ELEMENTARY FLUID DYNAMICS:
THE BERNOULLI EQUATIO

Bernoulli Along a Streamline

$$-\nabla p = \rho \mathbf{a} + g\hat{\mathbf{k}} \quad \text{(eqn 2.2)}$$

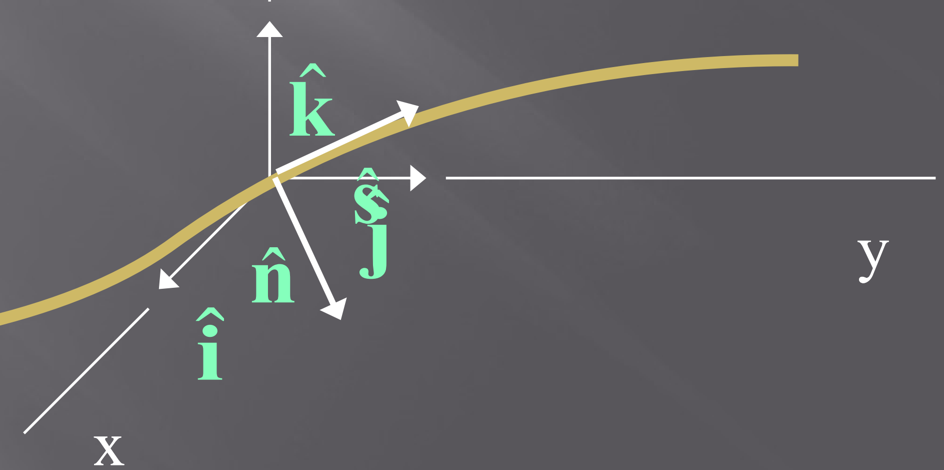
Separate acceleration due to gravity. Coordinate system may be in any orientation!

$$-\frac{\partial p}{\partial s} = \rho a_s + g\frac{dz}{ds}$$

Component of *g* in *s* direction

Note: No shear forces! Therefore flow must be frictionless.

Steady state (no change in *p* wrt time)



Bernoulli
Along a Streamline

$$-\frac{\partial p}{\partial s} = \rho a_s + \gamma \frac{dz}{ds}$$

Can we eliminate the partial derivative?

chain rule

$$a_s = \frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} = \frac{\partial V}{\partial s} V$$

Write acceleration as derivative wrt *s*

$$dp = \frac{\partial p}{\partial s} ds + \frac{\partial p}{\partial n} dn$$

0 (*n* is constant along streamline)

$\therefore \; dp/ds = \partial p/\partial s$ and $dV/ds = \partial V/\partial s$

$$-\frac{dp}{ds} = \rho V \frac{dV}{ds} + \gamma \frac{dz}{ds}$$

$$V\frac{dV}{ds} = \frac{1}{2}\frac{d\left(V^2\right)}{ds}$$

Integrate *F=ma* Along a Streamline

$$-\frac{dp}{ds} = \frac{1}{2}\rho\frac{d\left(V^2\right)}{ds} + \gamma\frac{dz}{ds}$$

Eliminate *ds*

$$dp + \frac{1}{2}\rho\, d\left(V^2\right) + \gamma\, dz = 0$$

Now let's integrate…
But density is a function of pressure.

$$\int\frac{dp}{\rho} + \frac{1}{2}\int d\left(V^2\right) + g\int dz = 0$$

$$\int\frac{dp}{\rho} + \frac{1}{2}V^2 + gz = C$$

If density is constant…

$$p + \frac{1}{2}\rho V^2 + \gamma z = C$$

Along a streamline

Bernoulli Equation

- Assumptions needed for Bernoulli Equation

  - Inviscid (frictionless)
  - Steady
  - Constant density (incompressible)
  - Along a streamline

- Eliminate the constant in the Bernoulli equation?

  Apply at two points along a streamline.

- Bernoulli equation does not include
  - Mechanical energy to thermal energy
  - Heat transfer, shaft work

Bernoulli Equation

The Bernoulli Equation is a statement of the conservation of Mechanical Energy

$$\frac{p}{\rho} + gz + \frac{1}{2}V^2 = C$$

p.e. k.e.

$$\frac{p}{\gamma} + z + \frac{V^2}{2g} = C$$

$\frac{p}{\gamma}$ = Pressure head

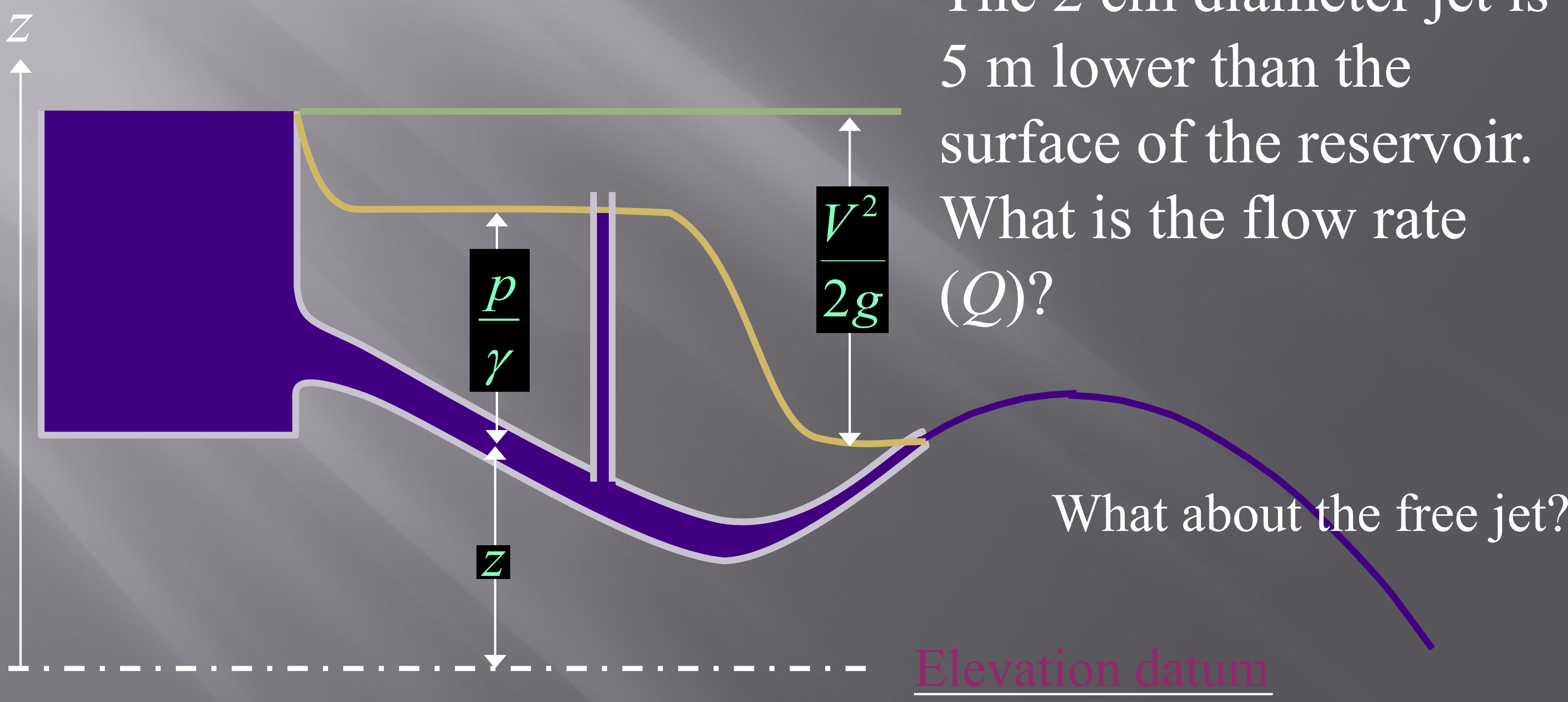
z = Elevation head

$\frac{V^2}{2g}$ = Velocity head

$\frac{p}{\gamma} + z$ = Piezometric head

$\frac{p}{\gamma} + z + \frac{V^2}{2g}$ = Total head

Hydraulic and Energy Grade Lines (neglecting losses for now)

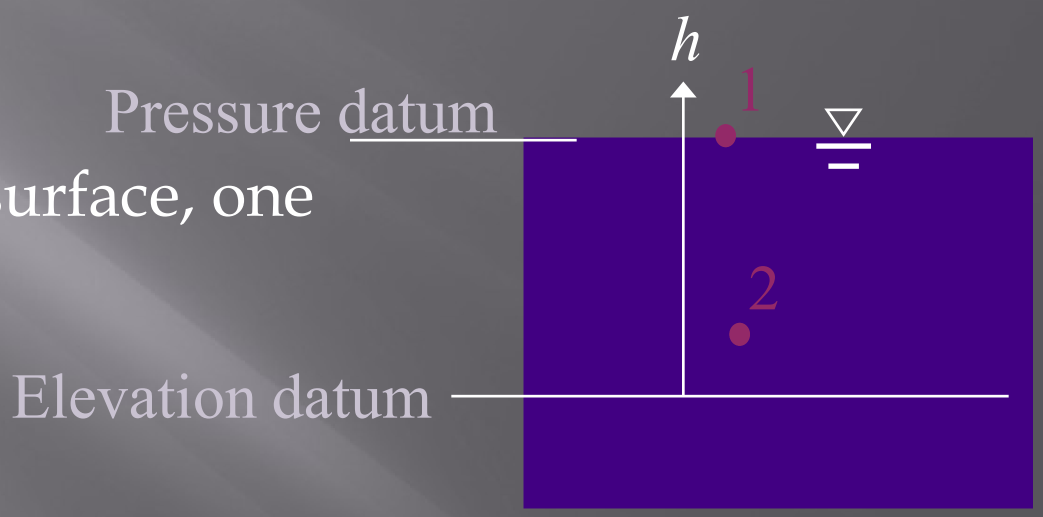


The 2 cm diameter jet is 5 m lower than the surface of the reservoir. What is the flow rate (Q)?

Pressure datum? Atmospheric pressure

Bernoulli Equation: Simple Case ($V = 0$)

- Reservoir ($V = 0$)
  - Put one point on the surface, one point anywhere else



$$\frac{p}{g} + z + \cancel{\frac{V^2}{2g}} = C$$

$$\cancel{\frac{p_1}{g}} + z_1 = \frac{p_2}{g} + z_2$$

We didn’t cross any streamlines so this analysis is okay!

$$z_1 - z_2 = \frac{p_2}{g}$$

Same as we found using statics

Bernoulli Equation: Simple Case (p = 0 or constant)

- What is an example of a fluid experiencing a change in elevation, but remaining at a constant pressure? _________

Free jet

$$\frac{p_1}{g} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{g} + z_2 + \frac{V_2^2}{2g}$$

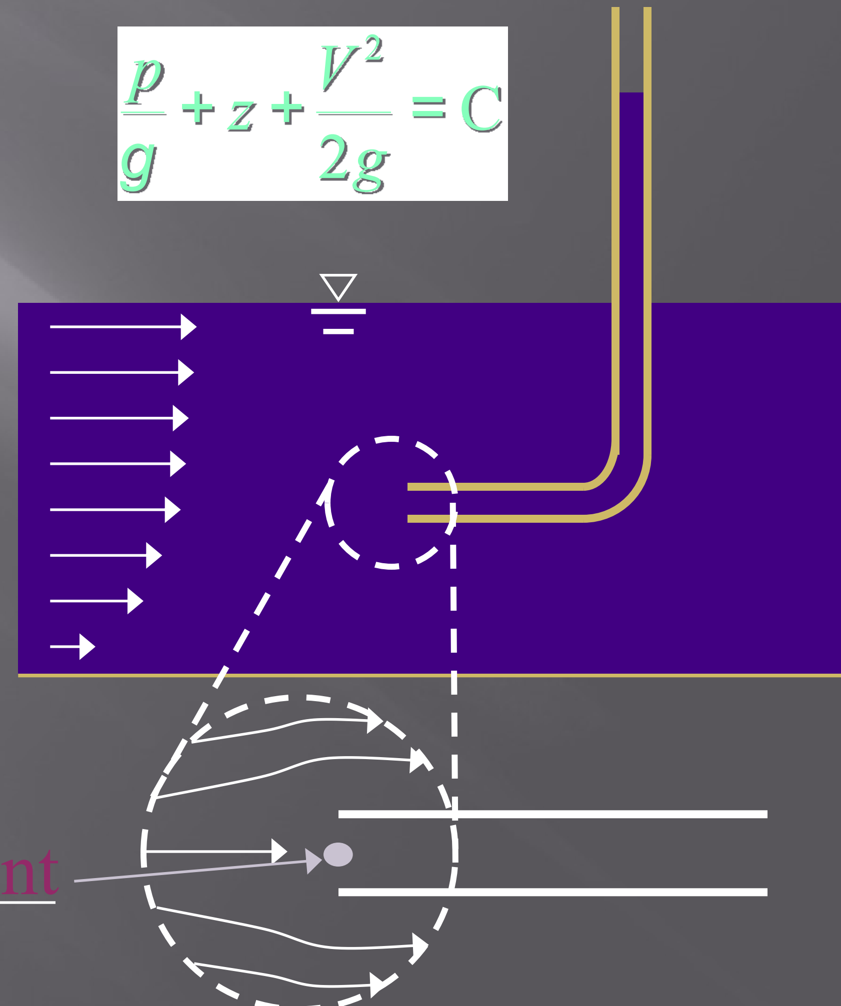
$$z_1 + \frac{V_1^2}{2g} = z_2 + \frac{V_2^2}{2g}$$

$$V_2 = \sqrt{2g(z_1 - z_2) + V_1^2}$$

Bernoulli Equation Application: Stagnation Tube

- What happens when the water starts flowing in the channel?
- Does the orientation of the tube matter? ________ Yes!
- How high does the water rise in the stagnation tube?
- How do we choose the points on the streamline?

$$\frac{p}{g} + z + \frac{V^2}{2g} = C$$

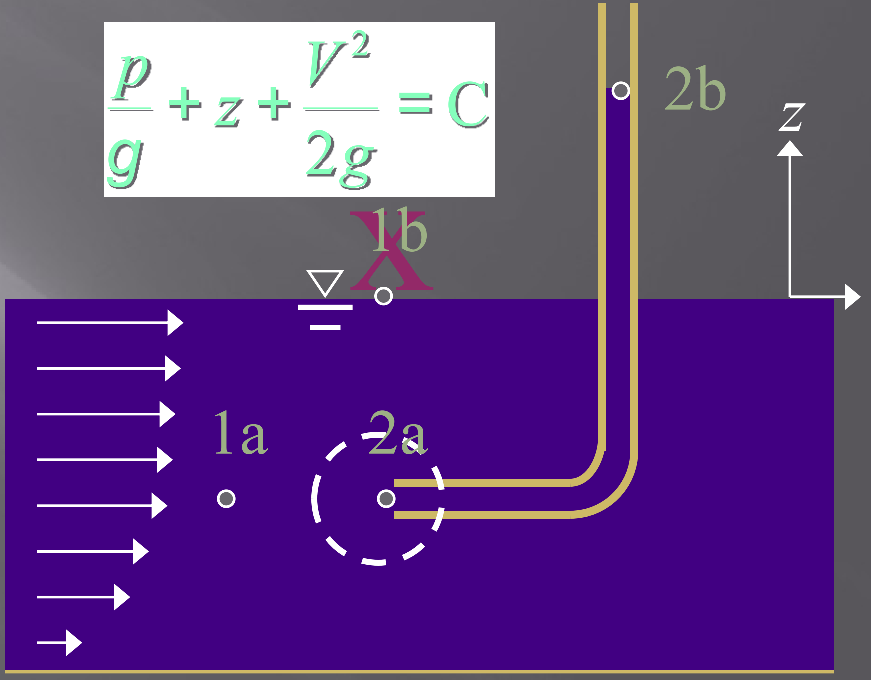


Stagnation point

Bernoulli Equation Application: Stagnation Tube

- 1a-2a
  - Same streamline
- 1b-2a
  - ________________ Crosses streamlines
- 1a-2b
  - ________________ Doesn't cross streamlines

$$\frac{p}{g} + z + \frac{V^2}{2g} = C$$



$$\frac{p_1}{g} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{g} + z_2 + \frac{V_2^2}{2g}$$

1. We can obtain V_1 if p_1 and (z_2-z_1) are known
2. z_2 is the total energy!

Stagnation Tube

- Great for measuring ____________________
- How could you measure *Q*?
- Could you use a stagnation tube in a pipeline?
  - What problem might you encounter?
  - **How could you modify the stagnation tube to solve** the problem?

Bernoulli
Normal to the Streamlines

$$-\nabla p = \rho \mathbf{a} + \gamma \hat{\mathbf{k}}$$

$$-\frac{\partial p}{\partial n} = \rho a_n + \gamma \frac{dz}{dn}$$

Separate acceleration due to gravity. Coordinate system may be in any orientation!

Component of *g* in *n* direction

Bernoulli
Normal to the Streamlines

$$-\frac{\partial p}{\partial n} = \rho\, a_n + \gamma \frac{dz}{dn}$$

$$a_n = \frac{V^2}{\mathcal{R}}$$

centrifugal force. $\mathcal{R}$ is local radius of curvature

n is toward the center of the radius of curvature

0 (s is constant along streamline)

$$dp = \frac{\partial p}{\partial s} ds + \frac{\partial p}{\partial n} dn$$

$\therefore \; dp/dn = \partial p/\partial n$ and $dV/dn = \partial V/\partial n$

$$-\frac{dp}{dn} = \rho \frac{V^2}{\mathcal{R}} + \gamma \frac{dz}{dn}$$

Integrate *F=ma* Normal to the Streamlines

$$-\frac{dp}{dn} = \rho\frac{V^2}{\Re} + \gamma\frac{dz}{dn}$$

Multiply by *dn*

$$\int\frac{dp}{\rho} + \int\frac{V^2}{\Re}dn + \int g\,dz = C$$

Integrate

$$\frac{p}{\rho} + \int\frac{V^2}{\Re}dn + gz = C$$

If density is constant…

$$p + \rho\int\frac{V^2}{\Re}dn + \gamma z = C$$

Normal to streamline

Pressure Change Across Streamlines

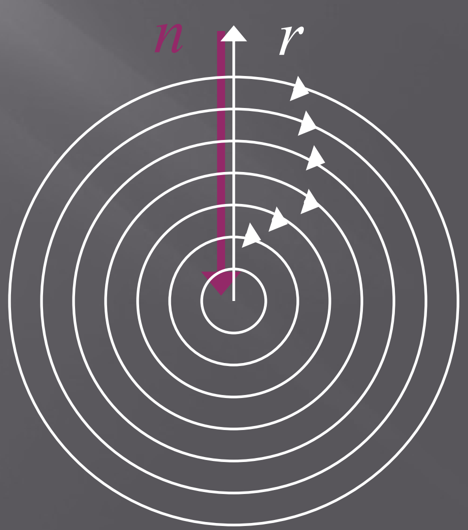
$$p + \rho \int \frac{V^2}{R} dn + \gamma z = C$$

If you cross streamlines that are straight and parallel, then $p + \gamma z = C$ and the pressure is hydrostatic.

$$p - \rho C_1^2 \int r dr + \gamma z = C$$

$$V(r) = C_1 r$$

$$dn = -dr$$



$$p - \frac{\rho C_1^2}{2} r^2 + \gamma z = C$$

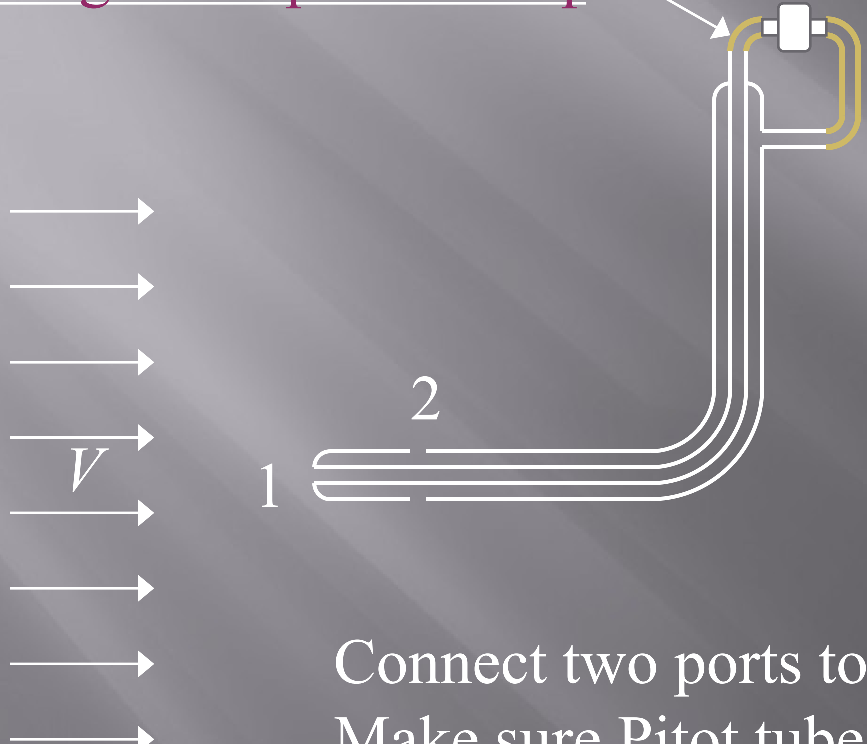
As r decreases p decreases

Pitot Tubes

- Used to measure air speed on airplanes
- Can connect a differential pressure transducer to directly measure $V^2/2g$
- Can be used to measure the flow of water in pipelines

Point measurement!

Pitot Tube



$$\frac{p_1}{g} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{g} + z_2 + \frac{V_2^2}{2g}$$

$V_1 = \underline{0}$

$z_1 = z_2$

$$V = \sqrt{\frac{2}{r}\left(p_1 - p_2\right)}$$

Connect two ports to differential pressure transducer.
Make sure Pitot tube is completely filled with the fluid that is being measured.
Solve for velocity as function of pressure difference

Relaxed Assumptions for Bernoulli Equation

- Frictionless
- Steady
- Constant density (incompressible)
- Along a streamline

Viscous energy loss must be small

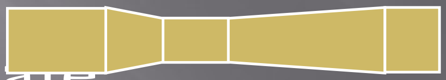
Or gradually varying

Small changes in density

Don’t cross streamlines

Bernoulli Equation Applications

- Stagnation tube
- Pitot tube
- Free Jets
- Orifice
- Venturi
- Sluice gate
- Sharp-crested weir

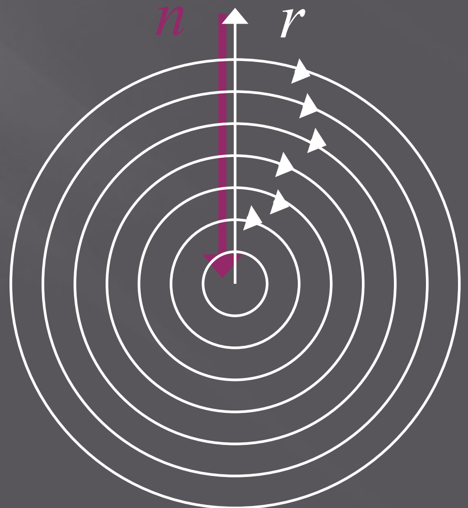


Applicable to contracting streamlines (accelerating flow).

Teams

Ping Pong Ball

Why does the ping pong ball try to return to the center of the jet?
What forces are acting on the ball when it is not centered on the jet?



How does the ball choose the distance above the source of the jet?

Summary

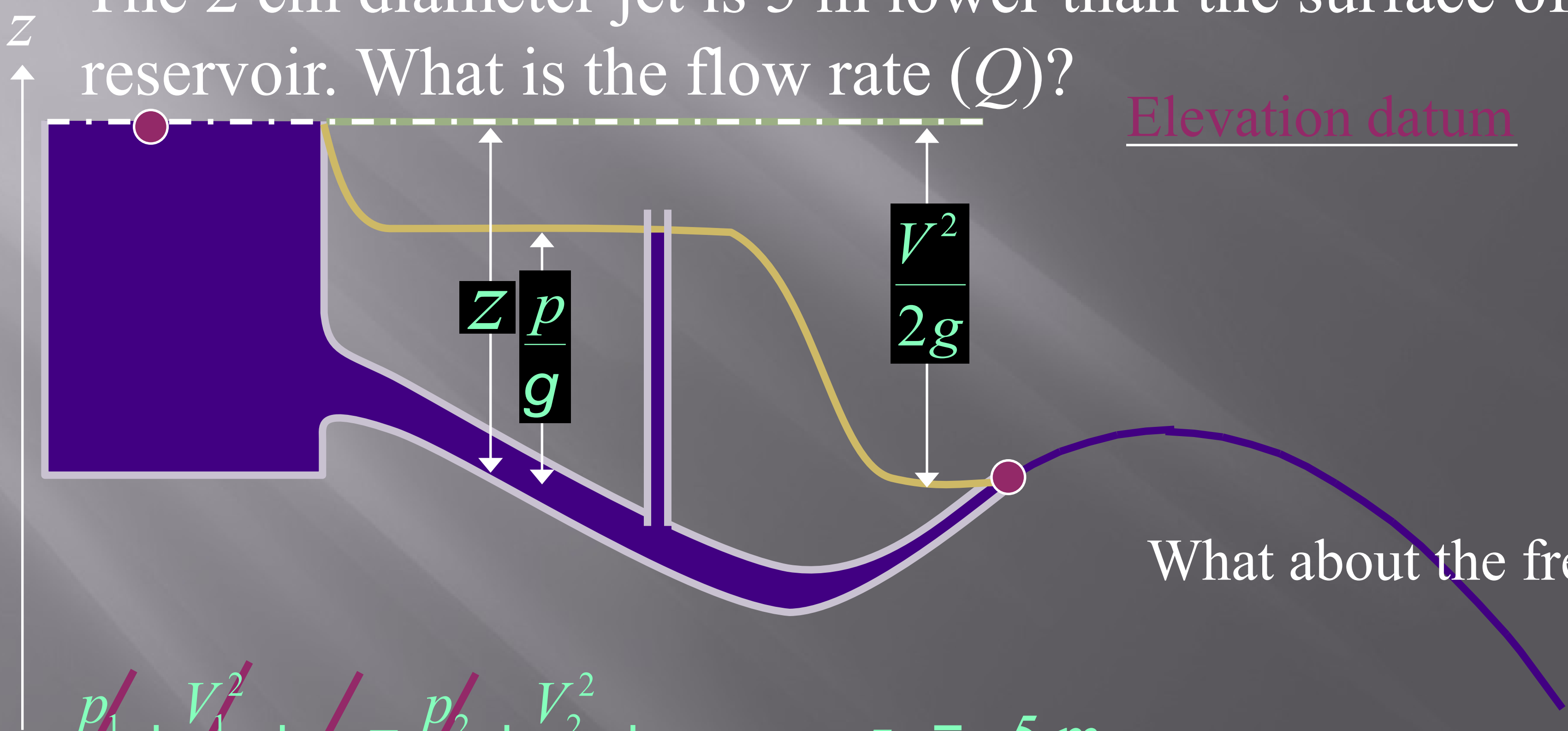
- By integrating *F=ma* along a streamline we found…
  - That energy can be converted between pressure, elevation, and velocity
  - That we can understand many simple flows by applying the Bernoulli equation
- However, the Bernoulli equation can not be applied to flows where viscosity is large or where mechanical energy is converted into thermal energy.

Jet Problem

- How could you choose your elevation datum to help simplify the problem?
- How can you pick 2 locations where you know enough of the parameters to actually find the velocity?
- You have one equation (so one unknown!)

Jet Solution

The 2 cm diameter jet is 5 m lower than the surface of the reservoir. What is the flow rate (Q)?

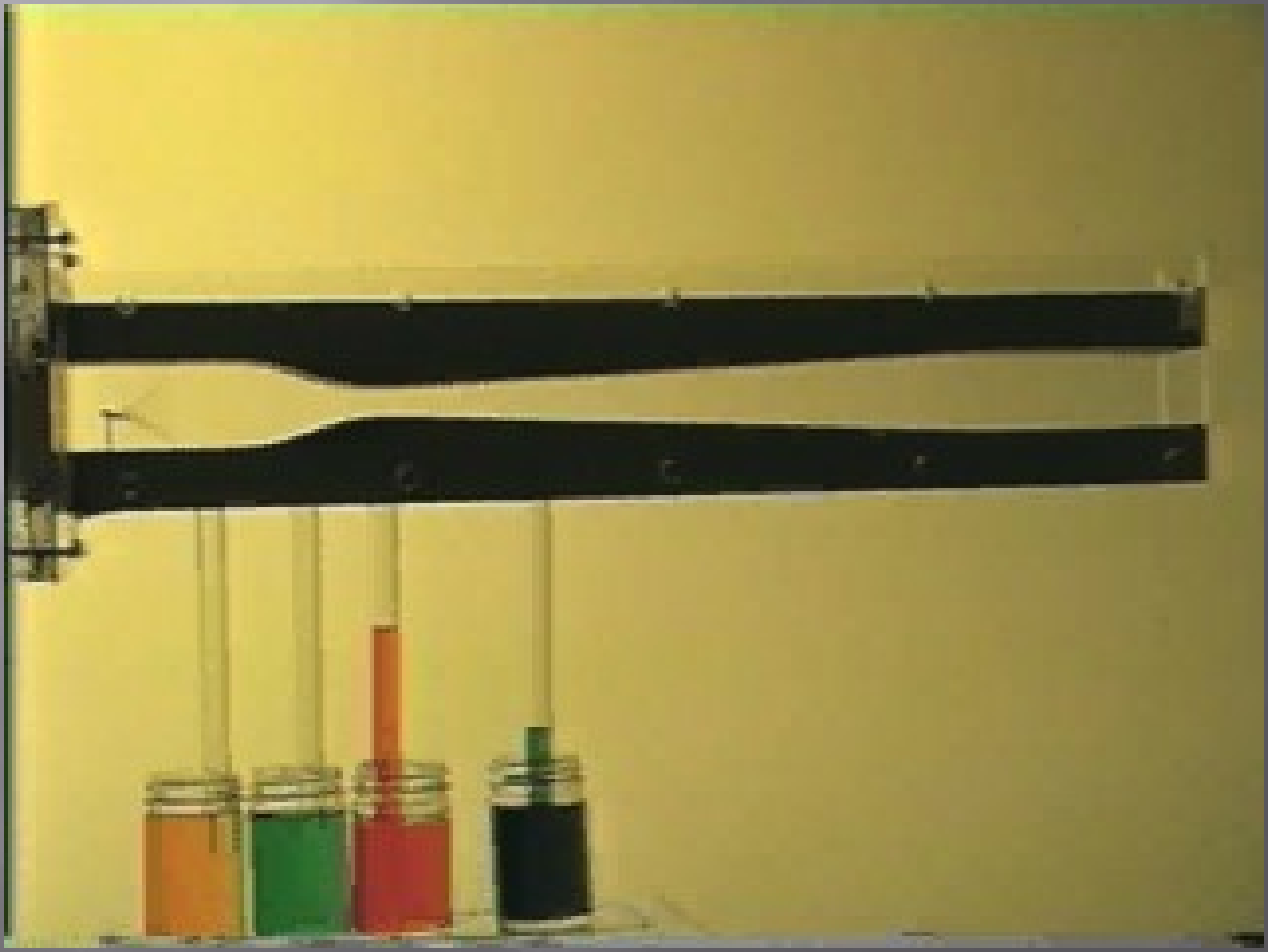


What about the free jet?

$$\cancel{\frac{p_1}{g}} + \cancel{\frac{V_1^2}{2g}} + \cancel{z_1} = \cancel{\frac{p_2}{g}} + \frac{V_2^2}{2g} + z_2$$

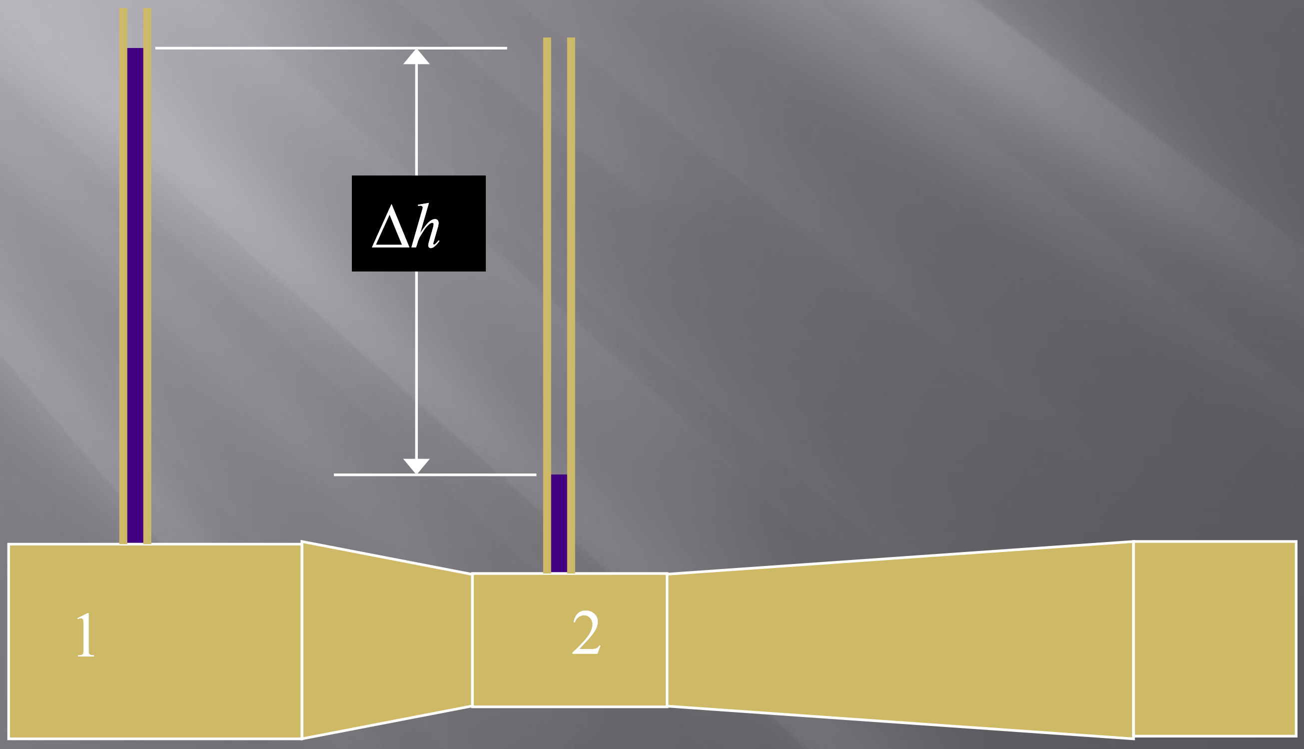
$z_2 = -5$ m

Example: Venturi



Example: Venturi

Find the flow (Q) given the pressure drop between point 1 and 2 and the diameters of the two sections. You may assume the head loss is negligible. Draw the EGL and the HGL.



Example Venturi

$$\frac{p_1}{g_1} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{g_2} + z_2 + \frac{V_2^2}{2g}$$

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} = \frac{V_2^2}{2g}\left[1 - \left(\frac{d_2}{d_1}\right)^4\right]$$

$$V_2 = \sqrt{\frac{2g(p_1 - p_2)}{\gamma\left[1 - (d_2/d_1)^4\right]}}$$

$$Q = C_v A_2 \sqrt{\frac{2g(p_1 - p_2)}{\gamma\left[1 - (d_2/d_1)^4\right]}}$$

$$Q = VA$$

$$V_1 A_1 = V_2 A_2$$

$$V_1 \frac{\pi d_1^2}{4} = V_2 \frac{\pi d_2^2}{4}$$

$$V_1 d_1^2 = V_2 d_2^2$$

$$V_1 = V_2 \frac{d_2^2}{d_1^2}$$