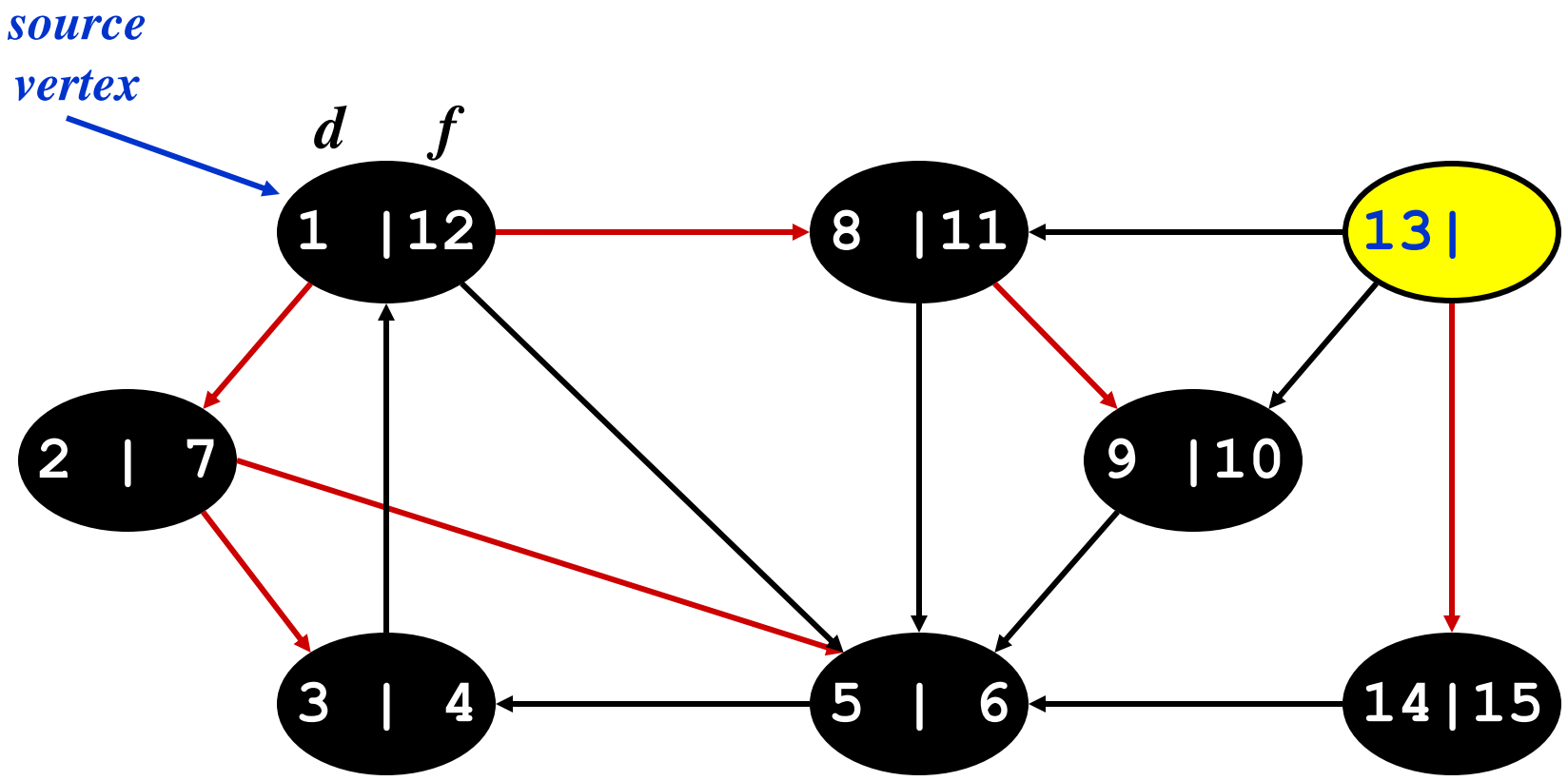

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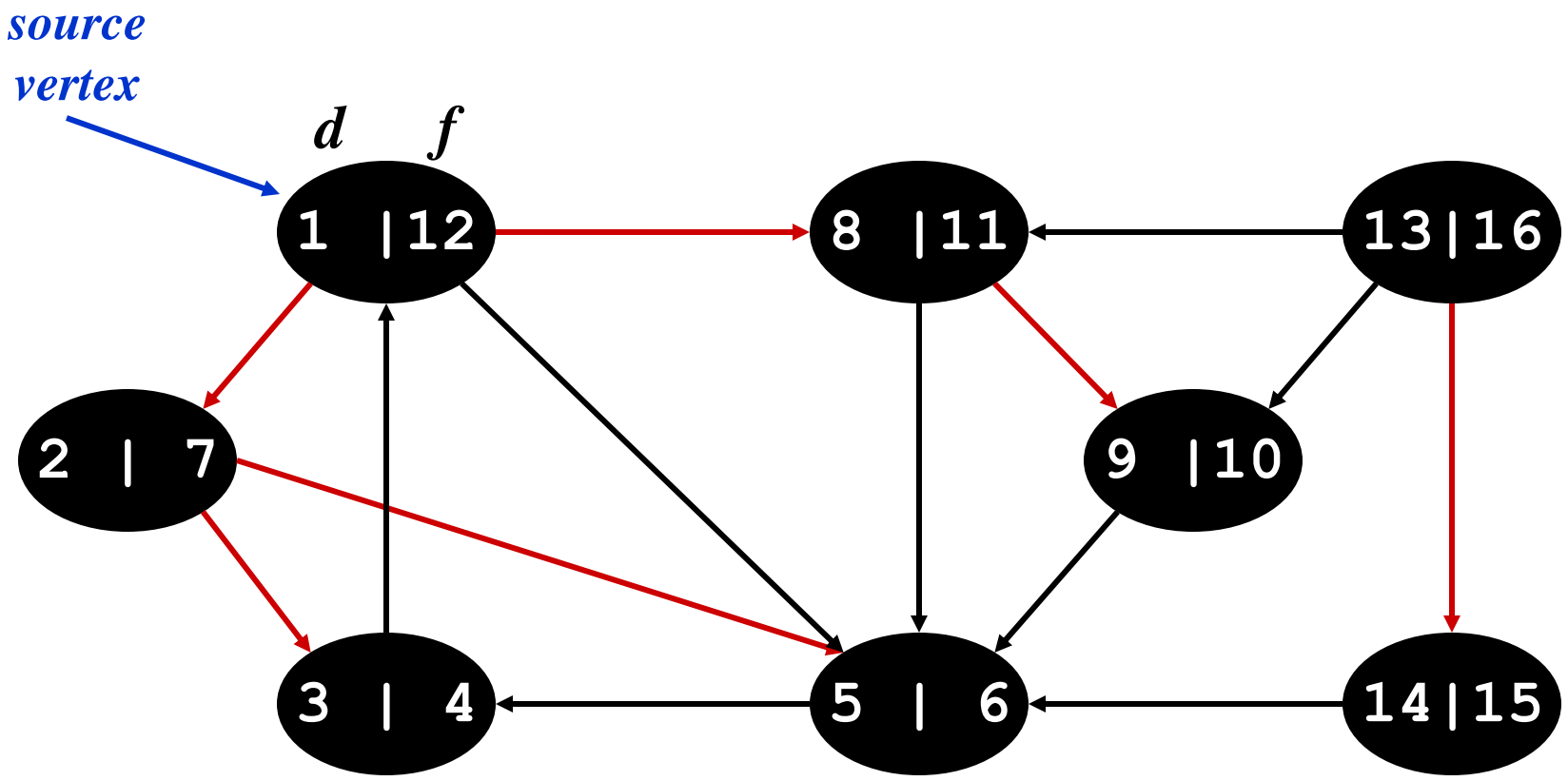
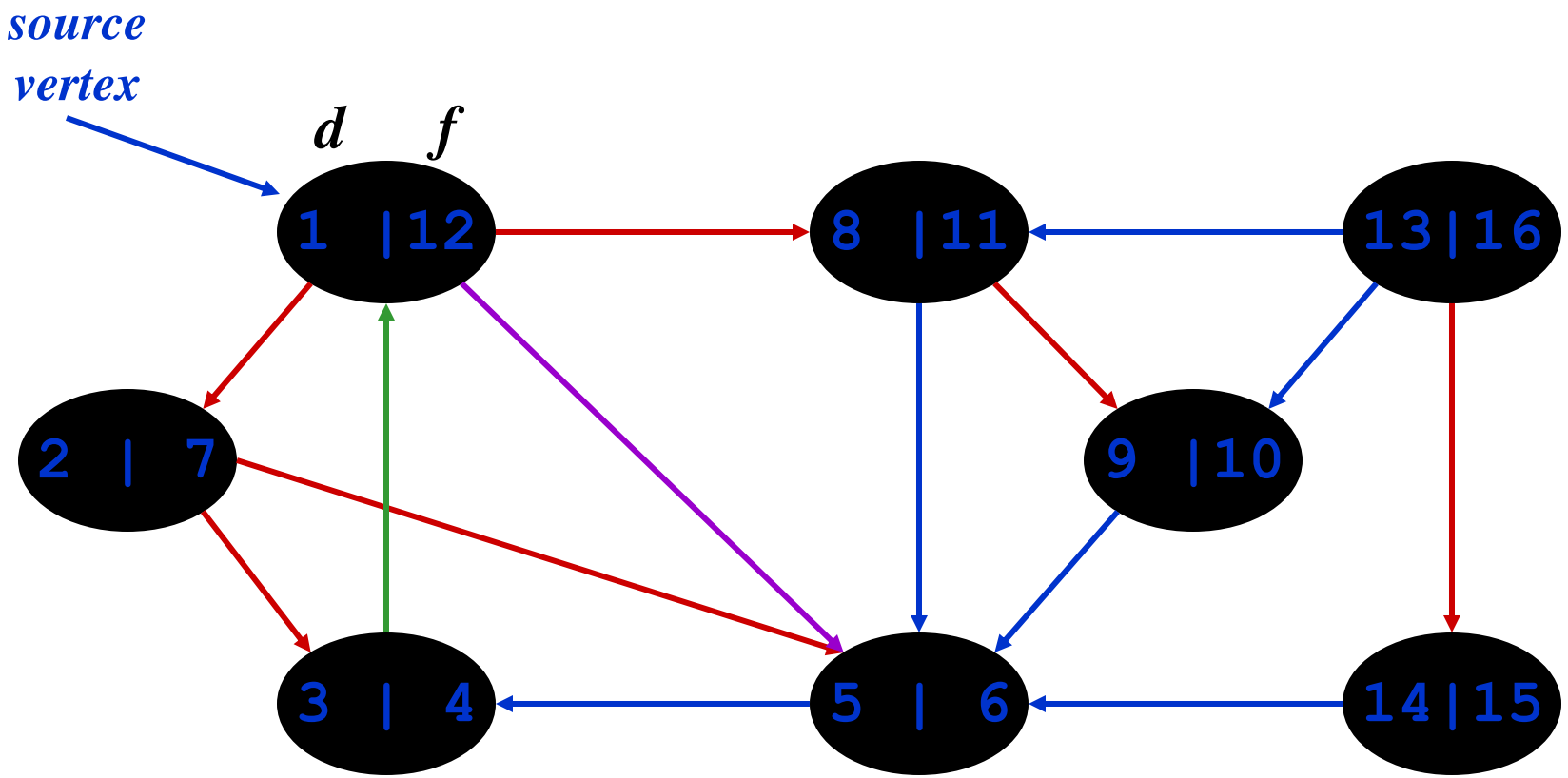


Review: Kinds Of Edges

- Thm: If G is undirected, a DFS produces only tree and back edges
- Thm: An undirected graph is *acyclic* iff a DFS yields no back edges
- Thus, can run DFS to find cycles

Review: Kinds of Edges



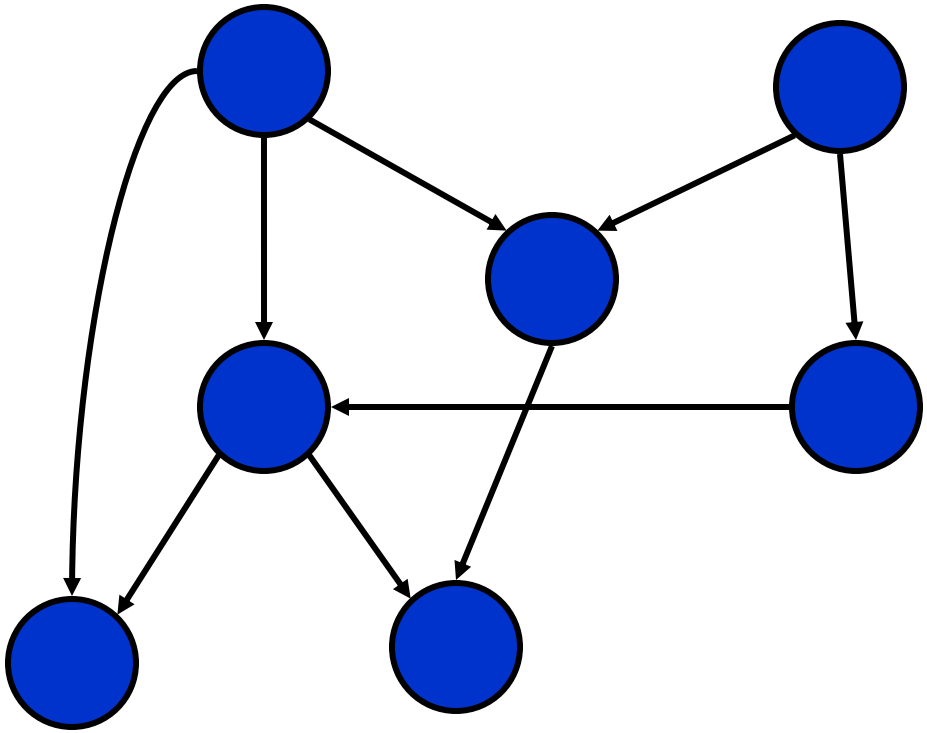
Tree edges *Back edges* *Forward edges* *Cross edges*

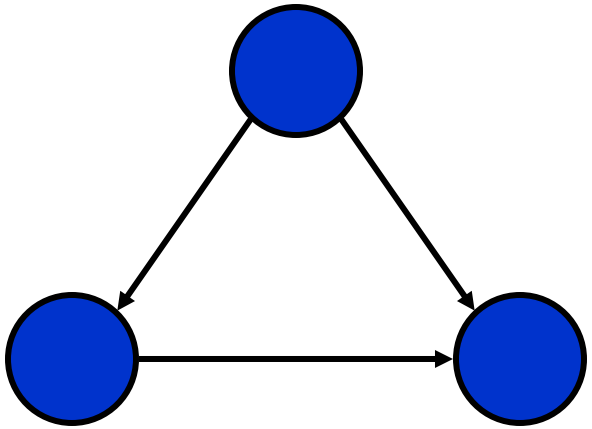
DFS And Cycles

- Running time: O(V+E)
- We can actually determine if cycles exist in O(V) time:
  - In an undirected acyclic forest, $|E| \leq |V| - 1$
  - So count the edges: if ever see |V| distinct edges, must have seen a back edge along the way
  - *Why not just test if $|E| < |V|$ and answer the question in constant time?*

Directed Acyclic Graphs

- A *directed acyclic graph* or *DAG* is a directed graph with no directed cycles:





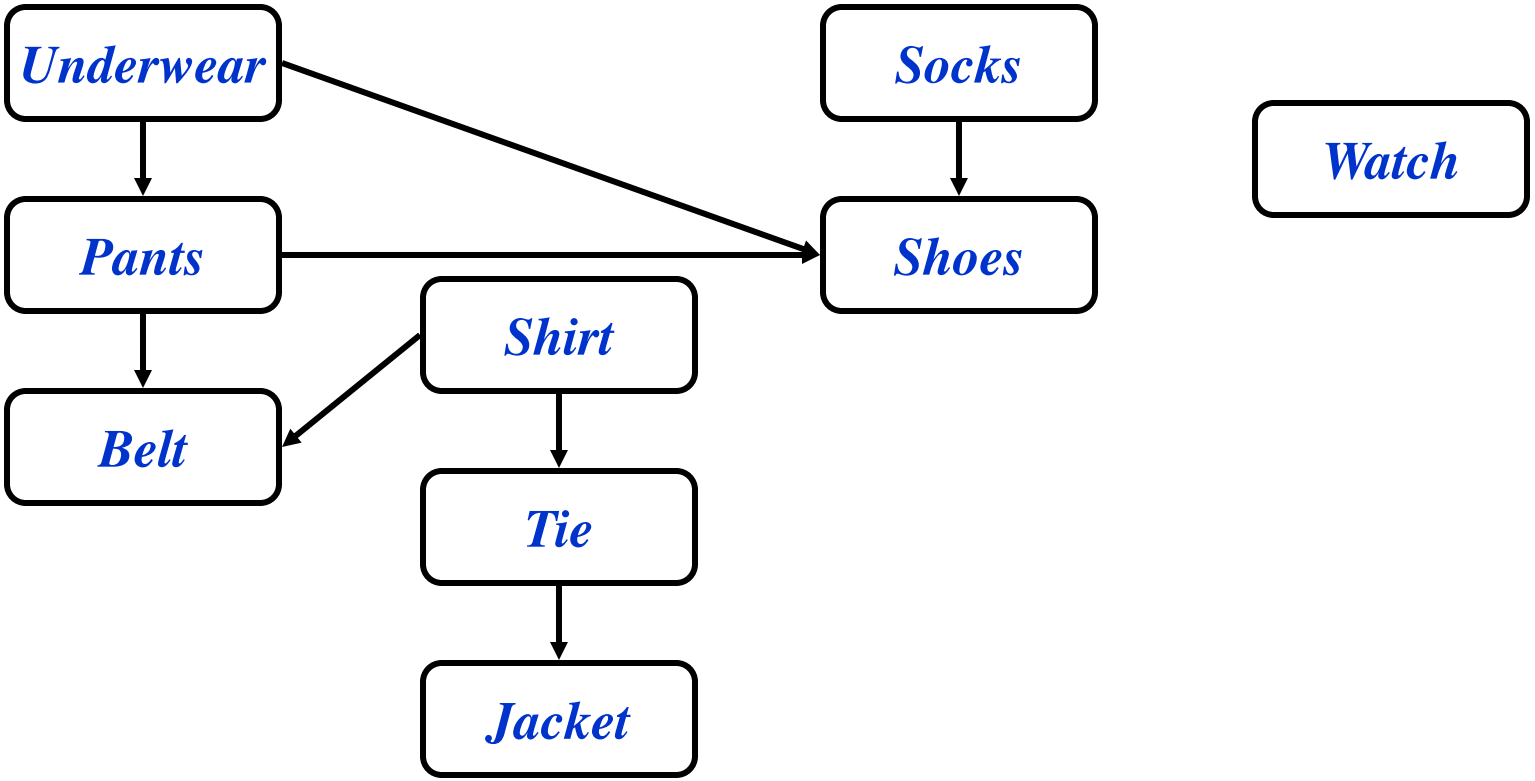
DFS and DAGs

- Argue that a directed graph G is acyclic iff a DFS of G yields no back edges:
  - Forward: if G is acyclic, will be no back edges
    - Trivial: a back edge implies a cycle
  - Backward: if no back edges, G is acyclic
    - Argue contrapositive: G has a cycle $\Rightarrow \exists$ a back edge
      - Let *v* be the vertex on the cycle first discovered, and *u* be the predecessor of *v* on the cycle
      - When *v* discovered, whole cycle is white
      - Must visit everything reachable from *v* before returning from DFS-Visit()
      - So path from u$\rightarrow$v is yellow$\rightarrow$yellow, thus (*u*, *v*) is a back edge

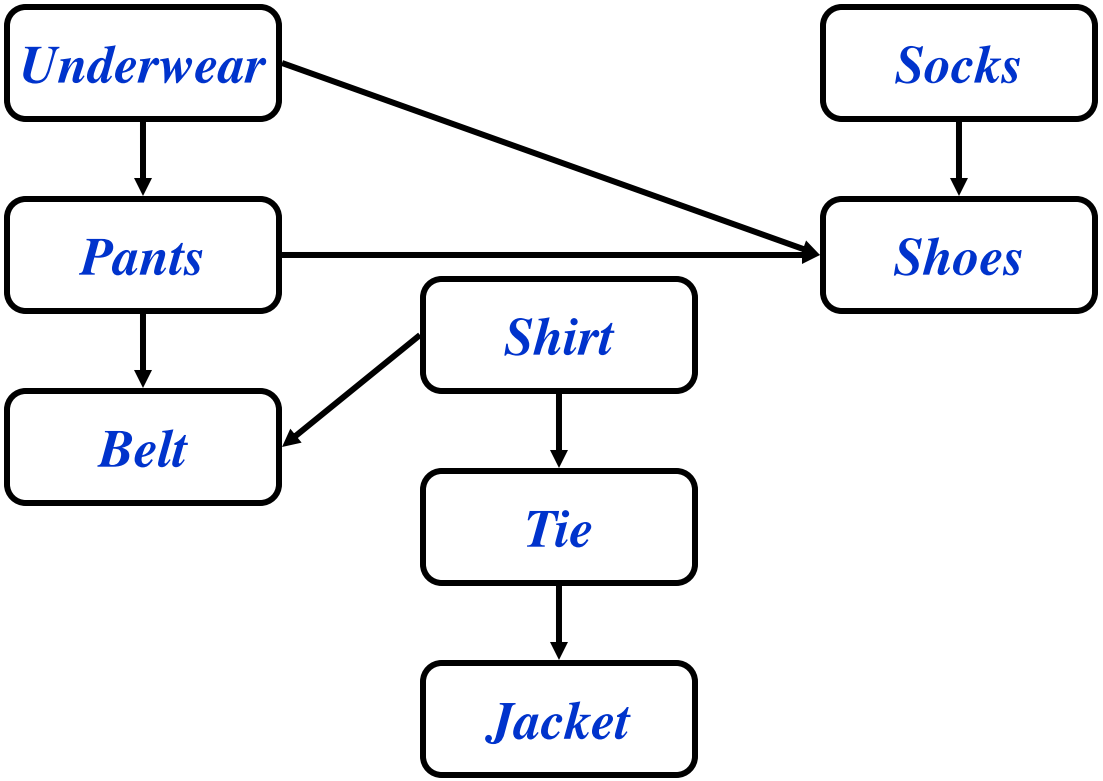
Topological Sort

- *Topological sort* of a DAG:
  - Linear ordering of all vertices in graph G such that vertex u comes before vertex v if edge $(u, v) \in G$
- Real-world example: getting dressed

Getting Dressed



Getting Dressed



Topological Sort Algorithm

```
Topological-Sort()
{
   Run DFS
   When a vertex is finished, output it
   Vertices are output in reverse
     topological order
}
```

- Time: O(V+E)
- Correctness: Want to prove that

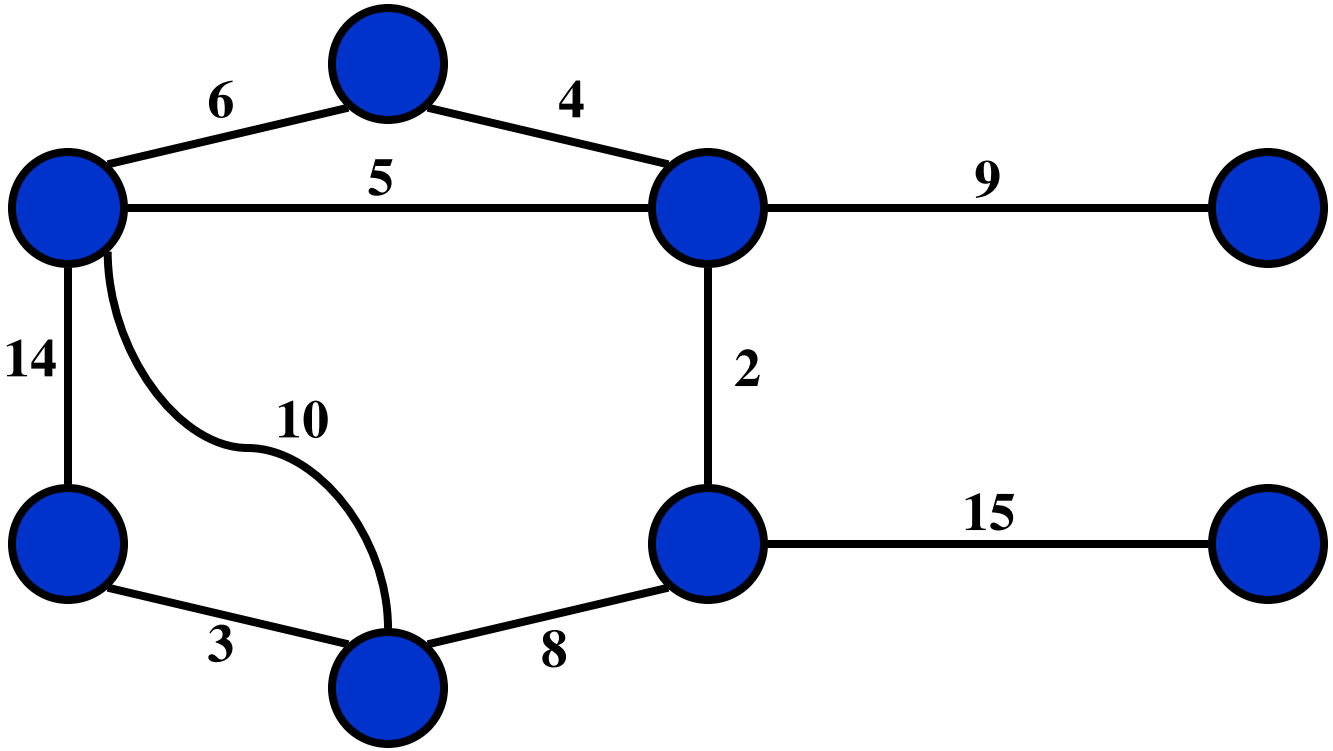
$$(u,v) \in \mathrm{G} \Rightarrow u{\rightarrow}\mathrm{f} > v{\rightarrow}\mathrm{f}$$

Correctness of Topological Sort

- Claim: $(u,v) \in G \Rightarrow u.f > v.f$
  - When (u,v) is explored, u is yellow
    - v = yellow $\Rightarrow (u,v)$ is back edge. Contradiction (*Why?*)
    - v = white $\Rightarrow$ v becomes descendent of $u \Rightarrow v.f < u.f$ (since must finish v before backtracking and finishing u)
    - v = black $\Rightarrow$ v already finished $\Rightarrow v.f < u.f$

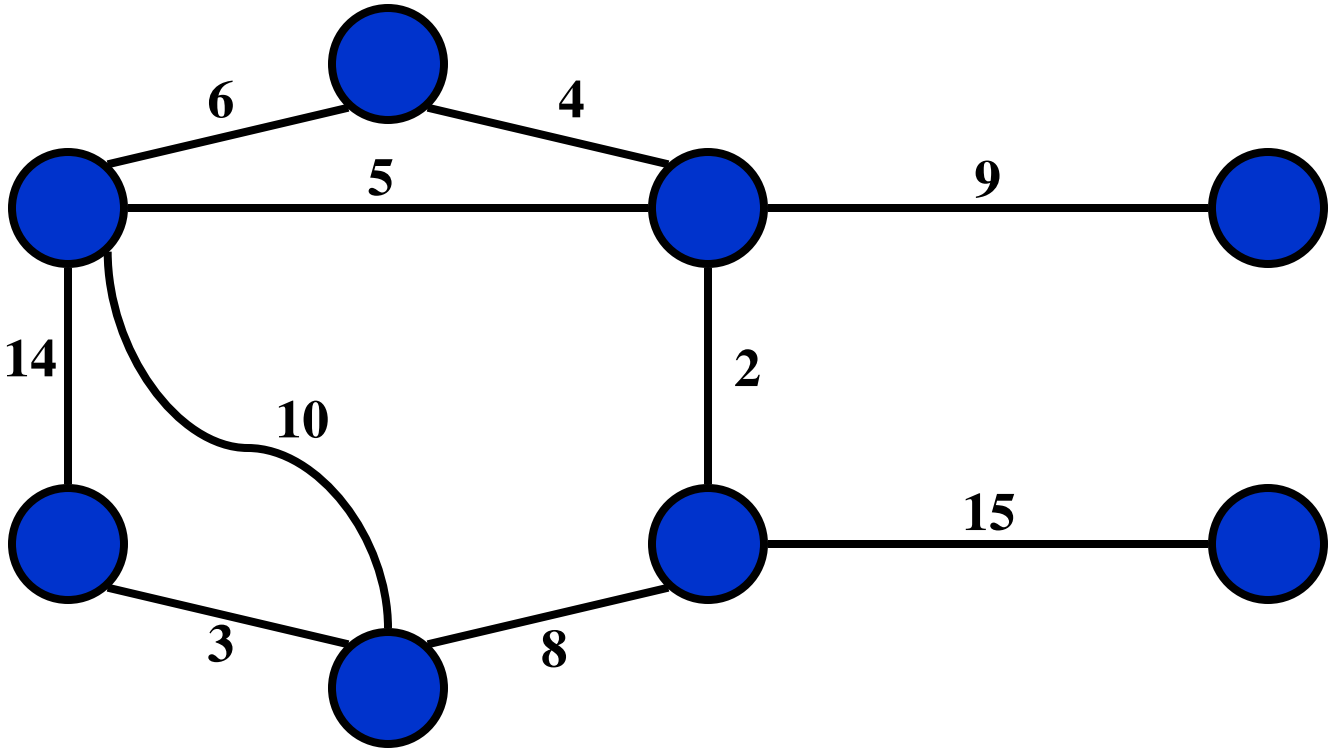
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph:



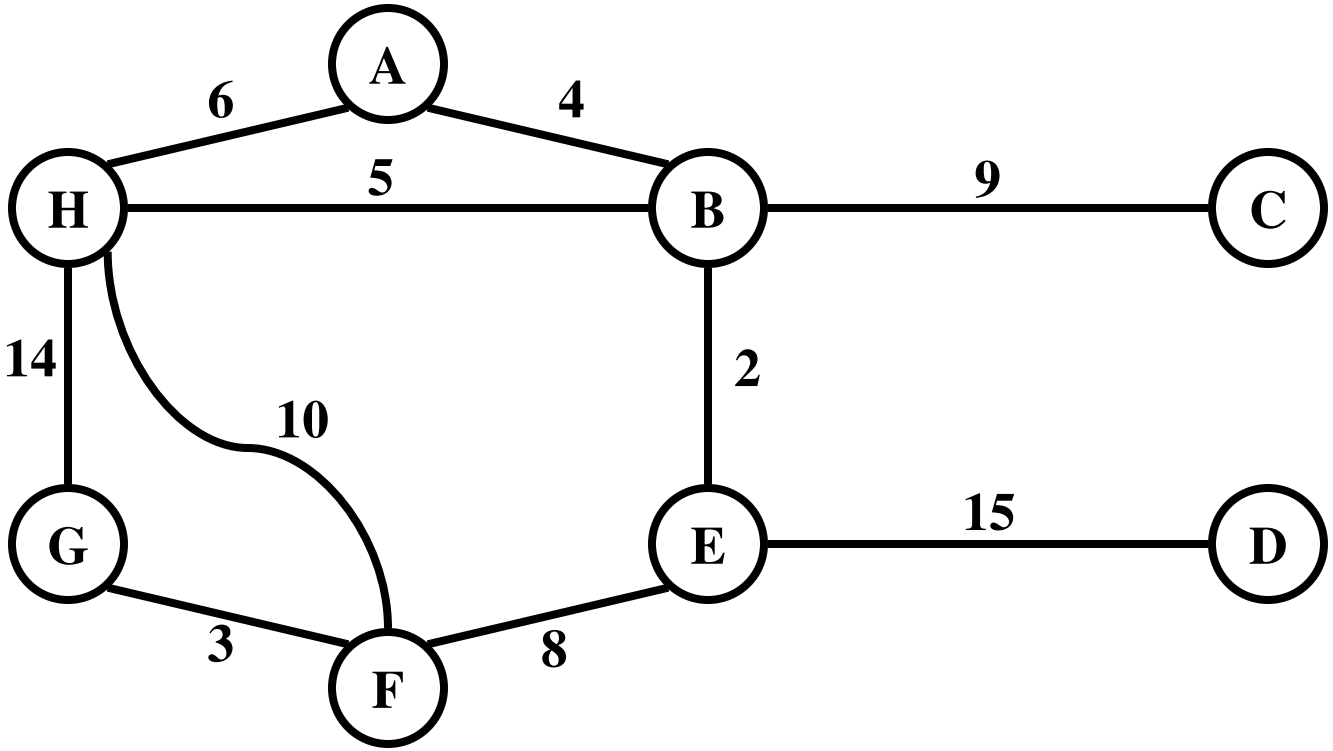
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph, find a *spanning tree* using edges that minimize the total weight



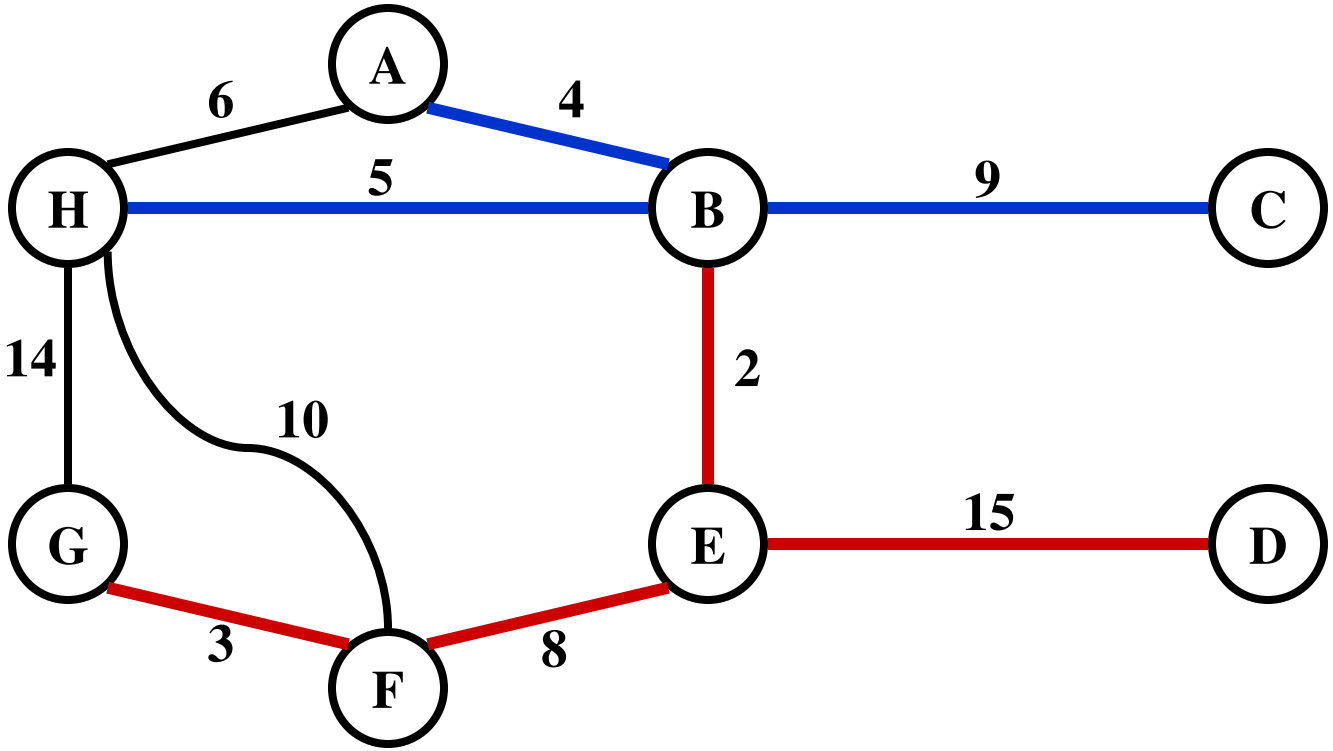
Minimum Spanning Tree

- *Which edges form the minimum spanning tree (MST) of the below graph?*



Minimum Spanning Tree

- Answer:



Minimum Spanning Tree

- MSTs satisfy the *optimal substructure* property: an optimal tree is composed of optimal subtrees
  - Let T be an MST of G with an edge (u,v) in the middle
  - Removing (u,v) partitions T into two trees T_1 and T_2
  - Claim: T_1 is an MST of $G_1 = (V_1,E_1)$, and T_2 is an MST of $G_2 = (V_2,E_2)$ (*Do V_1 and V_2 share vertices? Why?*)
  - Proof: $w(T) = w(u,v) + w(T_1) + w(T_2)$
    (There can't be a better tree than T_1 or T_2, or T would be suboptimal)

Minimum Spanning Tree

- Thm:
  - Let T be MST of G, and let A $\subseteq$ T be subtree of T
  - Let (u,v) be min-weight edge connecting A to V-A
  - Then $(u,v) \in$ T

Minimum Spanning Tree

- Thm:
  - Let T be MST of G, and let A $\subseteq$ T be subtree of T
  - Let (u,v) be min-weight edge connecting A to V-A
  - Then $(u,v) \in$ T
- Proof: in book (see Thm 24.1)

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```

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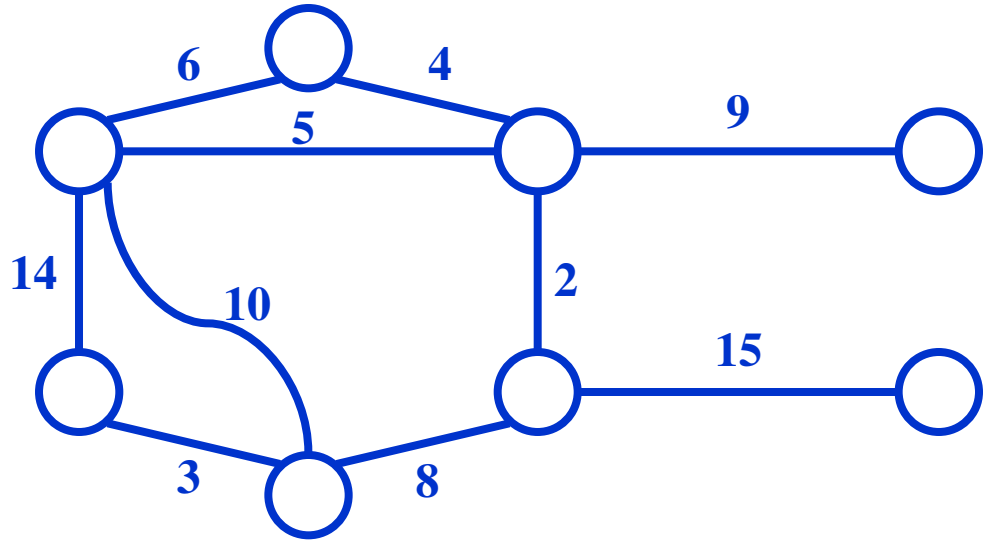

Run on example graph

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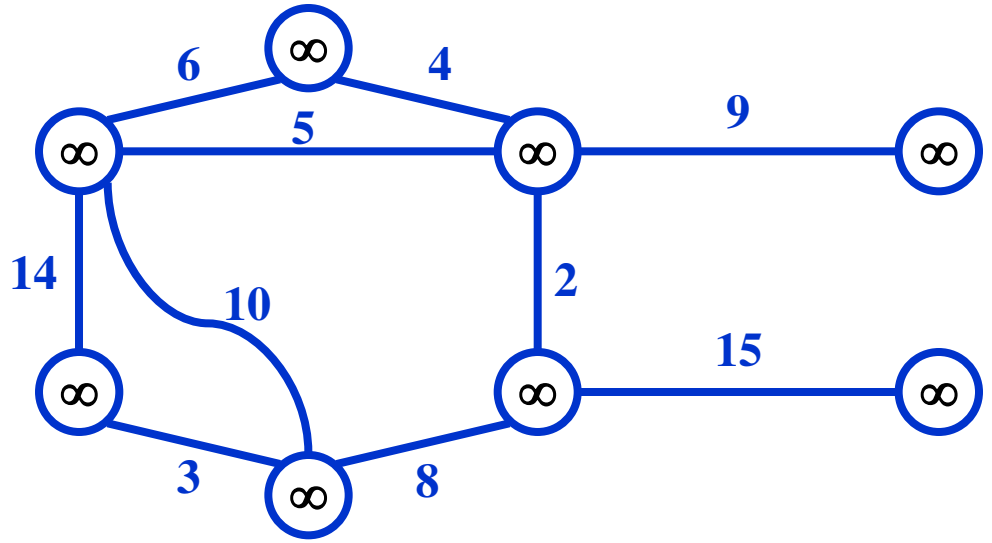

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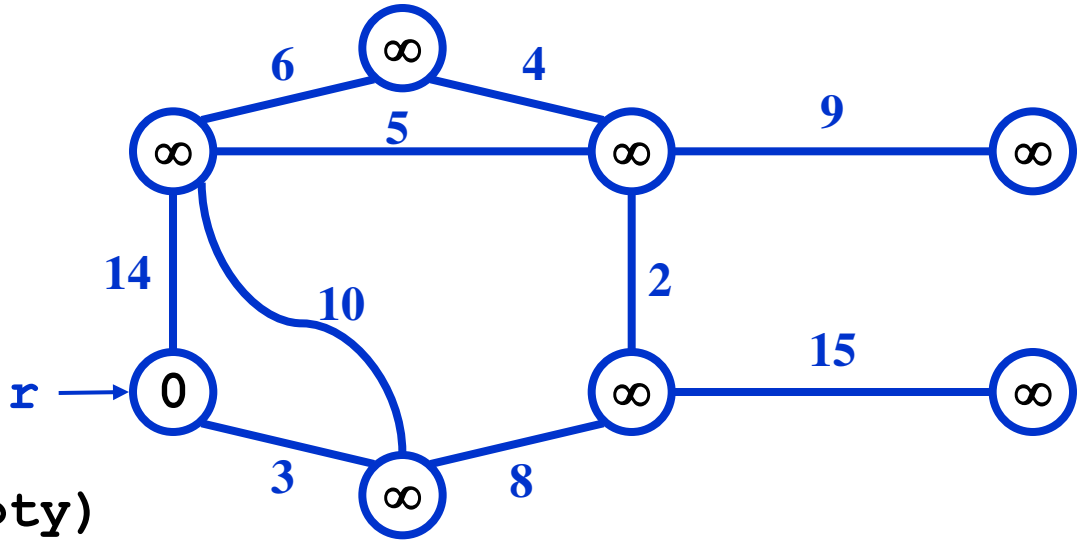

Pick a start vertex r

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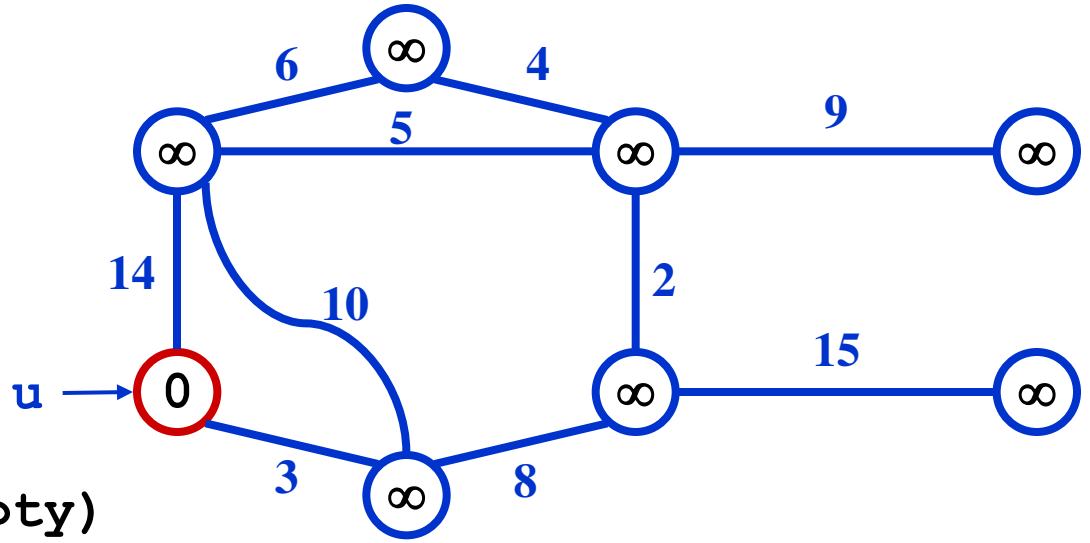

Red vertices have been removed from Q

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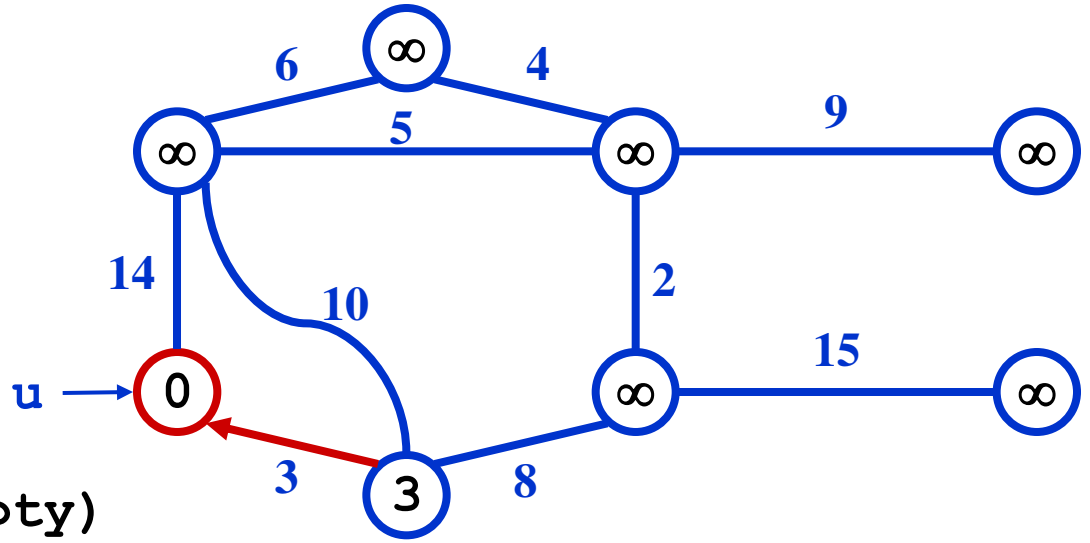

Red arrows indicate parent pointers

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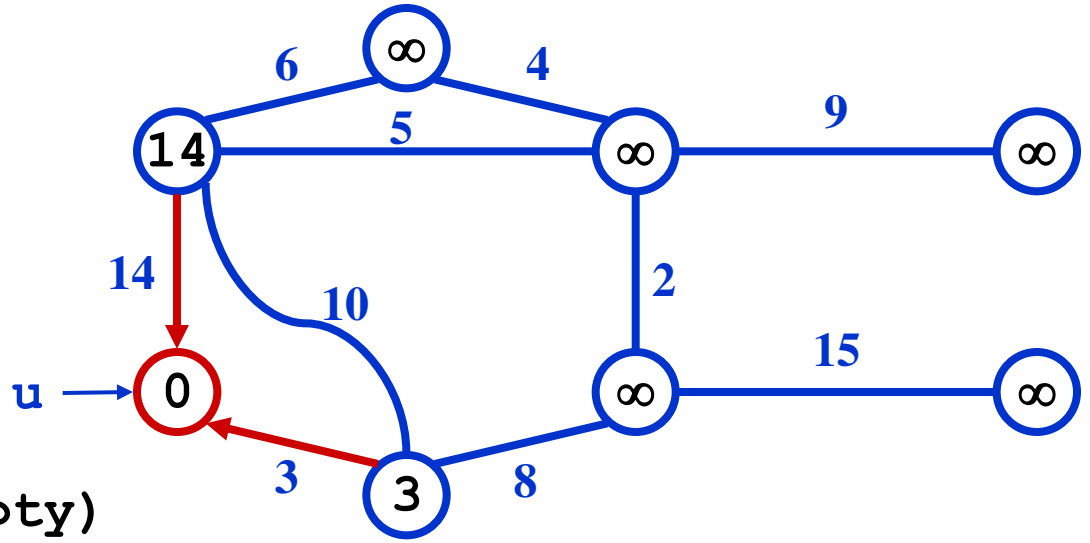

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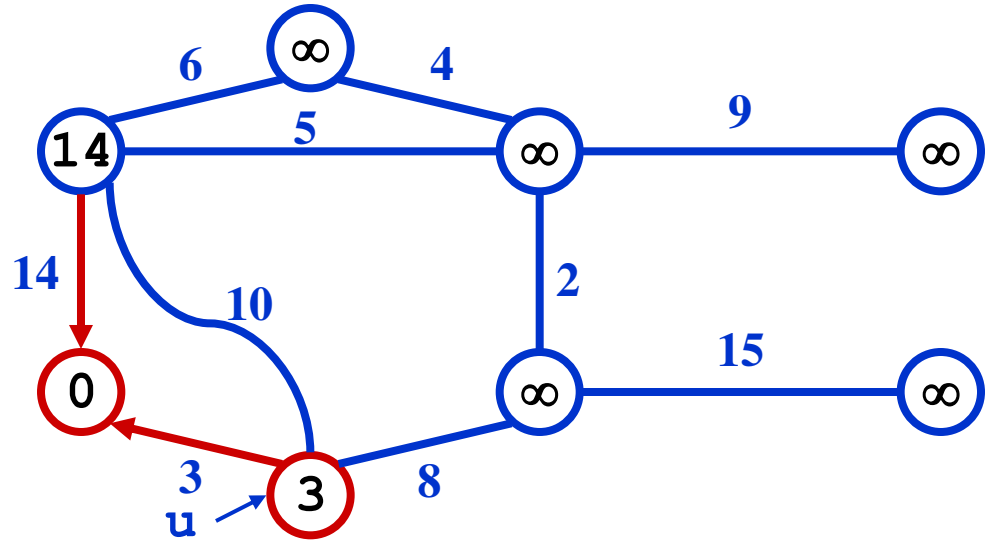

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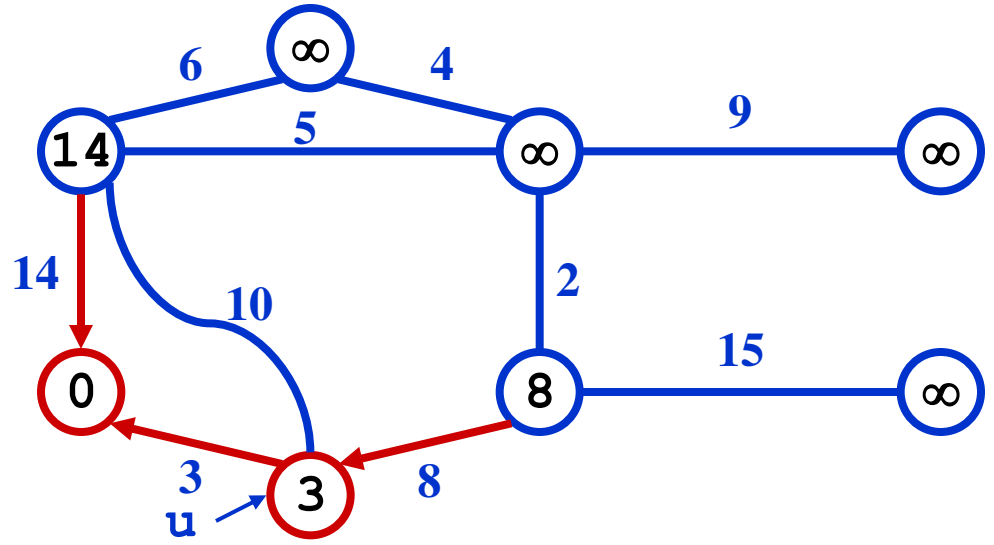

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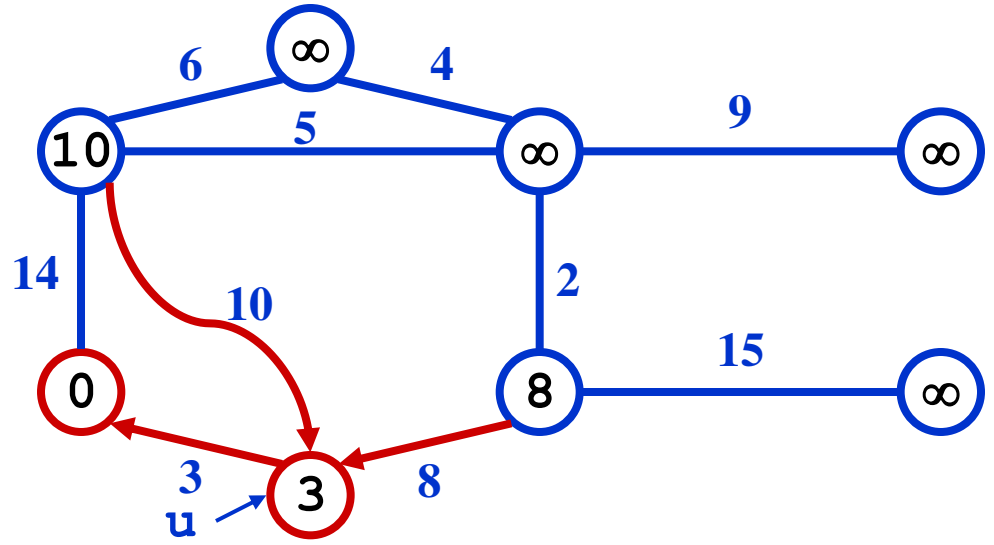

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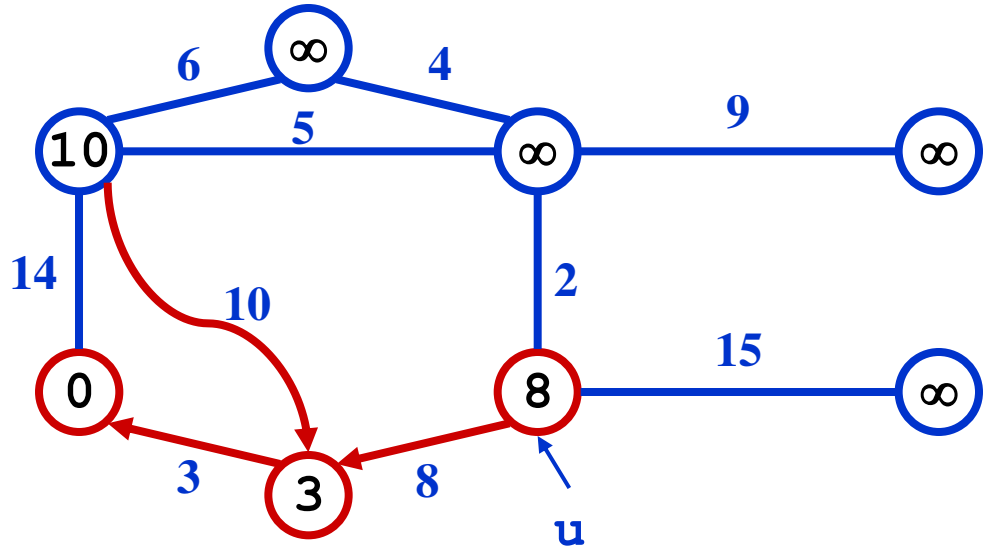

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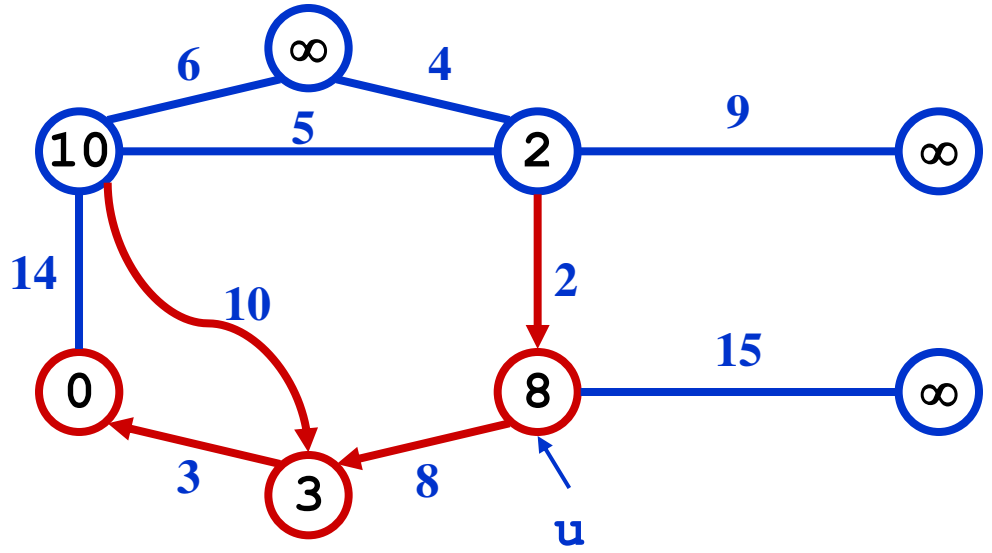

Prim's Algorithm

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    for each u ∈ Q
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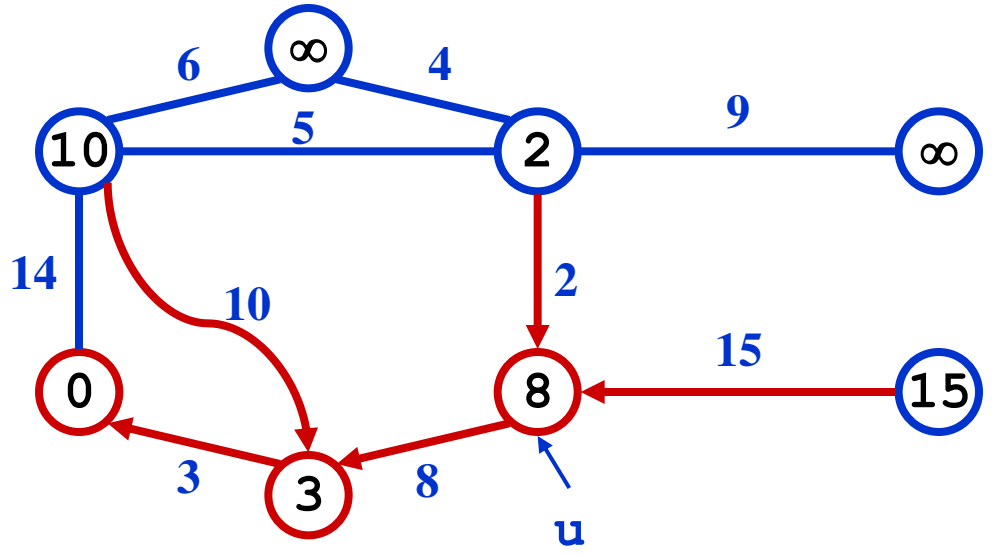

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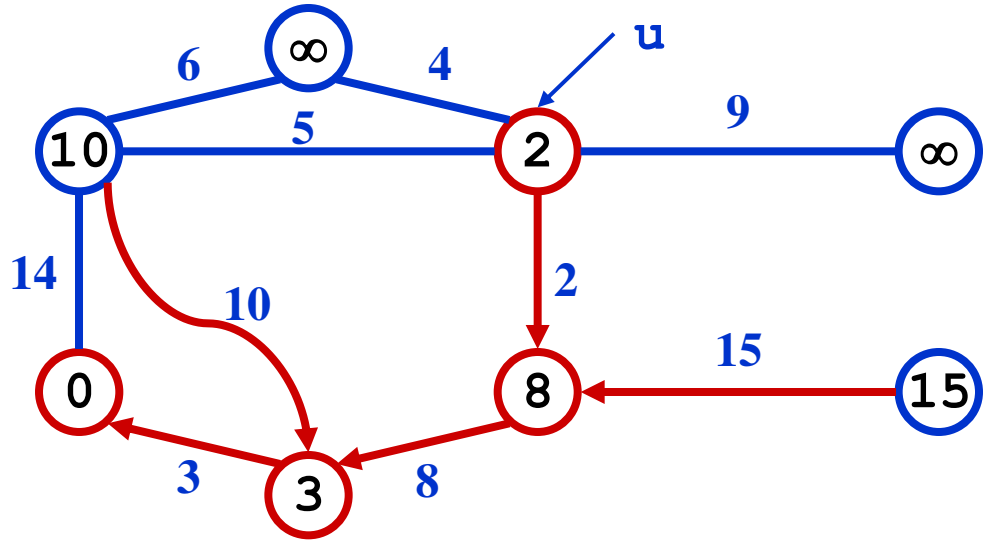

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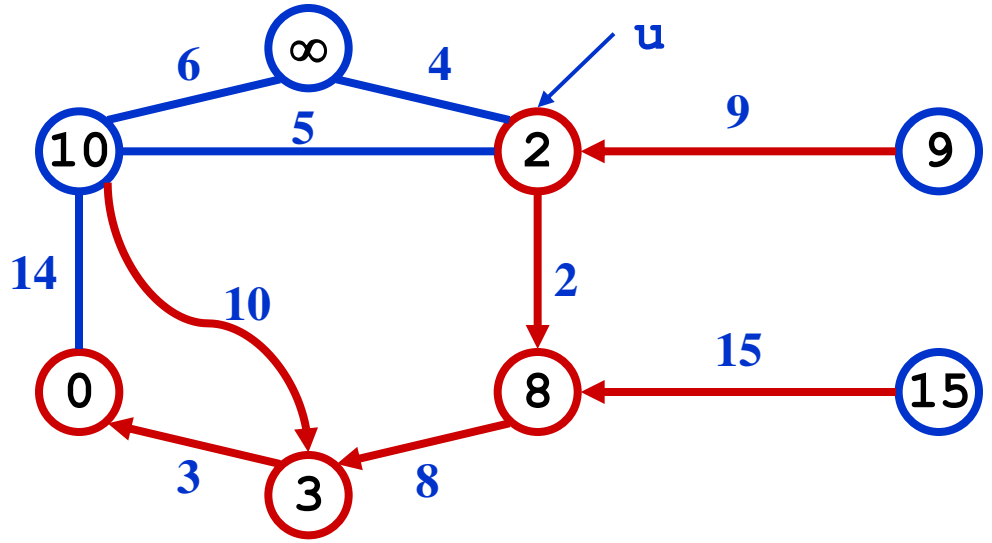

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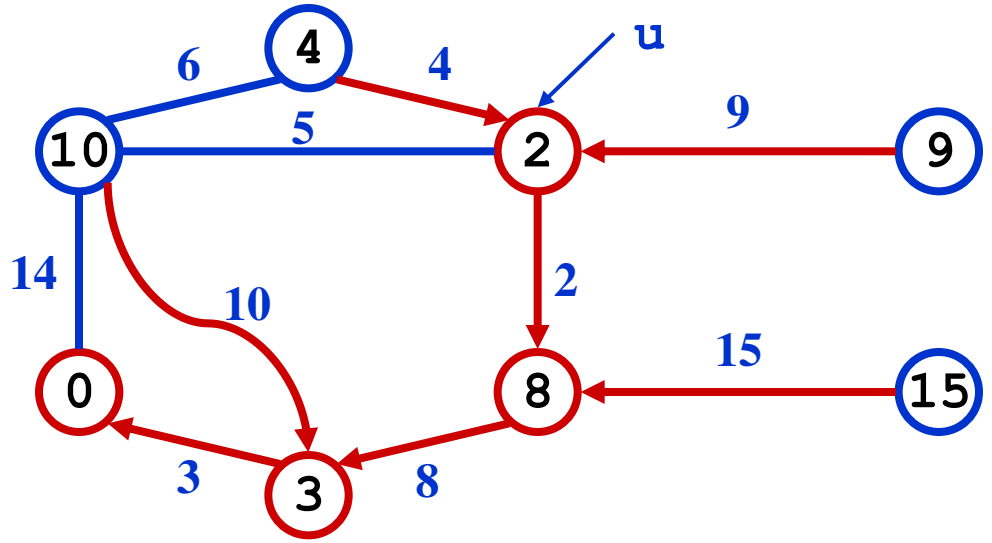

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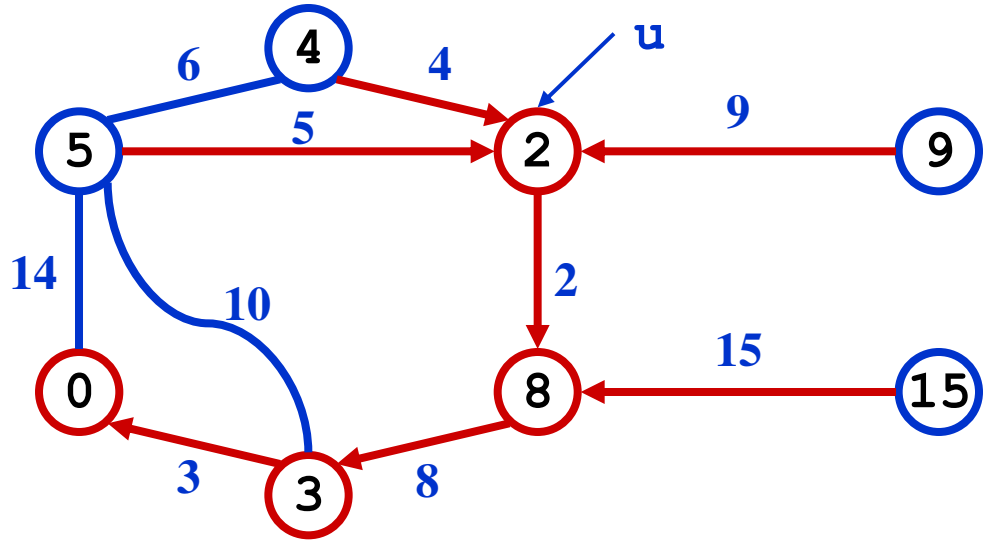

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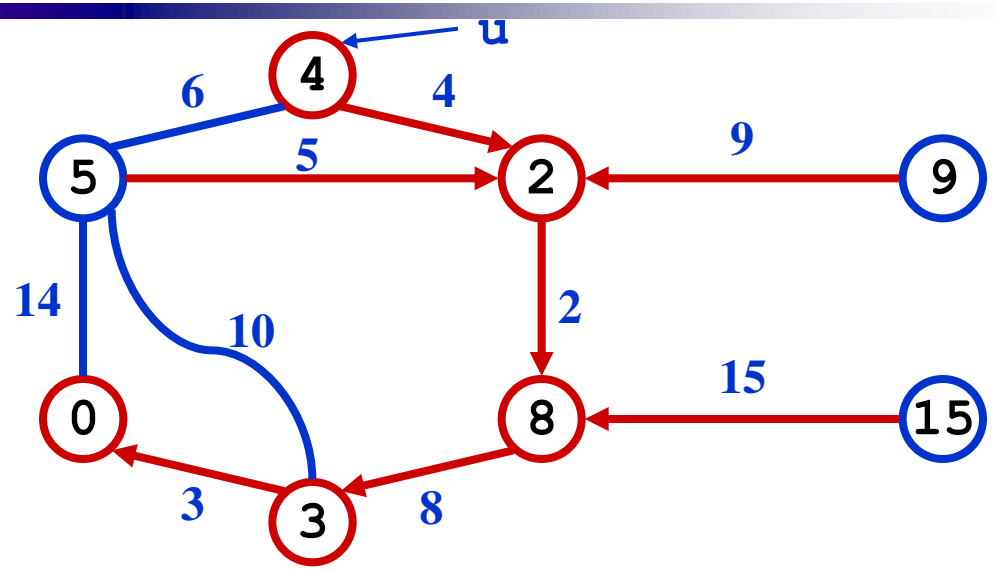

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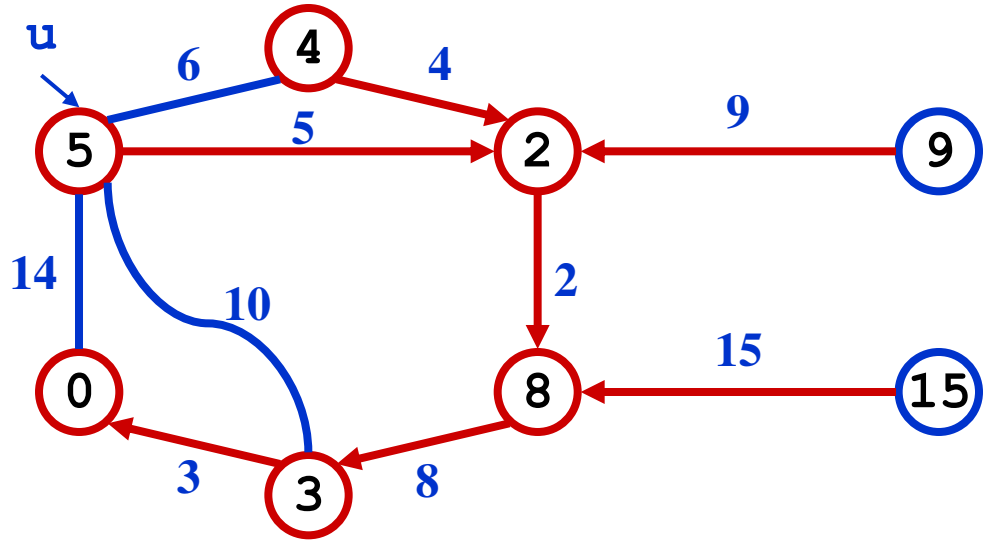

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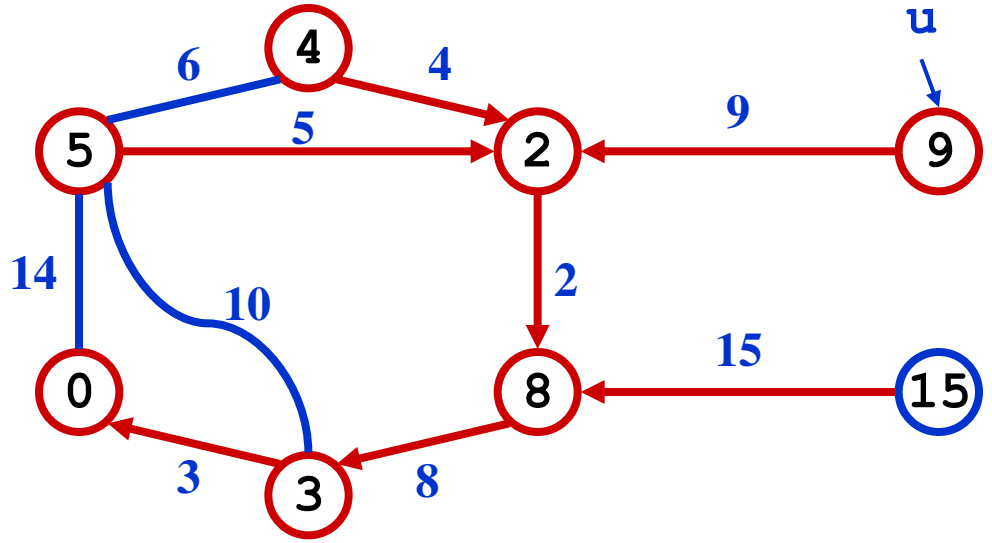

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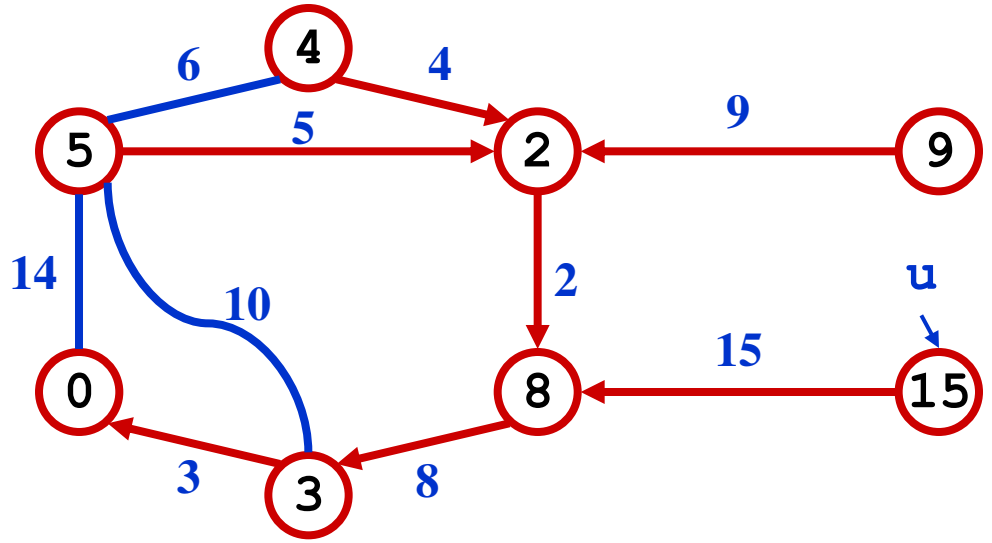

Review: Prim's Algorithm

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            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```

What is the hidden cost in this code?

Review: Prim's Algorithm

```
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    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
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    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                DecreaseKey(v, w(u,v));
```

Review: Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
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        key[u] = ∞;
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            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                DecreaseKey(v, w(u,v));
```

How often is ExtractMin() called?

How often is DecreaseKey() called?

Review: Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
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                p[v] = u;
                key[v] = w(u,v);
```

What will be the running time?

A: Depends on queue

binary heap: O(E lg V)

Fibonacci heap: O(V lg V + E)

Single-Source Shortest Path

- Problem: given a weighted directed graph G, find the minimum-weight path from a given source vertex s to another vertex v
  - "Shortest-path" = minimum weight
  - Weight of path is sum of edges
  - E.g., a road map: what is the shortest path from Chapel Hill to Charlottesville?