Algorithms

Dijkstra's Algorithm

Disjoint-Set Union

Review: Bellman-Ford Algorithm

```
BellmanFord()
    for each v ∈ V
        d[v] = ∞;
    d[s] = 0;
    for i=1 to |V|-1
        for each edge (u,v) ∈ E
            Relax(u,v, w(u,v));
    for each edge (u,v) ∈ E
        if (d[v] > d[u] + w(u,v))
            return "no solution";
```

Initialize d[], which will converge to shortest-path value δ

Relaxation: Make |V|-1 passes, relaxing each edge

Test for solution: have we converged yet? Ie, $\exists$ negative cycle?

```
Relax(u,v,w): if (d[v] > d[u]+w) then d[v]=d[u]+w
```

Review: DAG Shortest Paths

- Problem: finding shortest paths in DAG
  - Bellman-Ford takes O(VE) time.
  - Do better using topological sort.
    - Idea: if were lucky and processes vertices on each shortest path in order, B-F would be done in one pass
    - Every path in a dag is subsequence of topologically sorted vertex order, so processing verts in that order, we will do each path in forward order (will never relax edges out of vert before doing all edges into vert).
    - Thus: just one pass. Running time: O(V+E)

Review: Dijkstra's Algorithm

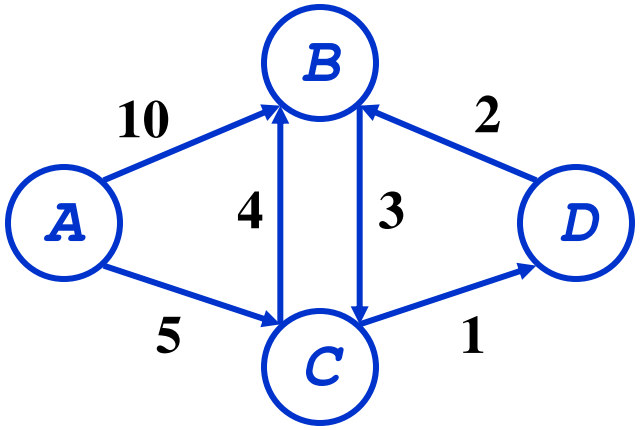
- If no negative edge weights, we can beat BF
- Similar to breadth-first search
  - Grow a tree gradually, advancing from vertices taken from a queue
- Also similar to Prim's algorithm for MST
  - Use a priority queue keyed on d[v]

Dijkstra's Algorithm

```
Dijkstra(G)
    for each v ∈ V
        d[v] = ∞;
    d[s] = 0; S = ∅; Q = V;
    while (Q ≠ ∅)
        u = ExtractMin(Q);
        S = S U {u};
        for each v ∈ u->Adj[]
            if (d[v] > d[u]+w(u,v))
                d[v] = d[u]+w(u,v);
```

Relaxation Step

Note: this is really a call to Q->DecreaseKey()



Ex: run the algorithm

Dijkstra's Algorithm

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How many times is ExtractMin() called?

How many times is DecreaseKey() called?

What will be the total running time?

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How many times is ExtractMin() called?

How many times is DecraseKey() called?

A: O(E lg V) using binary heap for Q

Can acheive O(V lg V + E) with Fibonacci heaps

Dijkstra's Algorithm

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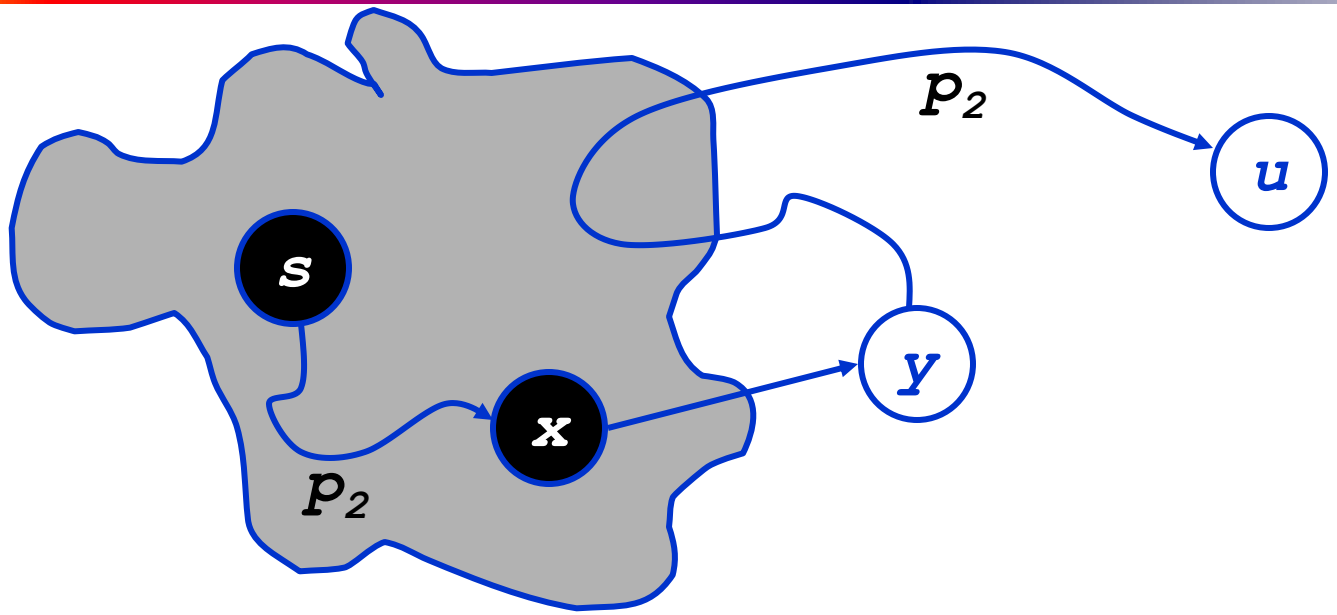
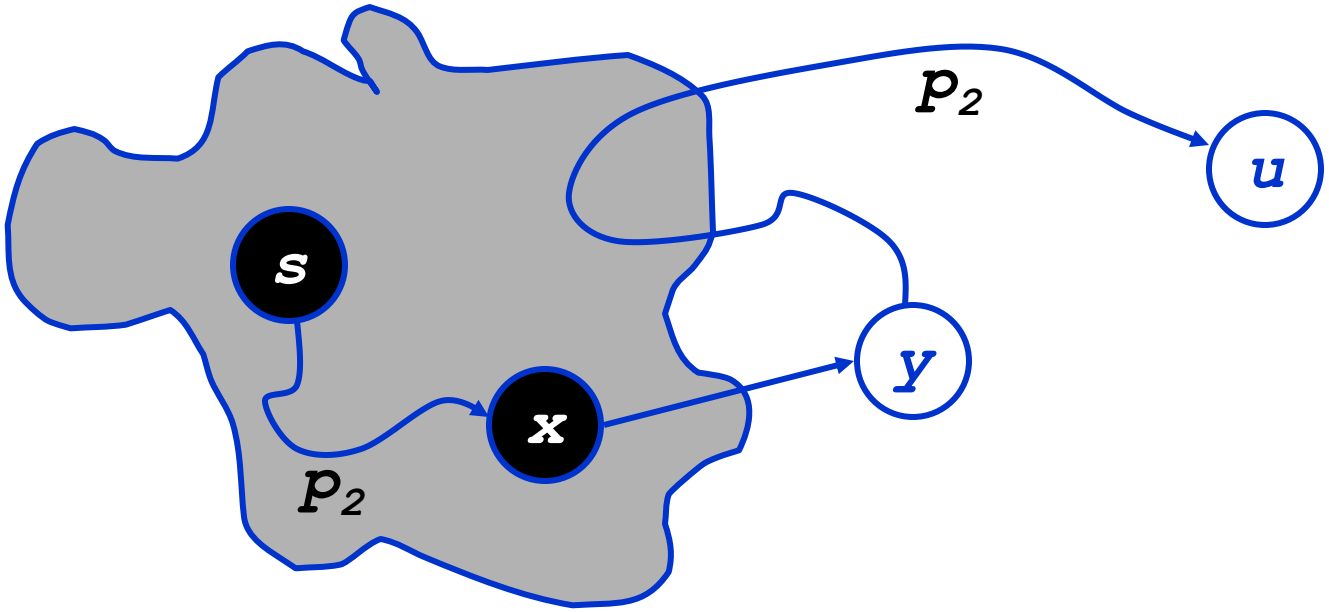
Correctness: we must show that when u is removed from Q, it has already converged

Correctness Of Dijkstra's Algorithm



- Note that $d[v] \geq \delta(s,v)\ \forall v$
- Let u be first vertex picked s.t. $\exists$ shorter path than d[u] $\Rightarrow d[u] > \delta(s,u)$
- Let y be first vertex $\in$ V-S on actual shortest path from s$\rightarrow$u $\Rightarrow d[y] = \delta(s,y)$
  - Because d[x] is set correctly for y's predecessor x $\in$ S on the shortest path, and
  - When we put x into S, we relaxed (x,y), giving d[y] the correct value

Correctness Of Dijkstra's Algorithm



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- $d[u] > \delta(s,u)$
  $= \delta(s,y) + \delta(y,u)$ (*Why?*)
  $= d[y] + \delta(y,u)$
  $\geq d[y]$

  But if d[u] > d[y], wouldn't have chosen u. Contradiction.

Disjoint-Set Union Problem

- Want a data structure to support disjoint sets
  - Collection of disjoint sets $S = \{S_i\}$, $S_i \cap S_j = \varnothing$
- Need to support following operations:
  - MakeSet(x): $S = S \cup \{\{x\}\}$
  - Union(S_i, S_j): $S = S - \{S_i, S_j\} \cup \{S_i \cup S_j\}$
  - FindSet(X): return $S_i \in S$ such that $x \in S_i$
- Before discussing implementation details, we look at example application: MSTs

Kruskal's Algorithm

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Kruskal()
{
    T = ∅;
    for each v ∈ V
        MakeSet(v);
    sort E by increasing edge weight w
    for each (u,v) ∈ E (in sorted order)
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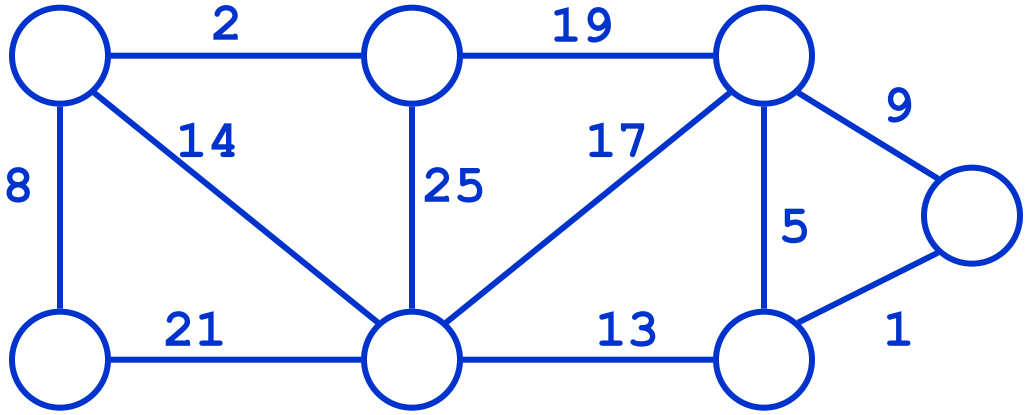

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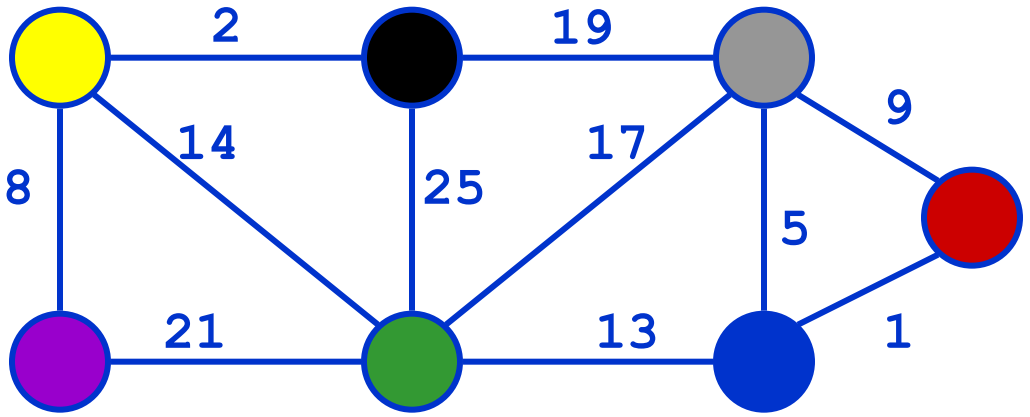

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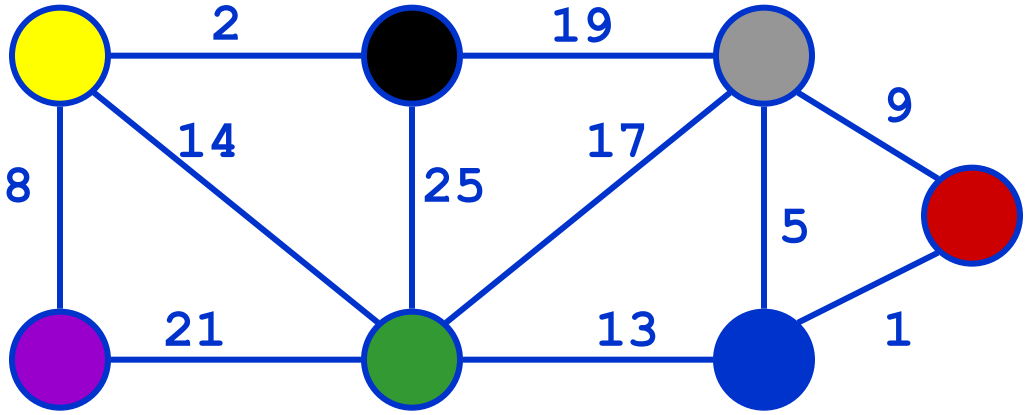

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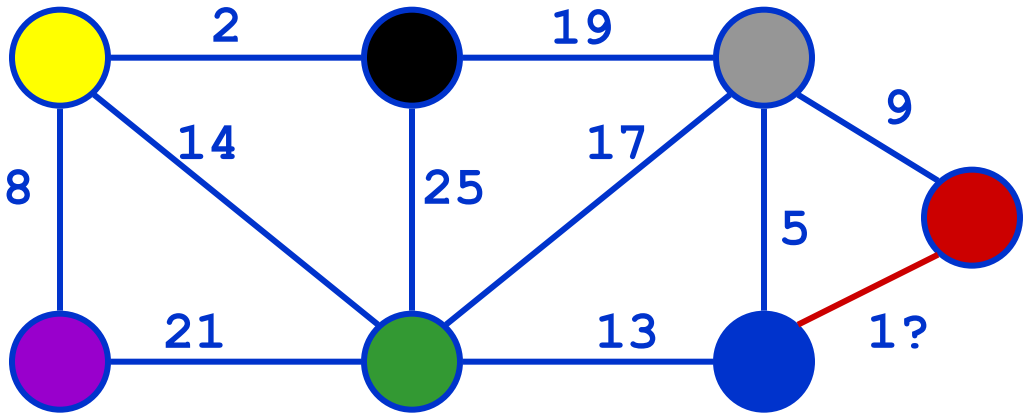

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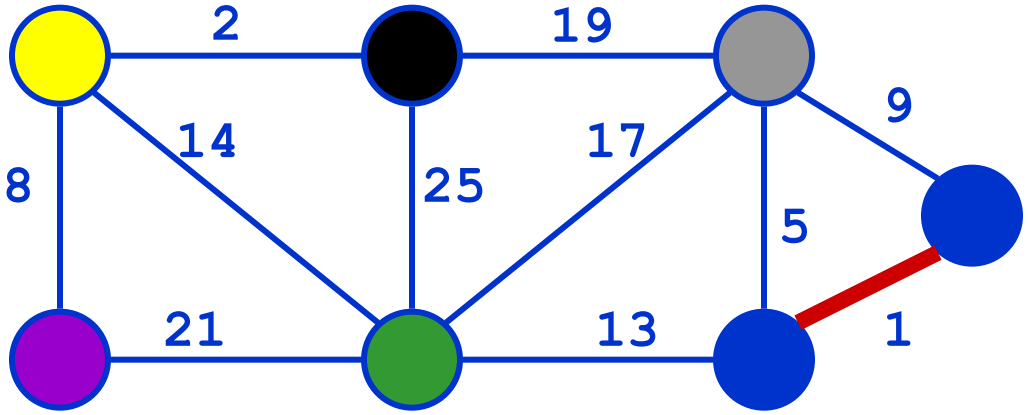

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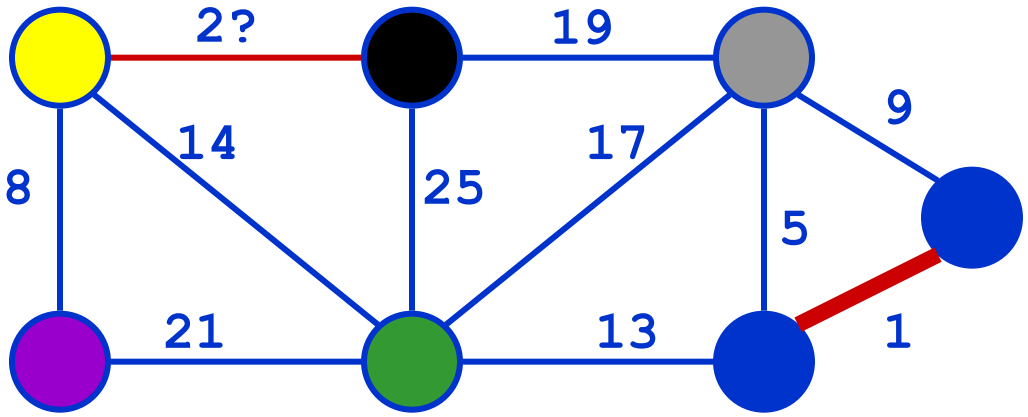

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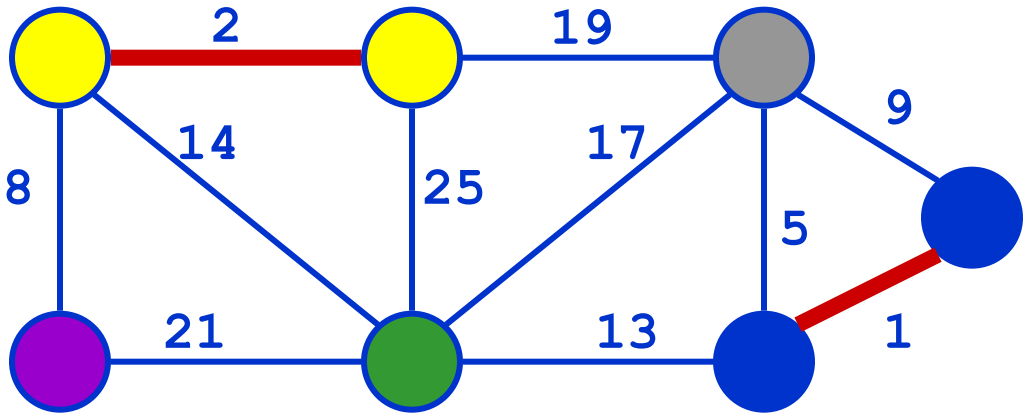

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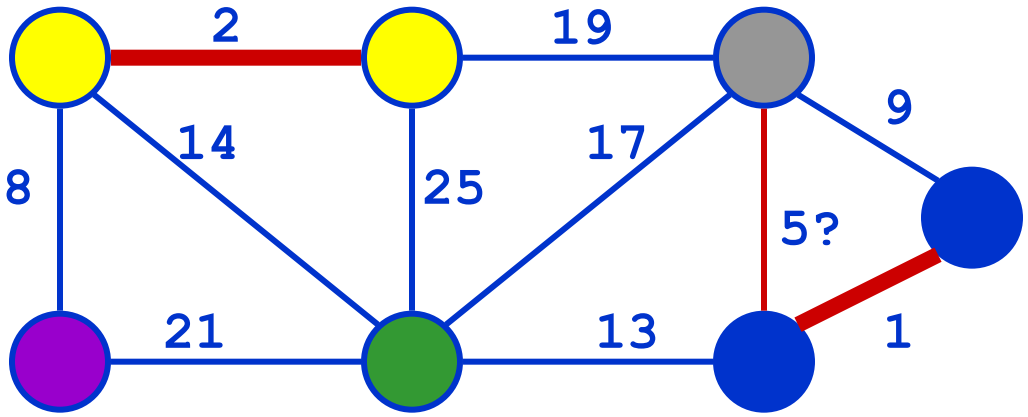

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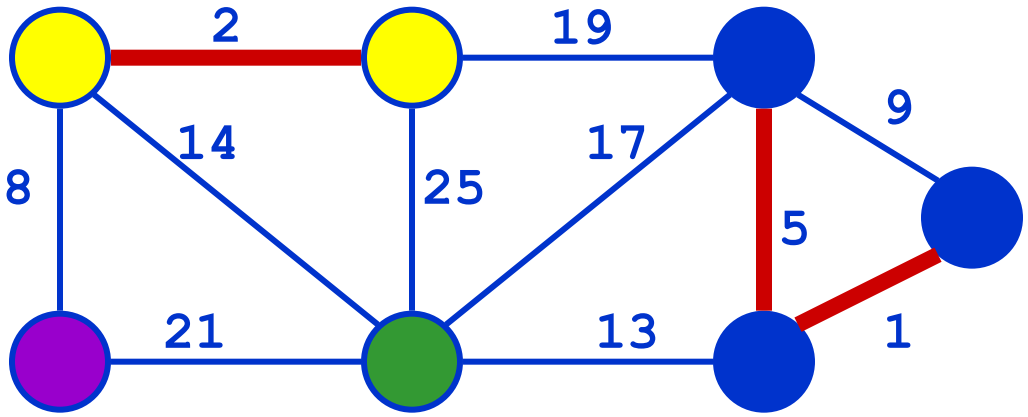

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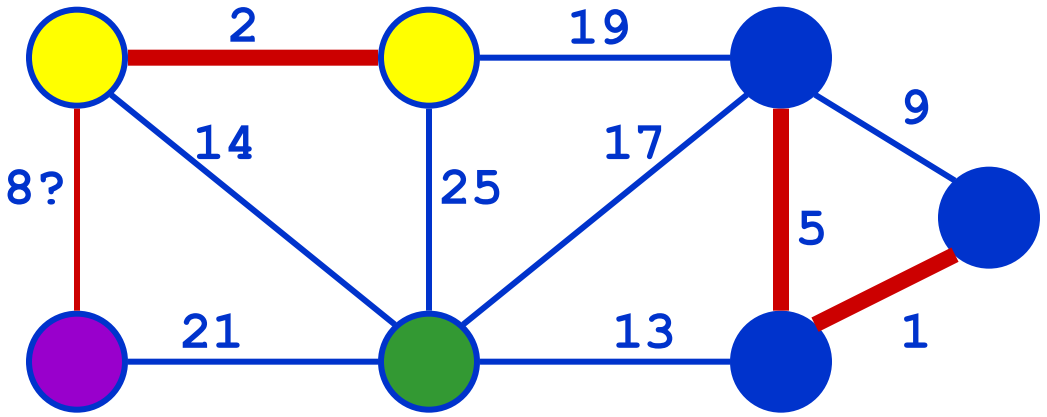

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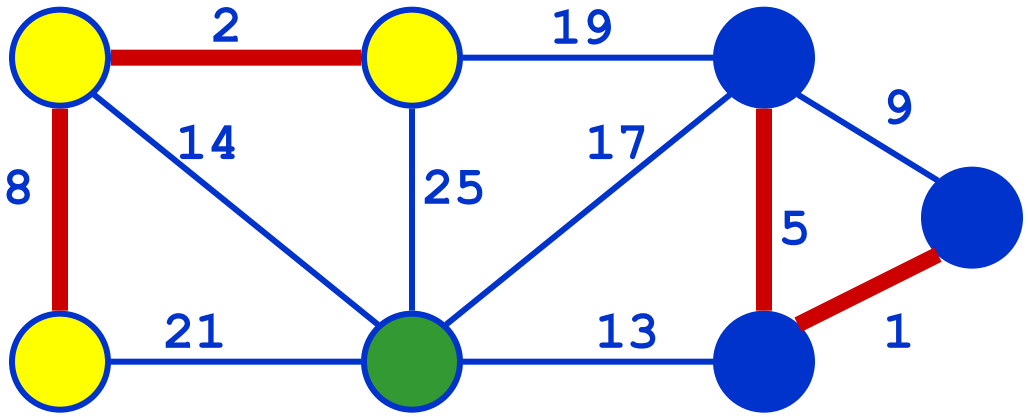

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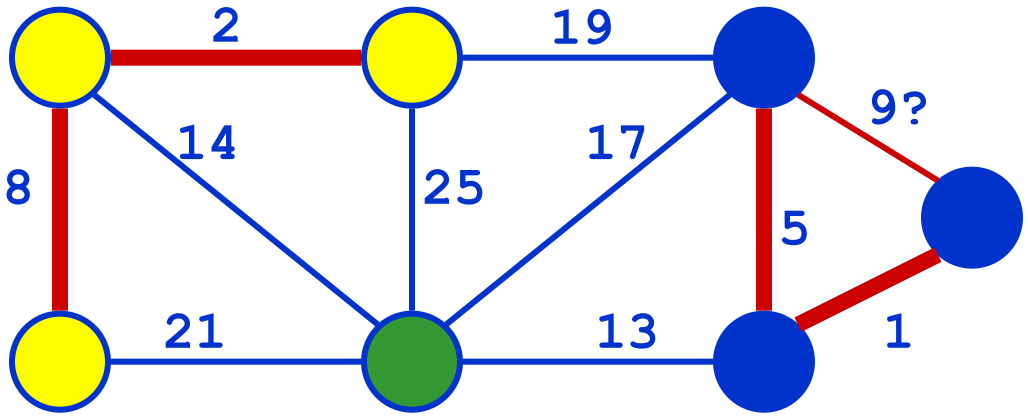

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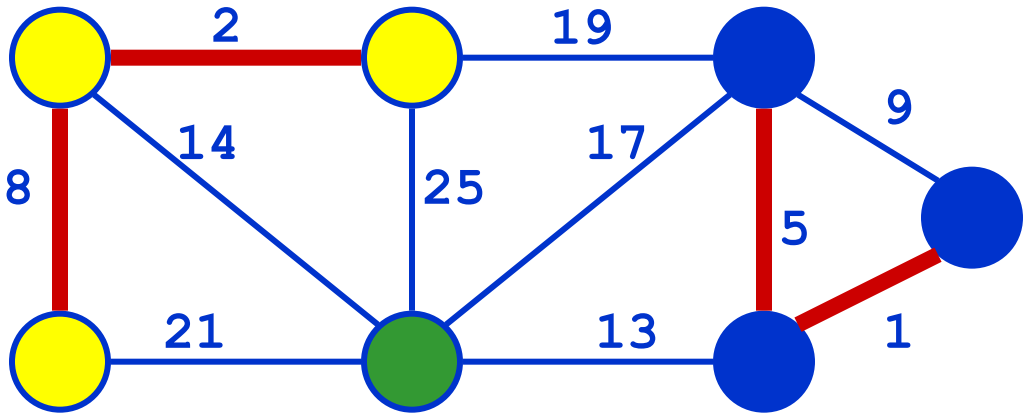

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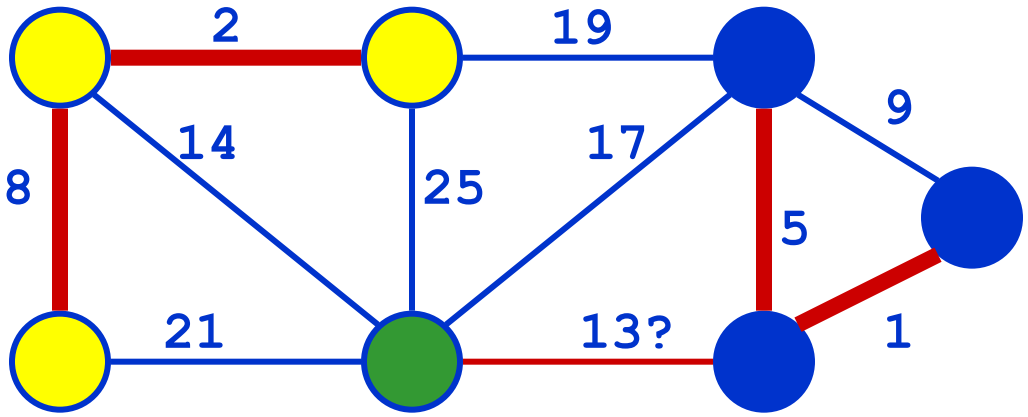

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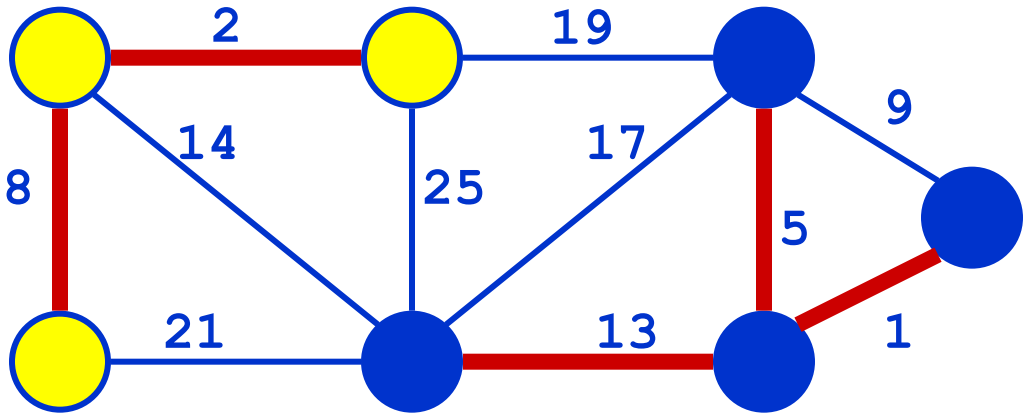

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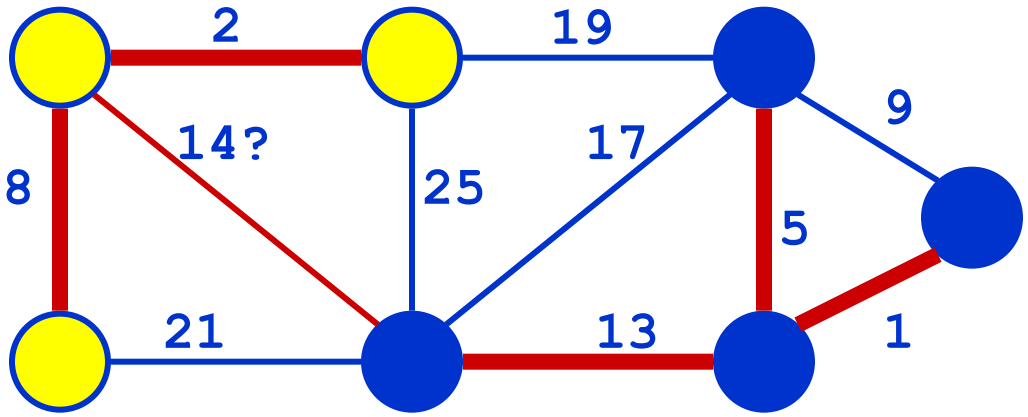

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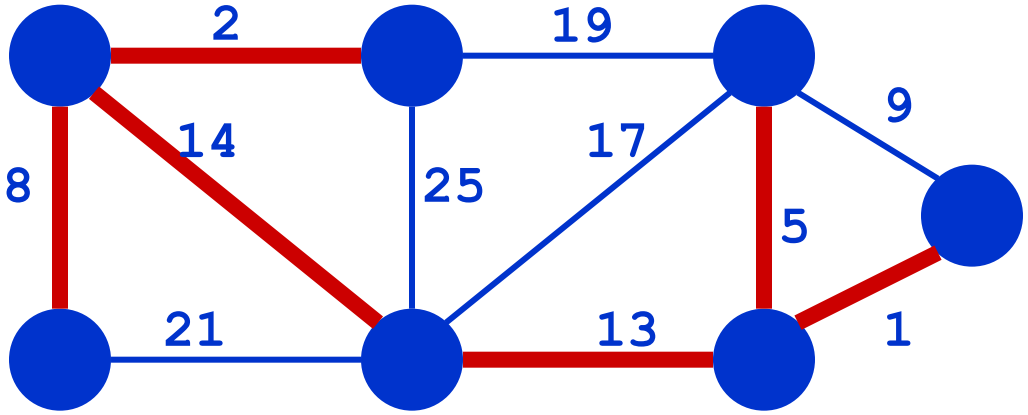

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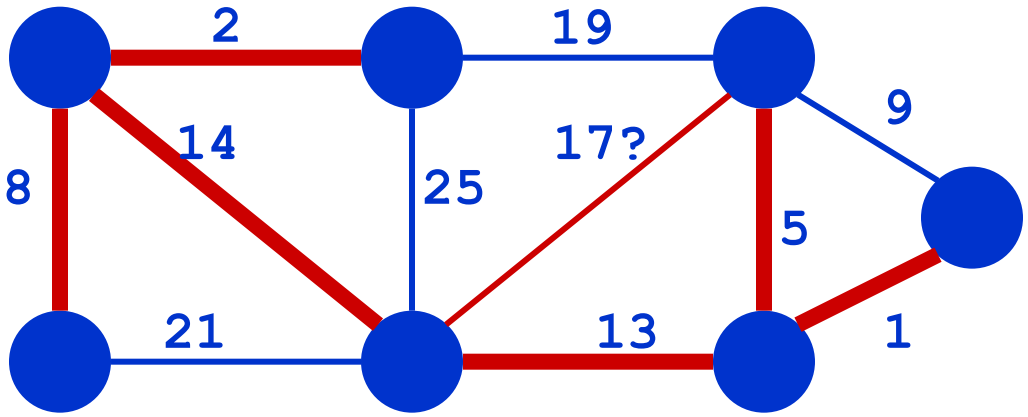

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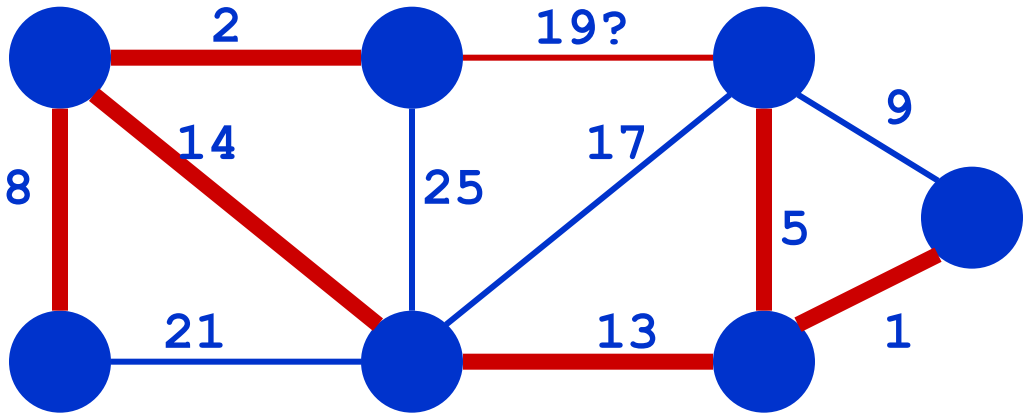

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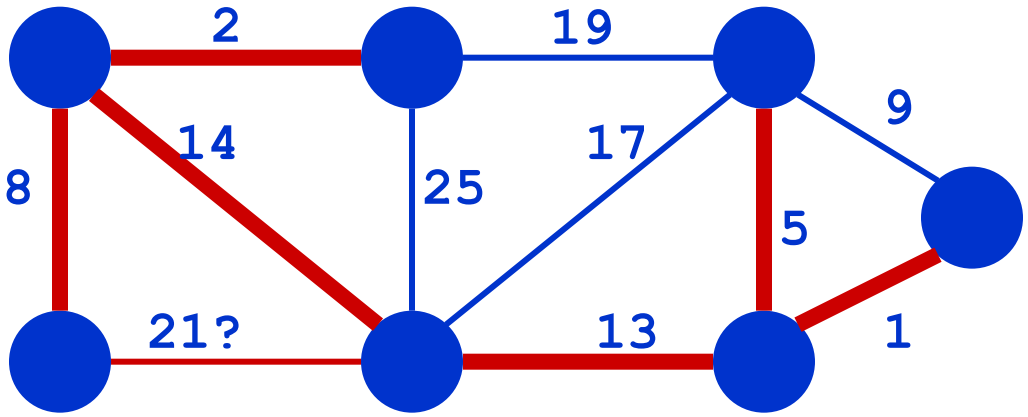

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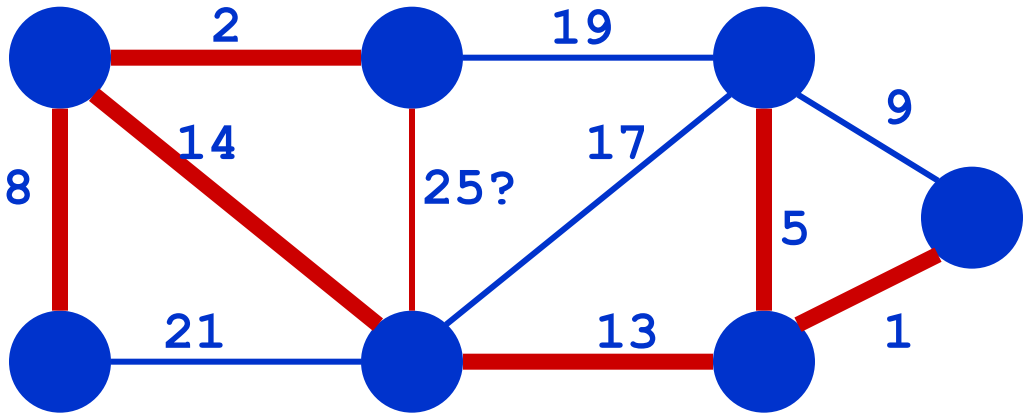

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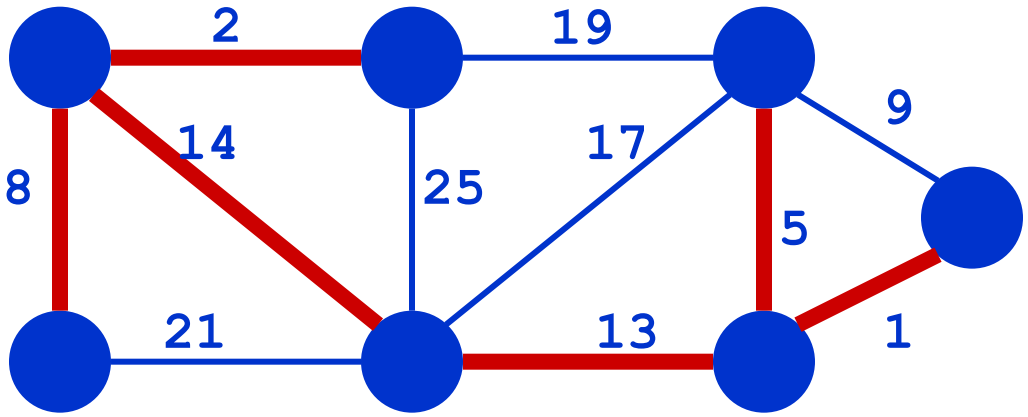

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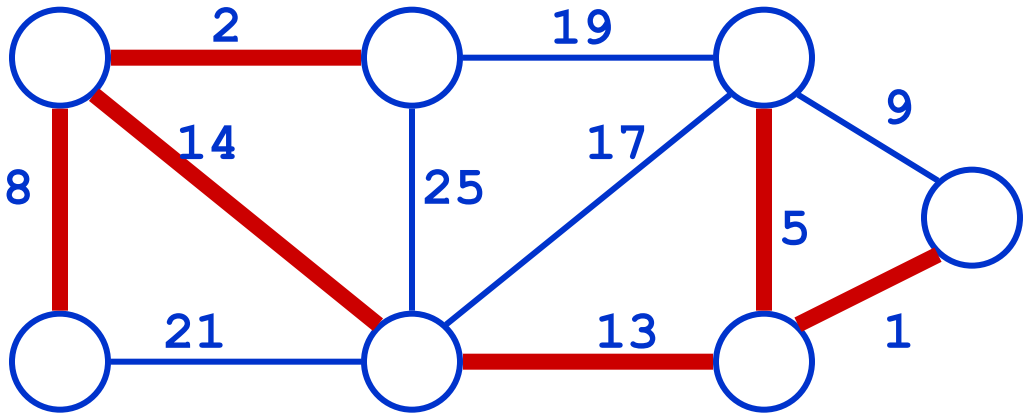

Correctness Of Kruskal's Algorithm

- Sketch of a proof that this algorithm produces an MST for T:
  - Assume algorithm is wrong: result is not an MST
  - Then algorithm adds a wrong edge at some point
  - If it adds a wrong edge, there must be a lower weight edge (cut and paste argument)
  - But algorithm chooses lowest weight edge at each step. Contradiction
- Again, important to be comfortable with cut and paste arguments

Kruskal's Algorithm

What will affect the running time?

```
Kruskal()
{
    T = ∅;
    for each v ∈ V
        MakeSet(v);
    sort E by increasing edge weight w
    for each (u,v) ∈ E (in sorted order)
        if FindSet(u) ≠ FindSet(v)
            T = T U {{u,v}};
            Union(FindSet(u), FindSet(v));
}
```

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What will affect the running time?

1 Sort

O(V) MakeSet() calls

O(E) FindSet() calls

O(V) Union() calls

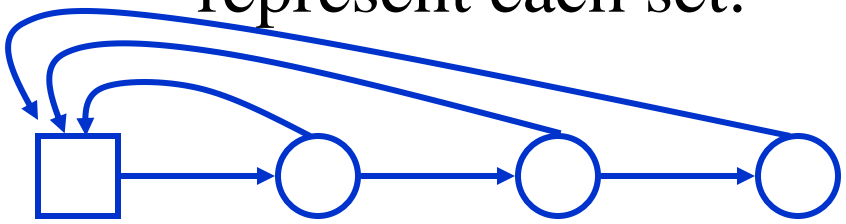
(***Exactly how many Union()s?***)

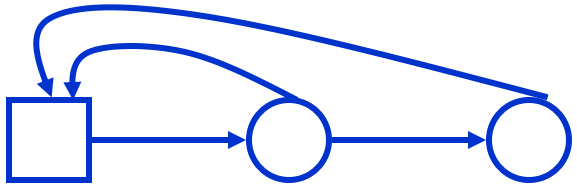
Kruskal's Algorithm: Running Time

- To summarize:
  - Sort edges: O(E lg E)
  - O(V) MakeSet()'s
  - O(E) FindSet()'s
  - O(V) Union()'s
- Upshot:
  - Best disjoint-set union algorithm makes above 3 operations take $O(E \cdot \alpha(E,V))$, α almost constant
  - Overall thus O(E lg E), almost linear w/o sorting

Disjoint Set Union

- So how do we implement disjoint-set union?
  - Naïve implementation: use a linked list to represent each set:





    - MakeSet(): *???* time
    - FindSet(): *???* time
    - Union(A,B): "copy" elements of A into B: *???* time

Disjoint Set Union

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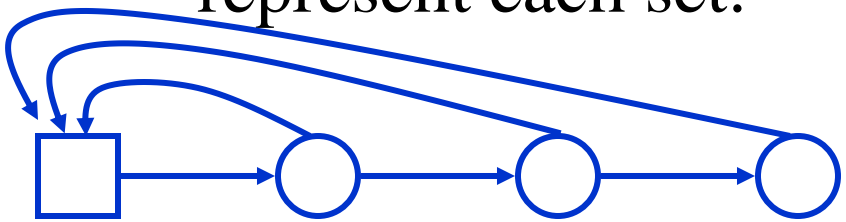
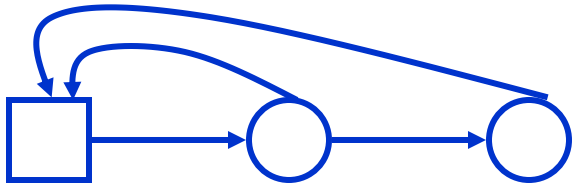
  - Naïve implementation: use a linked list to represent each set:





    - MakeSet(): O(1) time
    - FindSet(): O(1) time
    - Union(A,B): "copy" elements of A into B: O(A) time
  - *How long can a single Union() take?*
  - *How long will n Union()'s take?*

Disjoint Set Union: Analysis

- Worst-case analysis: $O(n^2)$ time for n Union's

Union(S_1, S_2)	"copy"	1 element
Union(S_2, S_3)	"copy"	2 elements
…		
Union(S_{n-1}, S_n)	"copy"	n-1 elements
		$O(n^2)$

- Improvement: always copy smaller into larger
  - *Why will this make things better?*
  - *What is the worst-case time of Union()?*
- But now n Union's take only O(n lg n) time!

Amortized Analysis of Disjoint Sets

- *Amortized analysis* computes average times without using probability
- With our new Union(), any individual element is copied at most *lg n* times when forming the complete set from 1-element sets
  - Worst case: Each time copied, element in smaller set

1st time	resulting set size	≥ 2
2nd time		≥ 4
…		
(lg n)th time		$\geq n$

Amortized Analysis of Disjoint Sets

- Since we have n elements each copied at most lg n times, n Union()'s takes O(n lg n) time
- We say that each Union() takes O(lg n) *amortized time*
  - Financial term: imagine paying $(lg n) per Union
  - At first we are overpaying; initial Union $O(1)
  - But we accumulate enough $ in bank to pay for later expensive O(n) operation.
  - Important: amount in bank never goes negative