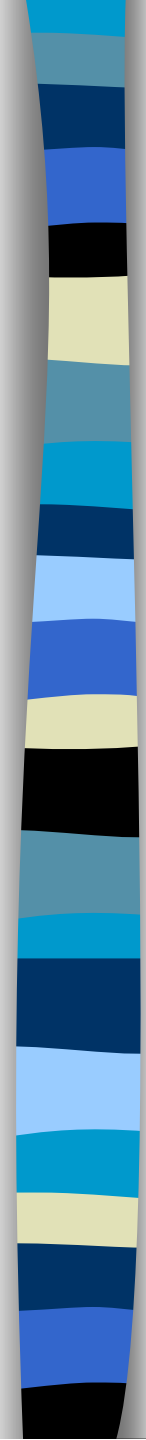


# Algorithms

## Dynamic programming

## 0-1 Knapsack problem

# Review: Dynamic programming

- DP is a method for solving certain kind of problems
- DP can be applied when the solution of a problem includes solutions to subproblems
- We need to find a recursive formula for the solution
- We can recursively solve subproblems, starting from the trivial case, and save their solutions in memory
- In the end we'll get the solution of the whole problem

# Properties of a problem that can be solved with dynamic programming

- Simple Subproblems
  - We should be able to break the original problem to smaller subproblems that have the same structure
- Optimal Substructure of the problems
  - The solution to the problem must be a composition of subproblem solutions
- Subproblem Overlap
  - Optimal subproblems to unrelated problems can contain subproblems in common

# Review: Longest Common Subsequence (LCS)

- Problem: how to find the longest pattern of characters that is common to two text strings X and Y
- Dynamic programming algorithm: solve subproblems until we get the final solution
- Subproblem: first find the LCS of *prefixes* of X and Y.
- this problem has *optimal substructure*: LCS of two prefixes is always a part of LCS of bigger strings

# Review: Longest Common Subsequence (LCS) continued

- Define $X_i$, $Y_j$ to be prefixes of X and Y of length i and j; m = |X|, n = |Y|
- We store the length of LCS($X_i$, $Y_j$) in c[i,j]
- Trivial cases: LCS($X_0$ , $Y_j$ ) and LCS($X_i$, $Y_0$) is empty (so c[0,j] = c[i,0] = 0 )
- Recursive formula for c[i,j]:

$$c[i,j] = \begin{cases} c[i-1, j-1]+1 & \text{if } x[i] = y[j], \\ \max(c[i, j-1], c[i-1, j]) & \text{otherwise} \end{cases}$$

c[m,n] is the final solution

# Review: Longest Common Subsequence (LCS)

- After we have filled the array *c[ ]*, we can use this data to find the characters that constitute the Longest Common Subsequence

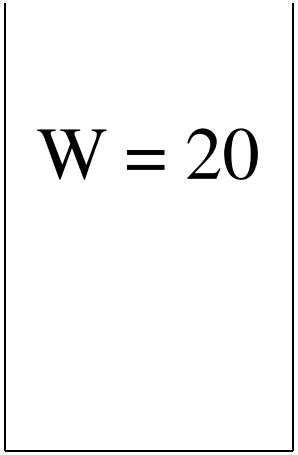
- Algorithm runs in *O(m*n)*, which is *much* better than the brute-force algorithm: *O(n $2^m$)*

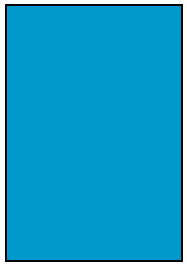
# 0-1 Knapsack problem

- Given a knapsack with maximum capacity $W$, and a set $S$ consisting of $n$ items
- Each item $i$ has some weight $w_i$ and benefit value $b_i$ (all $w_i$ , $b_i$ and $W$ are integer values)
- Problem: How to pack the knapsack to achieve maximum total value of packed items?

# 0-1 Knapsack problem: a picture

This is a knapsack
Max weight: W = 20




| Items | Weight $w_i$ | Benefit value $b_i$ |
|---|---|---|
|  | 2 | 3 |
|  | 3 | 4 |
|  | 4 | 5 |
|  | 5 | 8 |
|  | 9 | 10 |

# 0-1 Knapsack problem

- Problem, in other words, is to find

$$\max \sum_{i \in T} b_i \text{ subject to } \sum_{i \in T} w_i \leq W$$

- The problem is called a *"0-1"* problem, because each item must be entirely accepted or rejected.
- Just another version of this problem is the "*Fractional Knapsack Problem*", where we can take fractions of items.

# 0-1 Knapsack problem: brute-force approach

Let's first solve this problem with a straightforward algorithm

- Since there are $n$ items, there are $2^n$ possible combinations of items.
- We go through all combinations and find the one with the most total value and with total weight less or equal to $W$
- Running time will be $O(2^n)$

# 0-1 Knapsack problem: brute-force approach

- Can we do better?
- Yes, with an algorithm based on dynamic programming
- We need to carefully identify the subproblems

Let's try this:

If items are labeled *1..n*, then a subproblem would be to find an optimal solution for

$S_k$ = *{items labeled 1, 2, .. k}*

# Defining a Subproblem

If items are labeled *1..n*, then a subproblem would be to find an optimal solution for $S_k$ = *{items labeled 1, 2, .. k}*

- This is a valid subproblem definition.
- The question is: can we describe the final solution ($S_n$ ) in terms of subproblems ($S_k$)?
- Unfortunately, we can't do that. Explanation follows….

# Defining a Subproblem

| | | | | |
|---|---|---|---|---|
| $w_1 = 2$<br>$b_1 = 3$ | $w_2$<br>$= 4$<br>$b_2 = 5$ | $w_3 = 5$<br>$b_3 = 8$ | $w_4 = 3$<br>$b_4 = 4$ | |

?

Max weight: W = 20

**For $S_4$:**

Total weight: 14;

total benefit: 20

| Item # | Weight $w_i$ | Benefit $b_i$ |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 3 | 4 |
| 3 | 4 | 5 |
| 4 | 5 | 8 |
| 5 | 9 | 10 |

$S_4$

$S_5$

| | | | |
|---|---|---|---|
| $w_1 = 2$<br>$b_1 = 3$ | $w_2$<br>$= 4$<br>$b_2 = 5$ | $w_3 = 5$<br>$b_3 = 8$ | $w_4 = 9$<br>$b_4 = 10$ |

**For $S_5$:**

Total weight: 20

total benefit: 26

Solution for $S_4$ is not part of the solution for $S_5$!!!

# Defining a Subproblem (continued)

- As we have seen, the solution for $S_4$ is not part of the solution for $S_5$
- So our definition of a subproblem is flawed and we need another one!
- Let's add another parameter: $w$, which will represent the exact weight for each subset of items
- The subproblem then will be to compute $B[k,w]$

# Recursive Formula for subproblems

- Recursive formula for subproblems:

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{ B[k-1,w], B[k-1,w-w_k] + b_k \} & \text{else} \end{cases}$$

- It means, that the best subset of $S_k$ that has total weight $w$ is one of the two:

1) the best subset of $S_{k-1}$ that has total weight $w$, **or**

2) the best subset of $S_{k-1}$ that has total weight $w$-$w_k$ plus the item $k$

# Recursive Formula

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{ B[k-1,w], B[k-1,w-w_k] + b_k \} & \text{else} \end{cases}$$

- The best subset of $S_k$ that has the total weight $w$, either contains item $k$ or not.
- First case: $w_k > w$. Item $k$ can't be part of the solution, since if it was, the total weight would be $> w$, which is unacceptable
- Second case: $w_k <= w$. Then the item $k$ can be in the solution, and we choose the case with greater value

# 0-1 Knapsack Algorithm

```
for w = 0 to W
    B[0,w] = 0
for i = 0 to n
    B[i,0] = 0
    for w = 0 to W
        if w_i <= w // item i can be part of the solution
            if b_i + B[i-1,w-w_i] > B[i-1,w]
                B[i,w] = b_i + B[i-1,w- w_i]
            else
                B[i,w] = B[i-1,w]
        else B[i,w] = B[i-1,w]  // w_i > w
```

# Running time

```
for w = 0 to W
    B[0,w] = 0
for i = 0 to n
    B[i,0] = 0
    for w = 0 to W
        < the rest of the code >
```

*O(W)*

Repeat *n* times

*O(W)*

What is the running time of this algorithm?

O(n*W)

Remember that the brute-force algorithm takes $O(2^n)$

# Example

Let's run our algorithm on the following data:

n = 4 (# of elements)
W = 5 (max weight)
Elements (weight, benefit):
(2,3), (3,4), (4,5), (5,6)

# Example (2)

| W \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | | | | |
| 1 | 0 | | | | |
| 2 | 0 | | | | |
| 3 | 0 | | | | |
| 4 | 0 | | | | |
| 5 | 0 | | | | |

for w = 0 to W

B[0,w] = 0

# Example (3)

| W \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | | | | |
| 2 | 0 | | | | |
| 3 | 0 | | | | |
| 4 | 0 | | | | |
| 5 | 0 | | | | |

for i = 0 to n

B[i,0] = 0

# Example (4)

Items:

| |
|---|
| 1: (2,3) |
| 2: (3,4) |
| 3: (4,5) |
| 4: (5,6) |

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 → | 0 | | | |
| 2 | 0 | | | | |
| 3 | 0 | | | | |
| 4 | 0 | | | | |
| 5 | 0 | | | | |

$i=1$

$b_i=3$

$w_i=2$

$w=1$

$w-w_i=-1$

```
if w_i <= w // item i can be part of the solution
    if b_i + B[i-1,w-w_i] > B[i-1,w]
        B[i,w] = b_i + B[i-1,w- w_i]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

# Example (5)

Items:

| |
|---|
| 1: (2,3) |
| 2: (3,4) |
| 3: (4,5) |
| 4: (5,6) |

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | | | |
| 2 | 0 | **3** | | | |
| 3 | 0 | | | | |
| 4 | 0 | | | | |
| 5 | 0 | | | | |

$i=1$

$b_i=3$

$w_i=2$

$w=2$

$w-w_i=0$

```
if w_i <= w // item i can be part of the solution
   if b_i + B[i-1,w-w_i] > B[i-1,w]
      B[i,w] = b_i + B[i-1,w- w_i]
   else
      B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

# Example (6)

Items:

| |
|---|
| 1: (2,3) |
| 2: (3,4) |
| 3: (4,5) |
| 4: (5,6) |

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | | | |
| 2 | 0 | 3 | | | |
| 3 | 0 | **3** | | | |
| 4 | 0 | | | | |
| 5 | 0 | | | | |

$i=1$

$b_i=3$

$w_i=2$

$w=3$

$w-w_i=1$

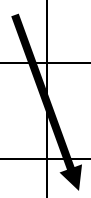
```
if w_i <= w // item i can be part of the solution
   if b_i + B[i-1,w-w_i] > B[i-1,w]
      B[i,w] = b_i + B[i-1,w- w_i]
   else
      B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

# Example (7)

Items:

| |
|---|
| 1: (2,3) |
| 2: (3,4) |
| 3: (4,5) |
| 4: (5,6) |

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | | | |
| 2 | 0 | 3 | | | |
| 3 | 0 | 3 | | | |
| 4 | 0 | **3** | | | |
| 5 | 0 | | | | |



$i=1$

$b_i=3$

$w_i=2$

$w=4$

$w-w_i=2$

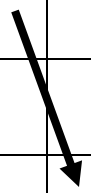
if $w_i <= w$ // item i can be part of the solution
  if $b_i + B[i-1,w-w_i] > B[i-1,w]$
    **$B[i,w] = b_i + B[i-1,w- w_i]$**
  else
    $B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

# Example (8)

Items:

| |
|---|
| 1: (2,3) |
| 2: (3,4) |
| 3: (4,5) |
| 4: (5,6) |

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | | | |
| 2 | 0 | 3 | | | |
| 3 | 0 | 3 | | | |
| 4 | 0 | 3 | | | |
| 5 | 0 | **3** | | | |



$i=1$

$b_i=3$

$w_i=2$

$w=5$

$w-w_i=2$

if $w_i <= w$ // item i can be part of the solution
  if $b_i + B[i-1,w-w_i] > B[i-1,w]$
    **$B[i,w] = b_i + B[i-1,w- w_i]$**
  else
    $B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

# Example (9)

Items:

| |
|---|
| 1: (2,3) |
| 2: (3,4) |
| 3: (4,5) |
| 4: (5,6) |

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | → **0** | | |
| 2 | 0 | 3 | | | |
| 3 | 0 | 3 | | | |
| 4 | 0 | 3 | | | |
| 5 | 0 | 3 | | | |

i=2

$b_i$=4

$w_i$=3

w=1

$w-w_i$=-2

```
if w_i <= w // item i can be part of the solution
    if b_i + B[i-1,w-w_i] > B[i-1,w]
        B[i,w] = b_i + B[i-1,w- w_i]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

# Example (10)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

| W \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | | |
| 2 | 0 | 3 | **3** | | |
| 3 | 0 | 3 | | | |
| 4 | 0 | 3 | | | |
| 5 | 0 | 3 | | | |

$i=2$

$b_i=4$

$w_i=3$

$w=2$

$w-w_i=-1$

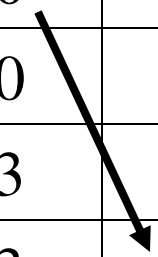
```
if w_i <= w // item i can be part of the solution
    if b_i + B[i-1,w-w_i] > B[i-1,w]
        B[i,w] = b_i + B[i-1,w- w_i]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

# Example (11)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | | |
| 2 | 0 | 3 | 3 | | |
| 3 | 0 | 3 | **4** | | |
| 4 | 0 | 3 | | | |
| 5 | 0 | 3 | | | |



$i=2$

$b_i=4$

$w_i=3$

$w=3$

$w-w_i=0$

if $w_i$ <= w // item i can be part of the solution
  if $b_i$ + B[i-1,w-$w_i$] > B[i-1,w]
    **B[i,w] = $b_i$ + B[i-1,w- $w_i$]**
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // $w_i$ > w

# Example (12)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | | |
| 2 | 0 | 3 | 3 | | |
| 3 | 0 | 3 | 4 | | |
| 4 | 0 | 3 | **4** | | |
| 5 | 0 | 3 | | | |

$i=2$

$b_i=4$

$w_i=3$

$w=4$

$w-w_i=1$

if $w_i <= w$ // item i can be part of the solution
  if $b_i + B[i-1,w-w_i] > B[i-1,w]$
    **$B[i,w] = b_i + B[i-1,w- w_i]$**
  else
    $B[i,w] = B[i-1,w]$
else $B[i,w] = B[i-1,w]$ // $w_i > w$

# Example (13)

Items:

| | |
|---|---|
| 1: | (2,3) |
| 2: | (3,4) |
| 3: | (4,5) |
| 4: | (5,6) |

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | | |
| 2 | 0 | 3 | 3 | | |
| 3 | 0 | 3 | 4 | | |
| 4 | 0 | 3 | 4 | | |
| 5 | 0 | 3 | **7** | | |

$i=2$

$b_i=4$

$w_i=3$

$w=5$

$w-w_i=2$

```
if w_i <= w // item i can be part of the solution
    if b_i + B[i-1,w-w_i] > B[i-1,w]
        B[i,w] = b_i + B[i-1,w- w_i]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

# Example (14)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 → | **0** | |
| 2 | 0 | 3 | 3 → | **3** | |
| 3 | 0 | 3 | 4 → | **4** | |
| 4 | 0 | 3 | 4 | | |
| 5 | 0 | 3 | 7 | | |

$i=3$

$b_i=5$

$w_i=4$

$w=1..3$

```
if w_i <= w // item i can be part of the solution
    if b_i + B[i-1,w-w_i] > B[i-1,w]
        B[i,w] = b_i + B[i-1,w- w_i]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

# Example (15)

Items:

| |
|---|
| 1: (2,3) |
| 2: (3,4) |
| 3: (4,5) |
| 4: (5,6) |

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | |
| 2 | 0 | 3 | 3 | 3 | |
| 3 | 0 | 3 | 4 | 4 | |
| 4 | 0 | 3 | 4 | **5** | |
| 5 | 0 | 3 | 7 | | |

$i=3$

$b_i=5$

$w_i=4$

$w=4$

$w- w_i=0$

```
if w_i <= w // item i can be part of the solution
    if b_i + B[i-1,w-w_i] > B[i-1,w]
        B[i,w] = b_i + B[i-1,w- w_i]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

# Example (15)

Items:

1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | |
| 2 | 0 | 3 | 3 | 3 | |
| 3 | 0 | 3 | 4 | 4 | |
| 4 | 0 | 3 | 4 | 5 | |
| 5 | 0 | 3 | 7 | → 7 | |

$i=3$

$b_i=5$

$w_i=4$

$w=5$

$w- w_i=1$

if $w_i <= w$ // item i can be part of the solution
  if $b_i + B[i-1,w-w_i] > B[i-1,w]$
    $B[i,w] = b_i + B[i-1,w- w_i]$
  else
    **B[i,w] = B[i-1,w]**
else B[i,w] = B[i-1,w] // $w_i > w$

# Example (16)

Items:

| |
|---|
| 1: (2,3) |
| 2: (3,4) |
| 3: (4,5) |
| 4: (5,6) |

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | **0** |
| 2 | 0 | 3 | 3 | 3 | **3** |
| 3 | 0 | 3 | 4 | 4 | **4** |
| 4 | 0 | 3 | 4 | 5 | **5** |
| 5 | 0 | 3 | 7 | 7 | |

$i=3$

$b_i=5$

$w_i=4$

$w=1..4$

```
if w_i <= w // item i can be part of the solution
    if b_i + B[i-1,w-w_i] > B[i-1,w]
        B[i,w] = b_i + B[i-1,w- w_i]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

# Example (17)

Items:

| |
|---|
| 1: (2,3) |
| 2: (3,4) |
| 3: (4,5) |
| 4: (5,6) |

| w \ i | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 3 | 3 | 3 | 3 |
| 3 | 0 | 3 | 4 | 4 | 4 |
| 4 | 0 | 3 | 4 | 5 | 5 |
| 5 | 0 | 3 | 7 | 7 → | **7** |

$i=3$

$b_i=5$

$w_i=4$

$w=5$

```
if w_i <= w // item i can be part of the solution
    if b_i + B[i-1,w-w_i] > B[i-1,w]
        B[i,w] = b_i + B[i-1,w- w_i]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

# Comments

- This algorithm only finds the max possible value that can be carried in the knapsack
- To know the items that make this maximum value, an addition to this algorithm is necessary
- Please see LCS algorithm from the previous lecture for the example how to extract this data from the table we built

# Conclusion

- Dynamic programming is a useful technique of solving certain kind of problems
- When the solution can be recursively described in terms of partial solutions, we can store these partial solutions and re-use them as necessary
- Running time (Dynamic Programming algorithm vs. naïve algorithm):
  - LCS: **O(m*n)** vs. **O(n * $2^m$)**
  - 0-1 Knapsack problem: **O(W*n)** vs. **O($2^n$)**