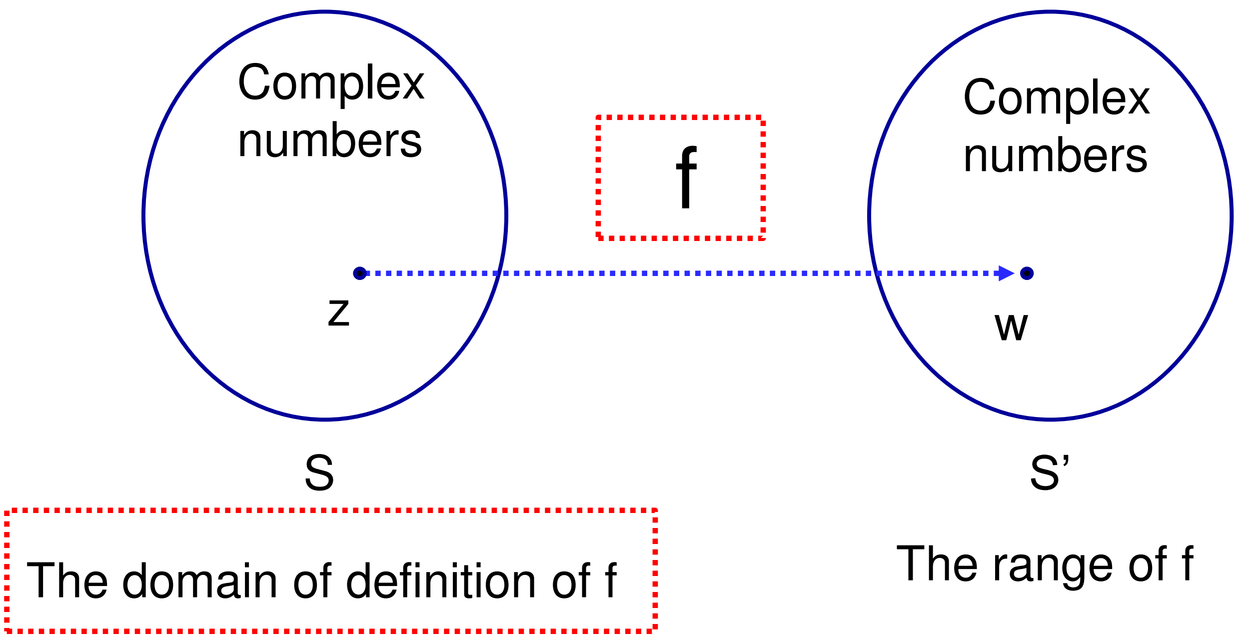
Analytic Functions

Functions of a Complex Variable

- Function of a complex variable

  Let s be a set complex numbers. A function f defined on S is a rule that assigns to each z in S a complex number w.



Functions of a Complex Variable

Suppose that w=u+iv is the value of a function *f* at z=x+iy, so that

$$u + iv = f(x + iy)$$

Thus each of real number u and v depends on the real variables x and y, meaning that

$$f(z) = u(x, y) + iv(x, y)$$

Similarly if the polar coordinates r and θ, instead of x and y, are used, we get

$$f(z) = u(r, \theta) + iv(r, \theta)$$

Functions of a Complex Variable

- Example 2

  If $f(z)=z^2$, then

  When v=0, f is a real-valued function.

  case #1: $z = x + iy$

$$f(z) = (x+iy)^2 = x^2 - y^2 + i2xy$$

$$\Rightarrow \quad u(x,y) = x^2 - y^2; v(x,y) = 2xy$$

  case #2: $z = re^{i\theta}$

$$f(z) = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2 \cos 2\theta + i r^2 \sin 2\theta$$

$$\Rightarrow \quad u(r,\theta) = r^2 \cos 2\theta; v(r,\theta) = r^2 \sin 2\theta$$

Functions of a Complex Variable

- Example 3

A real-valued function is used to illustrate some important concepts later in this chapter is

$$f(z) = |z|^2 = x^2 + y^2 + i0$$

- Polynomial function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

where n is zero or a positive integer and a_0, a_1, …a_n are complex constants, a_n is not 0;The domain of definition is the entire z plane

- Rational function

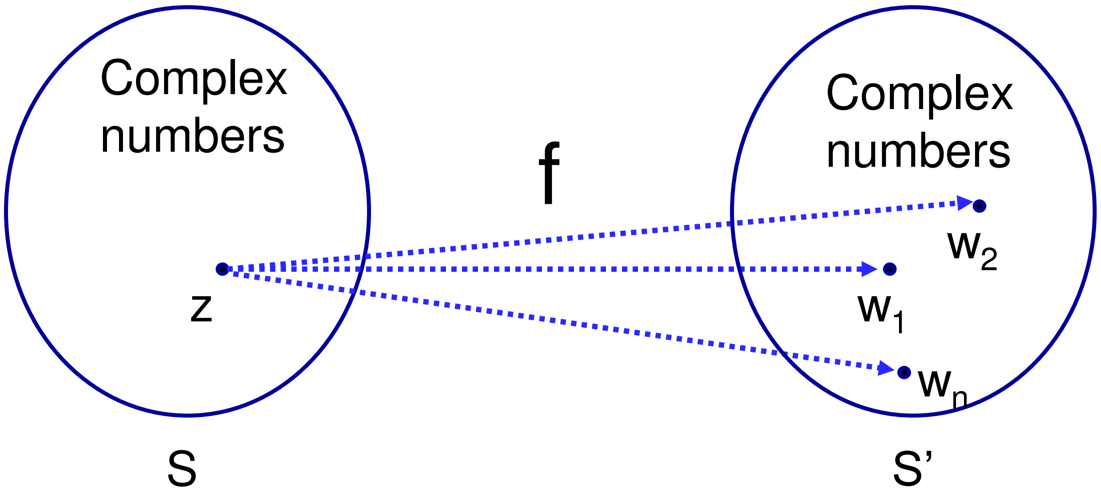
the quotients P(z)/Q(z) of polynomials

The domain of definition is Q(z)≠0

Functions of a Complex Variable

- Multiple-valued function

A generalization of the concept of function is a rule that assigns more than one value to a point z in the domain of definition.



Functions of a Complex Variable

- Example 4

  Let z denote any nonzero complex number, then $z^{1/2}$ has the two values

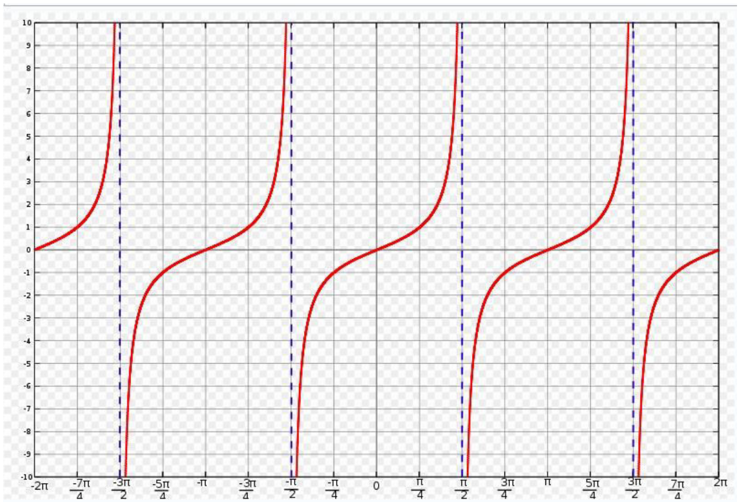
$$z^{1/2} = \pm\sqrt{r}\exp(i\frac{\theta}{2})$$ Multiple-valued function

If we just choose only the positive value of $\pm\sqrt{r}$

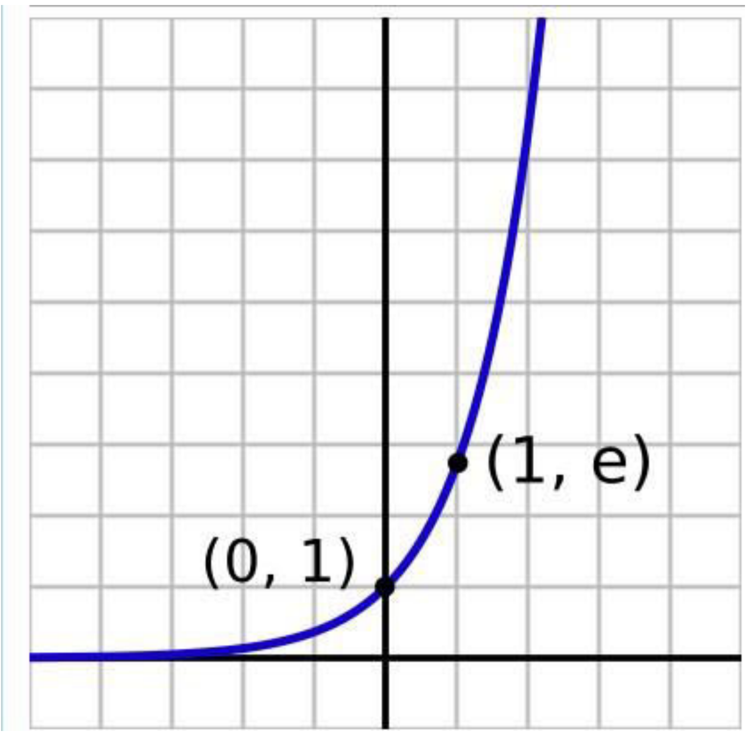
$$z^{1/2} = \sqrt{r}\exp(i\frac{\theta}{2}), r > 0$$ Single-valued function

Mappings

- Graphs of Real-value functions



f=tan(x)



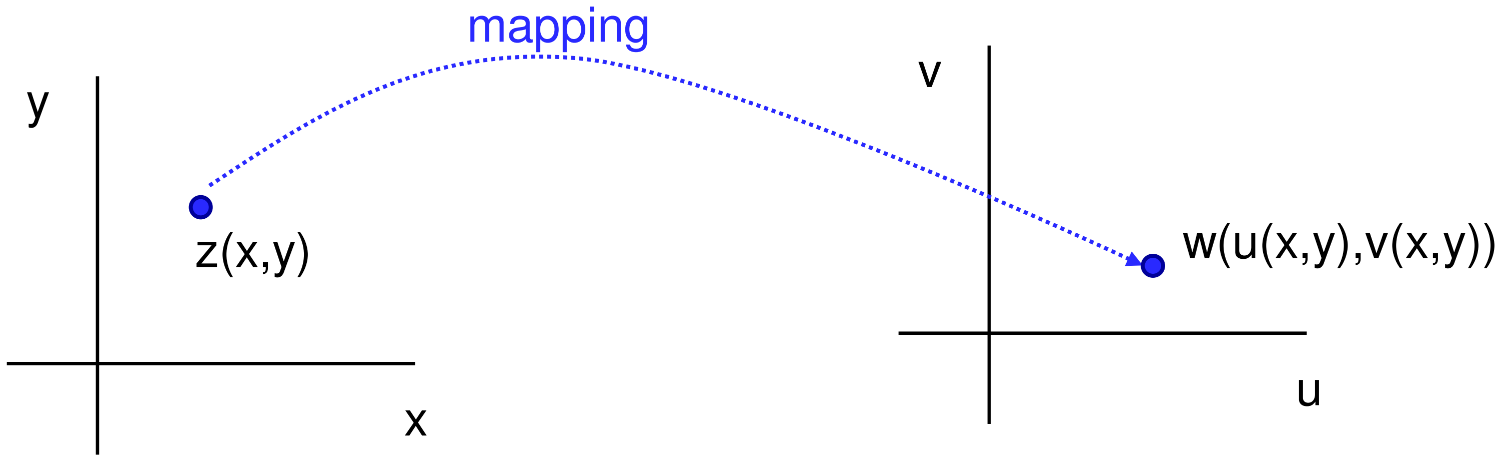
$f=e^x$

Note that both x and f(x) are real values.

Mappings

- Complex-value functions

$$f(z) = f(x + yi) = u(x, y) + iv(x, y)$$



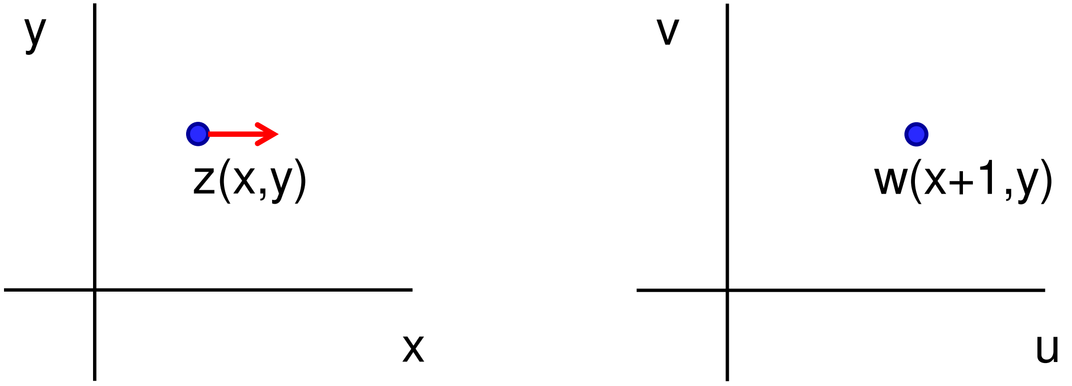
Note that here x, y, u(x,y) and v(x,y) are all real values.

Mappings

- Examples

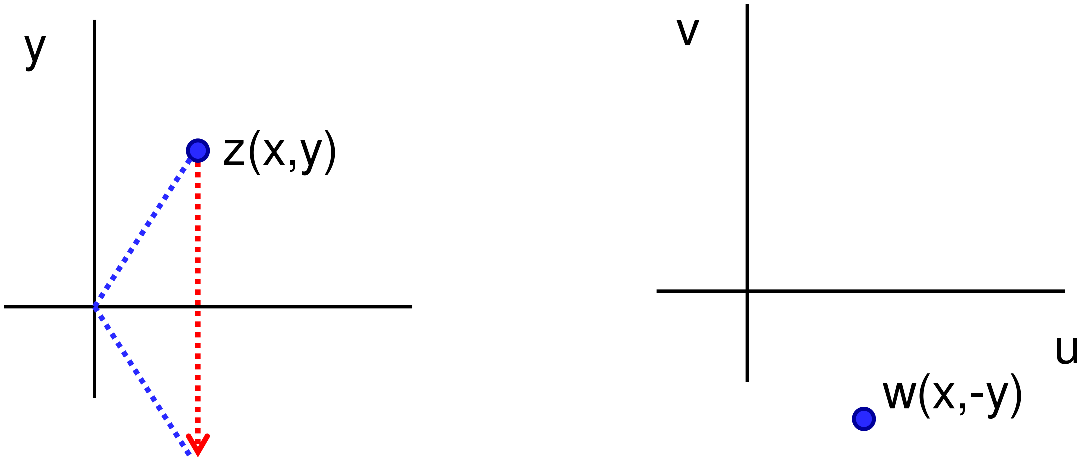
$$w = z + 1 = (x+1) + iy$$

Translation Mapping



$$w = \bar{z} = x - yi$$

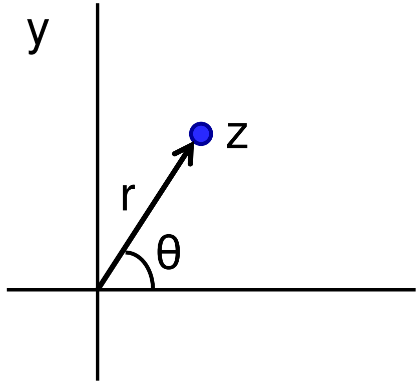
Reflection Mapping

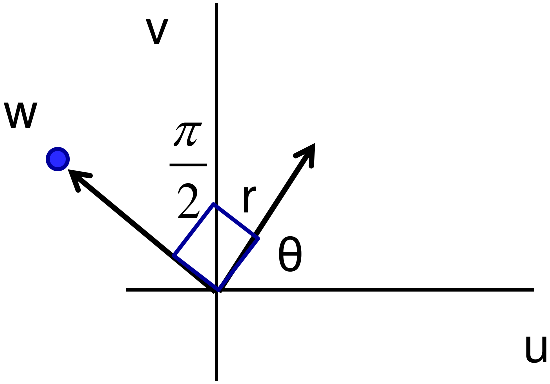


Mappings

- Example

$$w = iz = i(re^{i\theta}) = r\exp(i(\theta + \frac{\pi}{2}))$$ Rotation Mapping





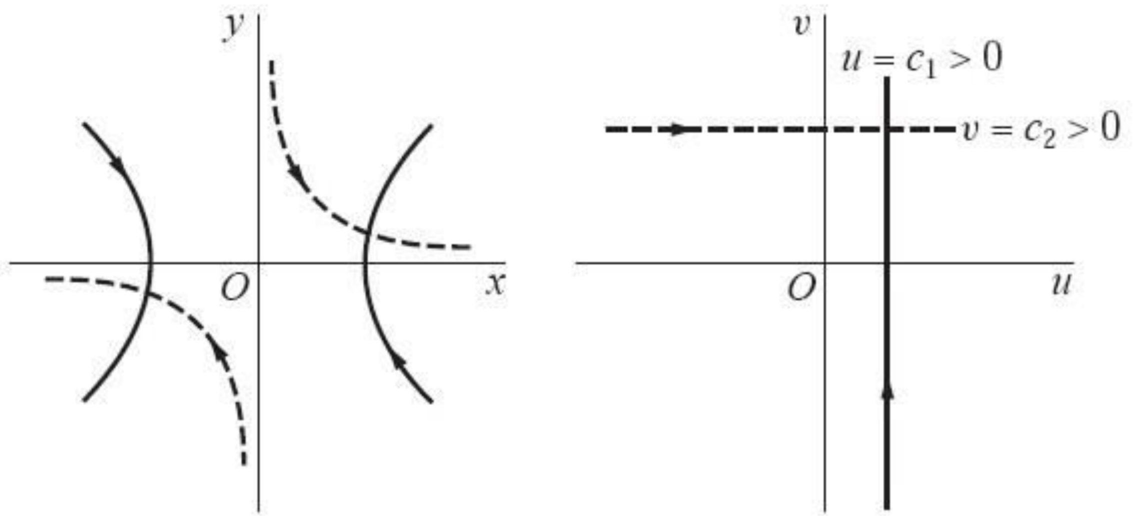
Mappings

- Example 1

$$w = z^2 \qquad u = x^2 - y^2, v = 2xy$$

Let $u=c_1>0$ in the w plane, then $x^2-y^2=c_1$ in the z plane

Let $v=c_2>0$ in the w plane, then $2xy=c_2$ in the z plane



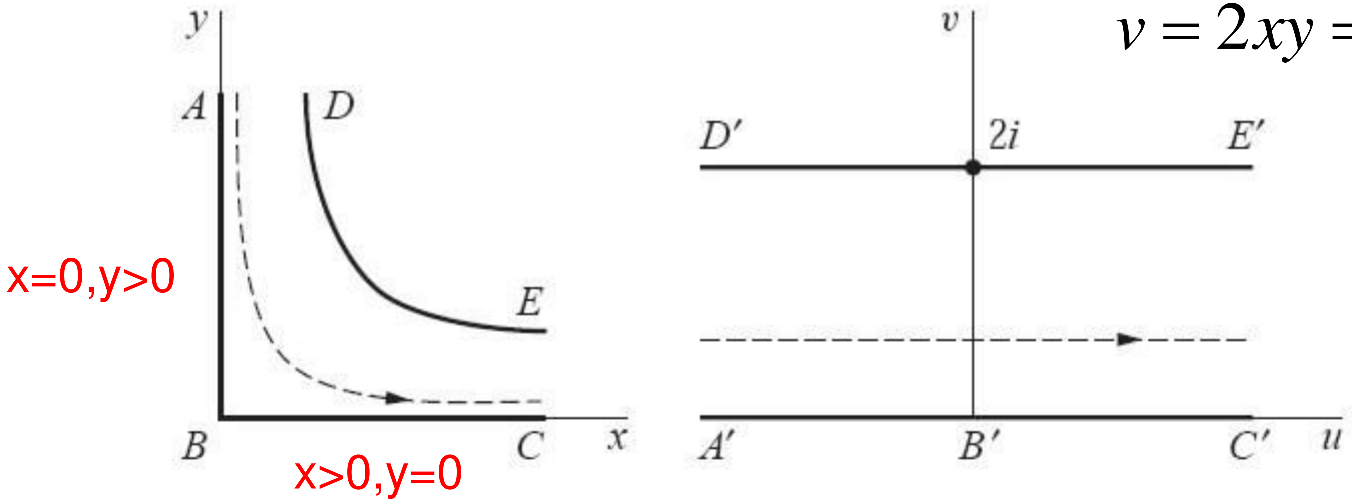
Mappings

- Example 2

The domain x>0, y>0, xy<1 consists of all points lying on the upper branches of hyperbolas

$$u = x^2 - y^2;$$

$$v = 2xy = 2 \Rightarrow xy = 1$$

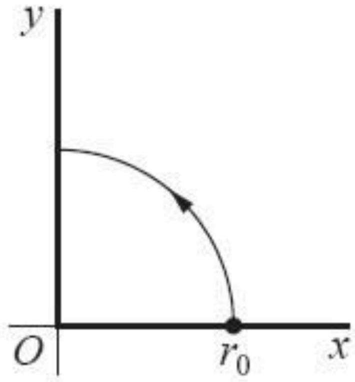


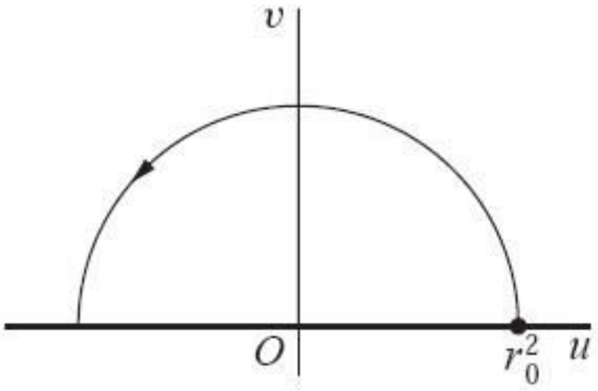
Mappings

- Example 3

$$w = z^2 = r^2 e^{i2\theta}$$

In polar coordinates



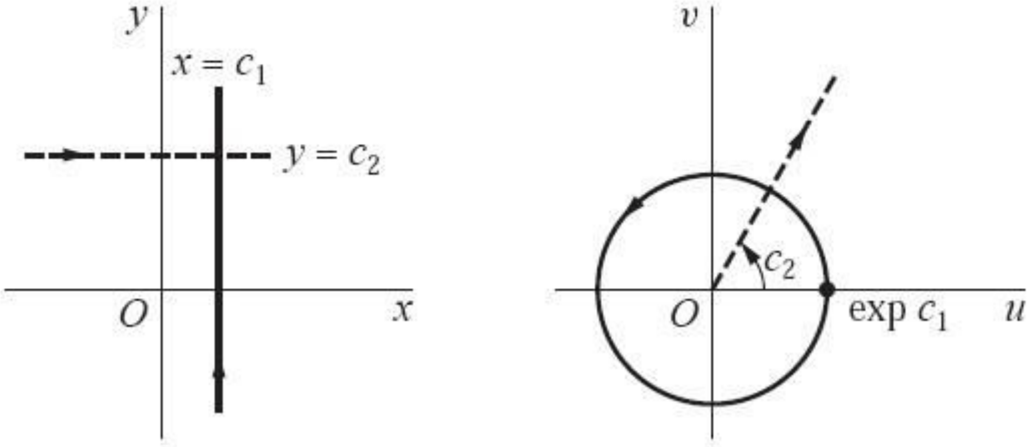


Mappings by the Exponential Function

- The exponential function

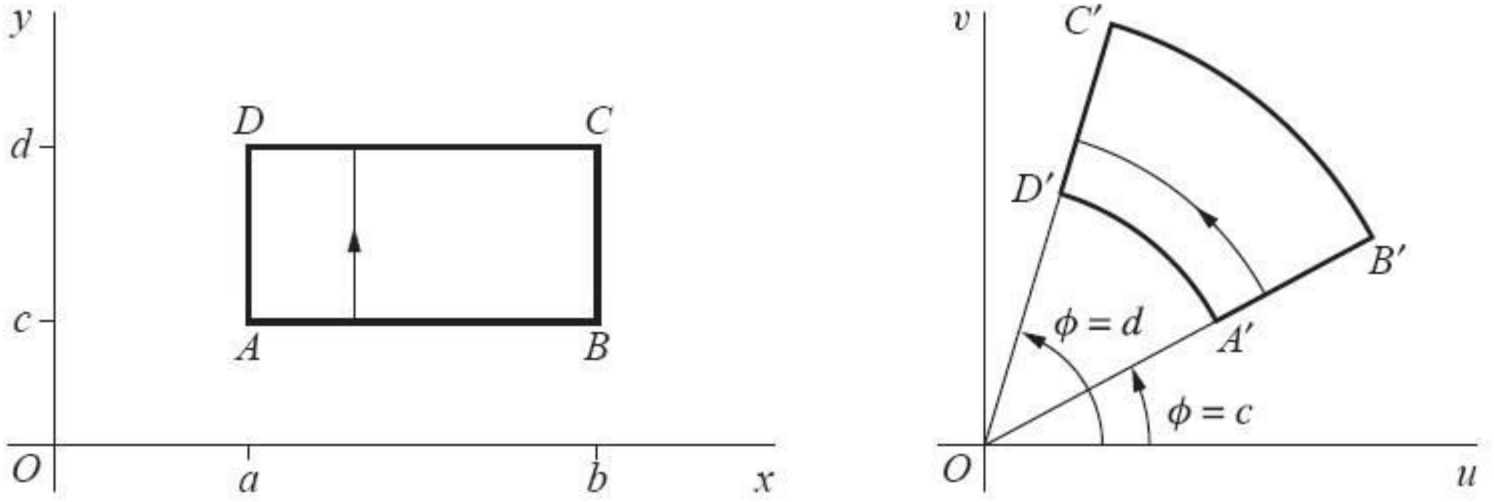
$$w = e^z = e^{x+iy} = e^x e^{iy}, z = x + iy$$

$\rho e^{i\theta}$ $\rho = e^x$, $\theta = y$



Mappings by the Exponential Function

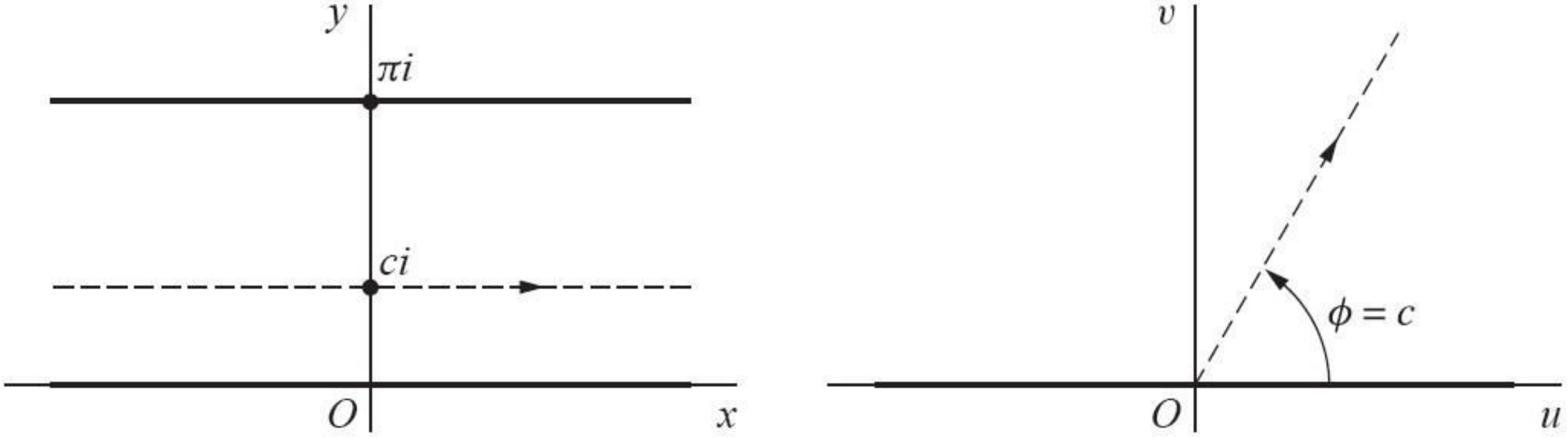
- Example 2



w=exp(z)

Mappings by the Exponential Function

- Example 3



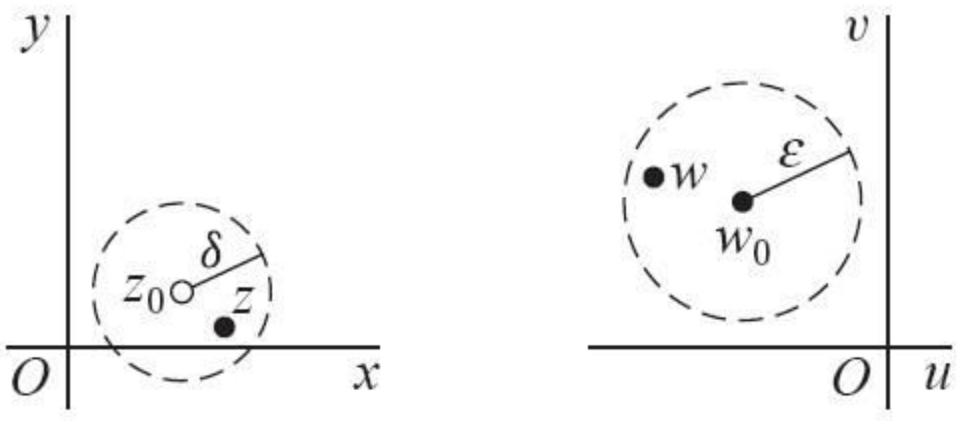
$w = \exp(z) = e^{x+yi}$

Limits

- For a given positive value ε, there exists a positive value δ (depends on ε) such that

  when $0 < |z-z_0| < \delta$, we have $|f(z)-w_0| < \varepsilon$

meaning the point w=f(z) can be made arbitrarily chose to w_0 if we choose the point z close enough to z_0 but distinct from it.



$$\lim_{z \to z_0} f(z) = w_0$$

Limits

- The uniqueness of limit

  If a limit of a function f(z) exists at a point z0, it is unique.

Proof: suppose that $\lim_{z\to z_0} f(z) = w_0$ & $\lim_{z\to z_0} f(z) = w_1$

then $\forall \varepsilon/2 > 0, \exists \delta_0 > 0, \exists \delta_1 > 0$

when $0 < |z - z_0| < \delta_0 \Longrightarrow |f(z) - w_0| < \varepsilon/2;$

$0 < |z - z_0| < \delta_1 \Longrightarrow |f(z) - w_1| < \varepsilon/2;$

Let $\delta = \min(\delta_0, \delta_1)$, when $0<|z-z_0|<\delta$, we have

$$\Rightarrow |w_1 - w_0| = |(f(z) - w_0) - (f(z) - w_1)|$$

$$\le |f(z) - w_0| + |f(z) - w_1| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Limits

- Example 1

Show that $f(z)=i\bar{z}/2$ in the open disk |z|<1, then

$$\lim_{z\to 1} f(z)=\frac{i}{2}$$

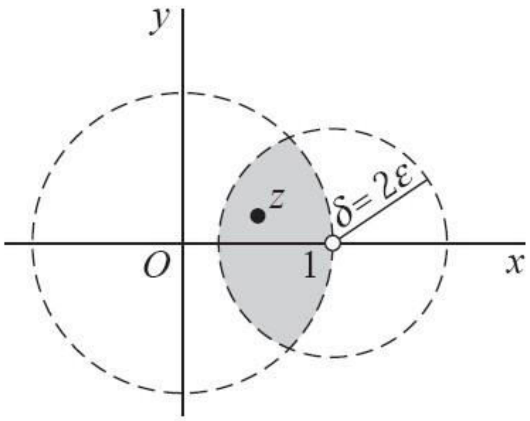
Proof:

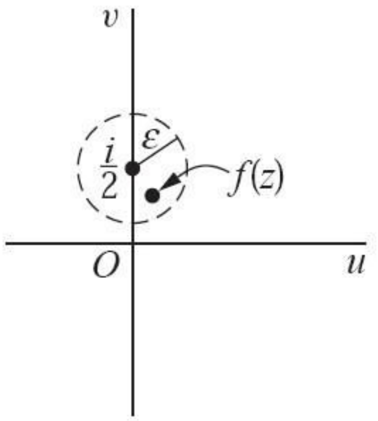
$$|f(z)-\frac{i}{2}|=|\frac{i\bar{z}}{2}-\frac{i}{2}|=\frac{|i||\bar{z}-1|}{2}=\frac{|z-1|}{2}$$

$\forall\varepsilon>0, \exists\delta=2\varepsilon, s.t.$

when $0<|z-1|<\delta(=2\varepsilon)$

$$\Rightarrow 0<\frac{|z-1|}{2}<\varepsilon \Rightarrow |f(z)-\frac{i}{2}|<\varepsilon$$





Limits

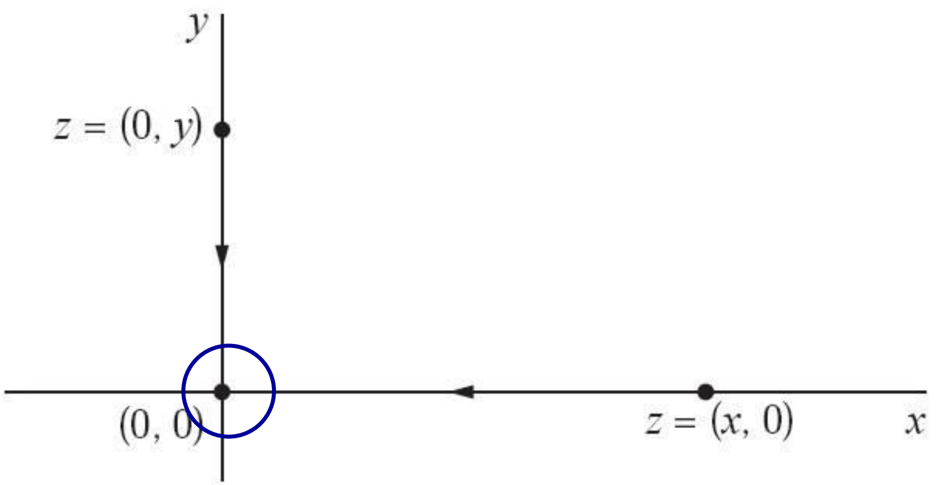
- Example 2

  If $f(z)=\frac{\bar{z}}{z}$ then the limit $\lim_{z\to 0} f(z)$ does not exist.

$$z=(x,0) \qquad \lim_{x\to 0}\frac{x+i0}{x-i0}=1$$

$$\neq$$

$$z=(0,y) \qquad \lim_{y\to 0}\frac{0+iy}{0-iy}=-1$$



Theorems on Limits

- Theorem 1

Let $f(z) = u(x, y) + iv(x, y) \quad z = x + iy$

and $z_0 = x_0 + iy_0; w_0 = u_0 + iv_0$

then

$$\lim_{z \to z_0} f(z) = w_0 \tag{a}$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0 \tag{b}$$

Theorems on Limits

- Proof: (b)➔(a)

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y)=u_0 \quad \& \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y)=v_0 \quad \Rightarrow \quad \lim_{z\to z_0} f(z)=w_0$$

$$\forall \varepsilon/2>0, \exists \delta_1>0, \exists \delta_2>0 \, s.t.$$

When $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta_1 \Rightarrow |u(x,y)-u_0|<\frac{\varepsilon}{2}$

$$0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta_2 \Rightarrow |v(x,y)-v_0|<\frac{\varepsilon}{2}$$

Let $\delta=\min(\delta_1,\delta_2)$ When $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta, i.e. 0<|z-z_0|<\delta$

$$|f(z)-w_0|=|(u(x,y)+iv(x,y))-(u_0+iv_0)| = |u(x,y)-u_0+i(v(x,y)-v_0)|$$

$$\le |u(x,y)-u_0|+|v(x,y)-v_0|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

Theorems on Limits

- Proof: (a)➔(b)

$$\lim_{z\to z_0} f(z) = w_0 \quad \Rightarrow \quad \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \& \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

$\forall \varepsilon > 0, \exists \delta > 0 \, s.t.$ When $0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z) - w_0| < \varepsilon$

$$|f(z) - w_0| = |u(x,y) + iv(x,y) - (u_0 + iv_0)|$$

$$= |(u(x,y) - u_0) + i(v(x,y) - v_0)| < \varepsilon$$

$$|u(x,y) - u_0| \le |(u(x,y) - u_0) + i(v(x,y) - v_0)| < \varepsilon$$

$$|v(x,y) - v_0| \le |(u(x,y) - u_0) + i(v(x,y) - v_0)| < \varepsilon$$

Thus $|u(x,y) - u_0| < \varepsilon; |v(x,y) - v_0| < \varepsilon$

When $(x,y)\to(x_0,y_0)$

Theorems on Limits

- Theorem 2

Let $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} F(z) = W_0$

then $$\lim_{z \to z_0}[f(z) \pm F(z)] = w_0 \pm W_0$$

$$\lim_{z \to z_0}[f(z)F(z)] = w_0 W_0$$

$$\lim_{z \to z_0}[\frac{f(z)}{F(z)}] = \frac{w_0}{W_0}, W_0 \neq 0$$

Theorems on Limits

$$\lim_{z\to z_0} f(z) = w_0 \quad \& \quad \lim_{z\to z_0} F(z) = W_0 \Rightarrow \lim_{z\to z_0}[f(z)F(z)] = w_0 W_0$$

Let $$f(z) = u(x, y) + iv(x, y), F(z) = U(x, y) + iV(x, y)$$

$$z_0 = x_0 + iy_0; w_0 = u_0 + iv_0; W_0 = U_0 + iV_0$$

$$f(z)F(z) = (uU - vV) + i(vU + uV)$$

$$\lim_{z\to z_0} f(z) = w_0$$

$$\lim_{z\to z_0} F(z) = W_0$$

$\Rightarrow$ When $(x,y)\to(x_0,y_0)$;
$u(x,y)\to u_0$; $v(x,y)\to v_0$; & $U(x,y)\to U_0$; $V(x,y)\to V_0$;

$\Downarrow$

Re(f(z)F(z)): $(u_0U_0 - v_0V_0)$

Im(f(z)F(z)): $(v_0U_0 + u_0V_0)$

w_0W_0

Theorems on Limits

It is easy to verify the limits

$$\lim_{z \to z_0} c = c \qquad \lim_{z \to z_0} z = z_0 \qquad \lim_{z \to z_0} z^n = z_0^n \, (n = 1, 2, \ldots)$$

For the polynomial

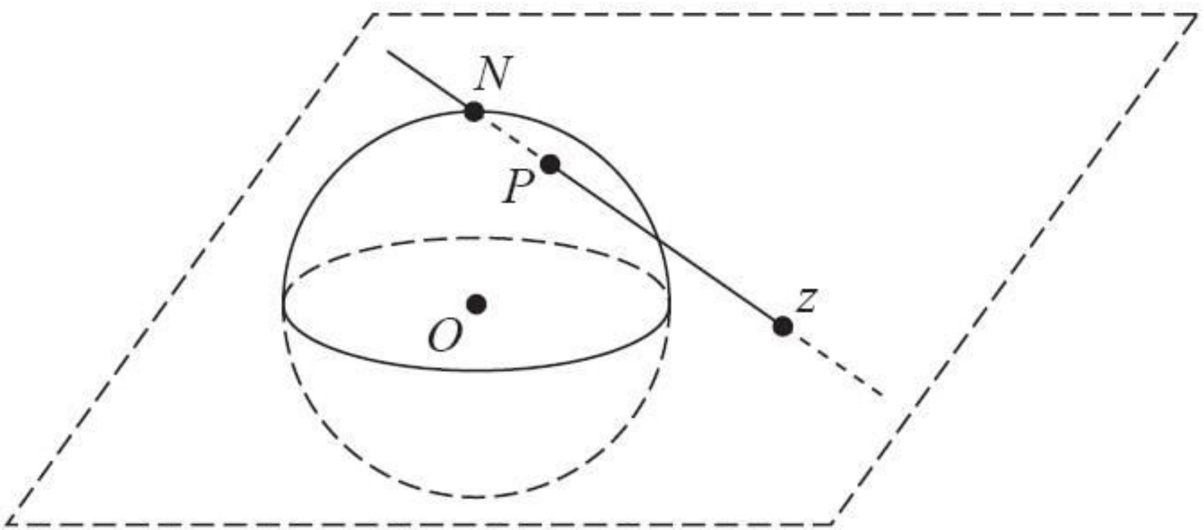
$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

We have that

$$\lim_{z \to z_0} P(z) = P(z_0)$$

Limits Involving the Point at Infinity

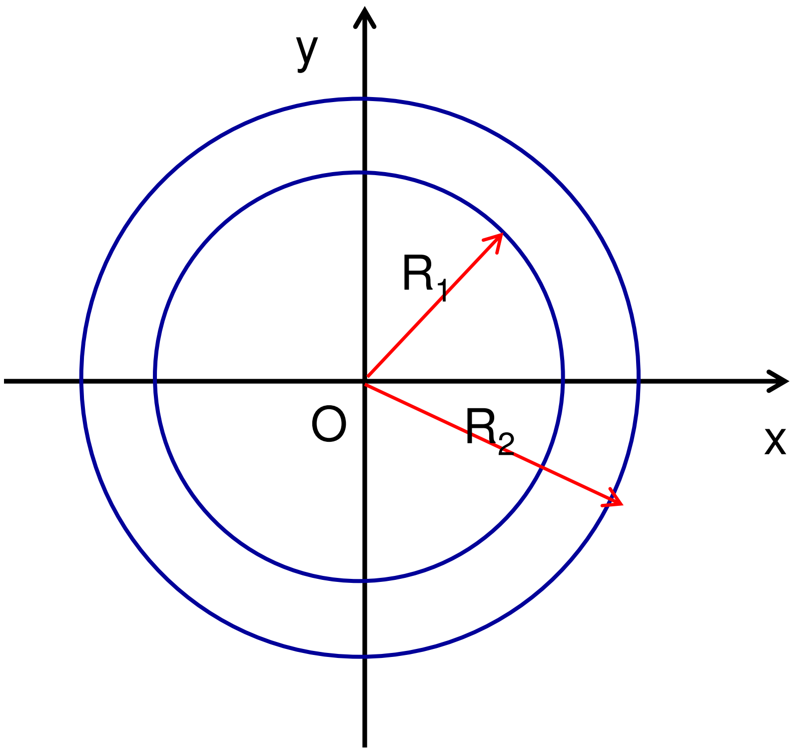
- Riemannsphere & Stereographic Projection



N: the north pole

Limits Involving the Point at Infinity

- The ε Neighborhood of Infinity



When the radius R is large enough

i.e. for each small positive number ε

R=1/ε

The region of |z|>R=1/ε is called the ε Neighborhood of Infinity(∞)

Limits Involving the Point at Infinity

- Theorem

If z_0 and w_0 are points in the z and w planes, respectively, then

$$\lim_{z\to z_0} f(z) = \infty \quad \text{iff} \quad \lim_{z\to z_0} \frac{1}{f(z)} = 0$$

$$\lim_{z\to\infty} f(z) = w_0 \quad \text{iff} \quad \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$$

$$\lim_{z\to\infty} f(z) = \infty \quad \text{iff} \quad \lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$$

Limits Involving the Point at Infinity

- Examples

$$\lim_{z\to -1} \frac{iz+3}{z+1} = \infty \quad \text{since} \quad \lim_{z\to -1} \frac{z+1}{iz+3} = 0$$

$$\lim_{z\to \infty} \frac{2z+i}{z+1} = 2 \quad \text{since} \quad \lim_{z\to 0} \frac{(2/z)+i}{(1/z)+1} = \lim_{z\to 0} \frac{2+iz}{1+z} = 2.$$

$$\lim_{z\to \infty} \frac{2z^3-1}{z^2+1} = \infty \quad \text{since} \quad \lim_{z\to 0} \frac{(1/z^2)+1}{(2/z^3)-1} = \lim_{z\to 0} \frac{z+z^3}{2-z^3} = 0.$$

Continuity

- Continuity

  A function is continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

  meaning that

1. the function f has a limit at point z_0 and
2. the limit is equal to the value of $f(z_0)$

For a given positive number ε, there exists a positive number δ, s.t.

When $|z - z_0| < \delta$ $\quad |f(z) - f(z_0)| < \varepsilon$

$$0 < |z - z_0| < \delta ?$$

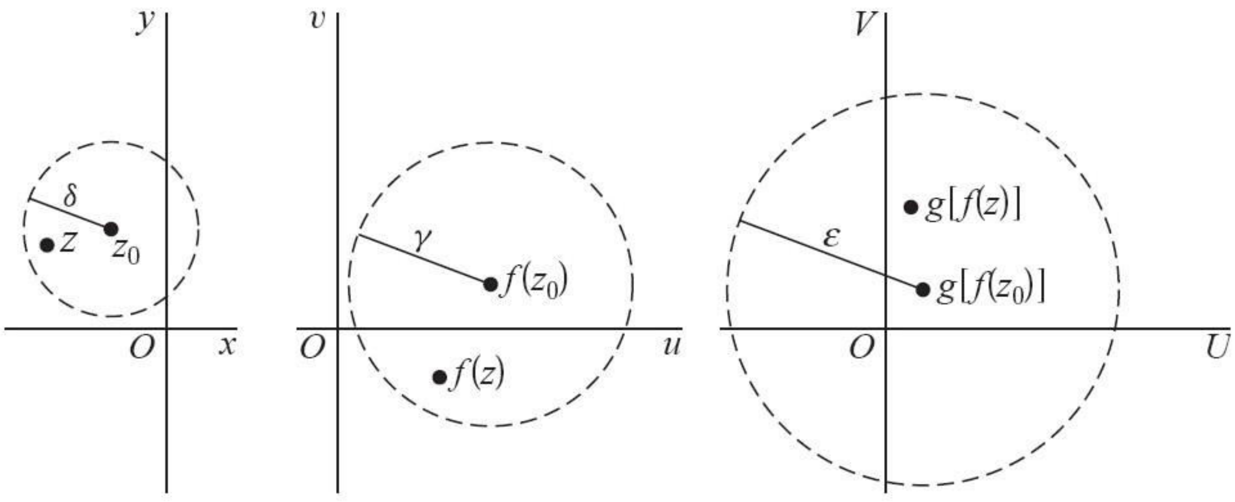
Continuity

- Theorem 1

  A composition of continuous functions is itself continuous.

  Suppose w=f(z) is a continuous at the point z_0;
  g=g(f(z)) is continuous at the point $f(z_0)$

  Then the composition g(f(z)) is continuous at the point z_0



Continuity

- Theorem 2

  If a function f (z) is continuous and nonzero at a point z_0, then f (z) ≠ 0 throughout some neighborhood of that point.

Proof $$\lim_{z\to z_0} f(z) = f(z_0) \neq 0$$

Why?

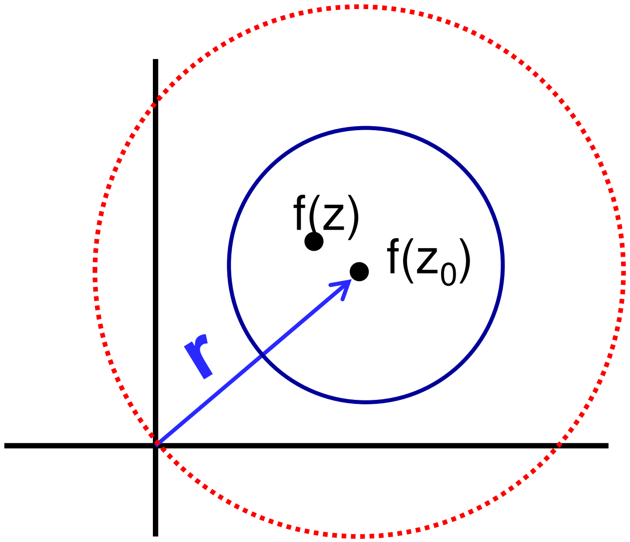
$$\forall \varepsilon = \frac{|f(z_0)|}{2} > 0, \exists \delta > 0, s.t.$$

When $|z - z_0| < \delta$

$$|f(z) - f(z_0)| < \varepsilon = \frac{|f(z_0)|}{2}$$

If f(z)=0, then $|f(z_0)| < \dfrac{|f(z_0)|}{2}$

Contradiction!



$$\forall \varepsilon \leq |f(z_0)|$$

Continuity

- Theorem 3

If a function f is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M such that

$$|f(z)| \leq M \qquad \text{for all points z in R}$$

where equality holds for at least one such z.

Note: $|f(z)| = \sqrt{u^2(x,y) + v^2(x,y)}$

where u(x,y) and v(x,y) are continuous real functions

Derivatives

- Derivative

Let f be a function whose domain of definition contains a neighborhood $|z-z_0|<\varepsilon$ of a point z_0. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

And the function f is said to be differentiable at z_0 when $f'(z_0)$ exists.

Derivatives

- Illustration of Derivative

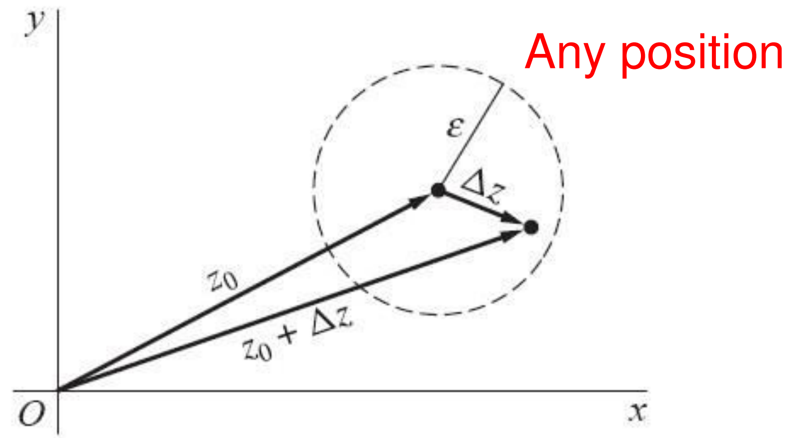
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

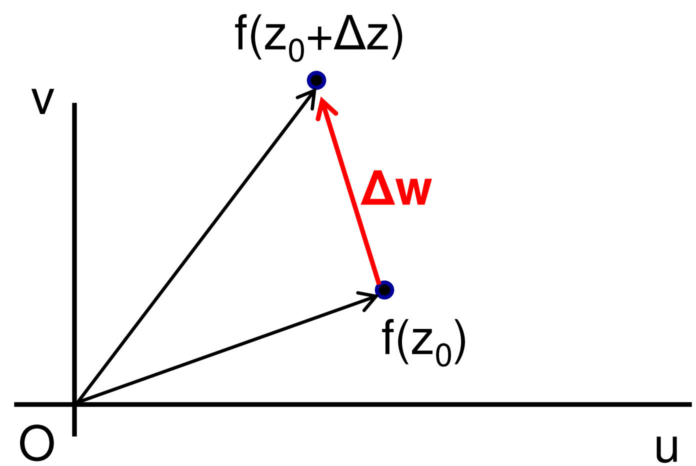
$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$z = z_0 + \Delta z$$

$$\Delta w = f(z_0 + \Delta z) - f(z_0)$$

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$





Derivatives

- Example 1

Suppose that $f(z)=z^2$. At any point z

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

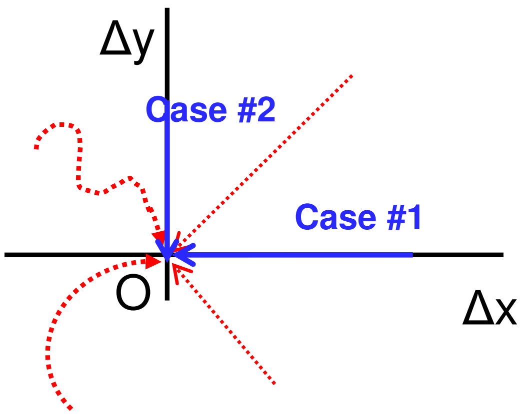
since $2z + \Delta z$ is a polynomial in Δz. Hence $dw/dz=2z$ or $f'(z)=2z$.

Derivatives

- Example 2

If f(z)=$\bar{z}$, then $\frac{\Delta w}{\Delta z} = \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$

$\Delta z = (\Delta x, \Delta y) \to (0,0)$ In any direction



Case #1: Δx→0, Δy=0

$$\lim_{\Delta x \to 0} \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i0}{\Delta x + i0} = 1$$

Case #2: Δx=0, Δy→0

$$\lim_{\Delta x \to 0} \frac{\overline{\Delta z}}{\Delta z} = \frac{0 - i\Delta y}{0 + i\Delta y} = -1$$

Since the limit is unique, this function does not exist anywhere

Derivatives

- Example 3

  Consider the real-valued function f(z)=|z|^2. Here

$$\frac{\Delta w}{\Delta z}=\frac{|z+\Delta z|^2-|z|^2}{\Delta z}=\frac{(z+\Delta z)(\bar{z}+\overline{\Delta z})-z\bar{z}}{\Delta z}=\bar{z}+\overline{\Delta z}+z\frac{\overline{\Delta z}}{\Delta z}$$

Case #1: Δx→0, Δy=0

$$\lim_{\Delta x\to 0}(\bar{z}+\overline{\Delta z}+z\frac{\overline{\Delta z}}{\Delta z})=\lim_{\Delta x\to 0}(\bar{z}+\Delta x+z\frac{\Delta x-i0}{\Delta x+i0})=\bar{z}+z$$

Case #2: Δx=0, Δy→0

$$\lim_{\Delta y\to 0}(\bar{z}+\overline{\Delta z}+z\frac{\overline{\Delta z}}{\Delta z})=\lim_{\Delta y\to 0}(\bar{z}-i\Delta y+z\frac{0-i\Delta y}{0+i\Delta y})=\bar{z}-z$$

$$\bar{z}+z=\bar{z}-z\Rightarrow z=0$$

dw/dz can not exist when z is not 0

Derivatives

- Continuity & Derivative

Continuity $\not\Longrightarrow$ Derivative

For instance,
f(z)=|z|^2 is continuous at each point, however, dw/dz does not exists when z is not 0

Derivative $\Longrightarrow$ Continuity

$$\lim_{z\to z_0}[f(z)-f(z_0)] = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\lim_{z\to z_0}(z-z_0) = f'(z_0)0 = 0$$

Note: The existence of the derivative of a function at a point implies the continuity of the function at that point.

Differentiation Formulas

- Differentiation Formulas

$$\frac{d}{dz}c = 0; \frac{d}{dz}z = 1; \frac{d}{dz}[cf(z)] = cf'(z)$$

$$\frac{d}{dz}[z^n] = nz^{n-1}$$

Refer to pp.7 (13)

$$\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z)$$

$$\frac{d}{dz}[f(z) \bullet g(z)] = f(z) \bullet g'(z) + f'(z) \bullet g(z)$$

$$\frac{d}{dz}[\frac{f(z)}{g(z)}] = \frac{f'(z) \bullet g(z) - f(z) \bullet g'(z)}{[g(z)]^2}$$

$$F(z) = g(f(z))$$

$$F'(z_0) = g'(f(z_0))f'(z_0)$$

$$\frac{dW}{dz} = \frac{dW}{dw}\frac{dw}{dz}$$

Differentiation Formulas

- Example

  To find the derivative of $(2z^2+i)^5$, write $w=2z^2+i$ and $W=w^5$. Then

$$\frac{d}{dz}(2z^2+i)^5 = (5w^4)w' = 5(2z^2+i)^4(4z) = 20z(2z^2+i)^4$$

Analytic Function

- Analytic at a point z_0

  A function f of the complex variable z is analytic at a point z_0 if it has a derivative at each point in some neighborhood of z_0.

Note that if *f* is analytic at a point z_0, it must be analytic at each point in some neighborhood of z_0

- Analytic function

  A function f is analytic in an open set if it has a derivative everywhere in that set.

Note that if *f* is analytic in a set S which is not open, it is to be understood that *f* is analytic in an open set containing S.

Analytic Function

- Analytic vs. Derivative
  - For a point

    Analytic → Derivative ✓

    Derivative → Analytic ✗

  - For all points in an open set

    Analytic → Derivative ✓

    Derivative → Analytic ✓

f is analytic in an open set D iff f is derivative in D

Analytic Function

- Singular point (singularity)

 If function *f* fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is called a singular point.

For instance, the function $f(z)=1/z$ is analytic at every point in the finite plane except for the point of (0,0). Thus (0,0) is the singular point of function $1/z$.

- Entire Function

 An entire function is a function that is analytic at each point in the entire finite plane.

For instance, the polynomial is entire function.

Analytic Function

- Property 1

  If two functions are analytic in a domain D, then

  - their sum and product are both analytic in D
  - their quotient is analytic in D provided the function in the denominator does not vanish at any point in D

- Property 2

  From the chain rule for the derivative of a composite function, a composition of two analytic functions is analytic.

$$\frac{d}{dz} g(f(z)) = g'[f(z)]f'(z)$$

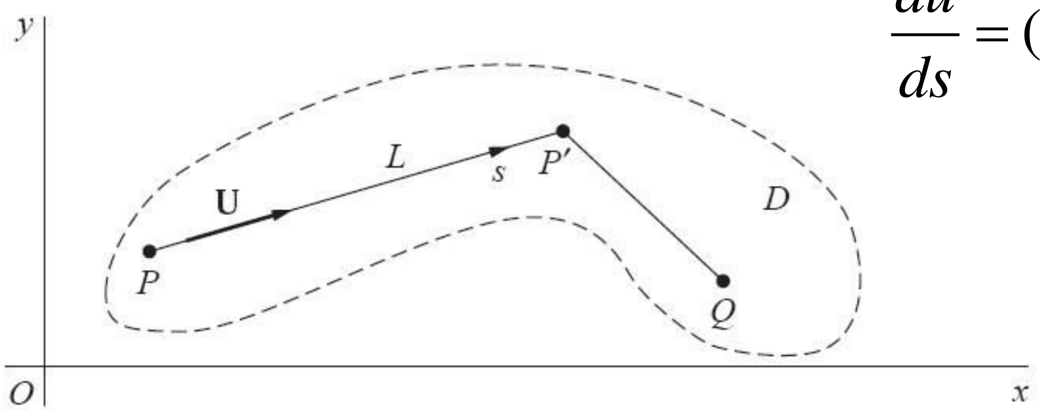
Analytic Function

- Theorem

If f ’(z) = 0 everywhere in a domain D, then f (z) must be constant throughout D.

$$f'(z) = u_x + iv_x = v_y - iu_y = 0$$

$$u_x = u_y = 0 \;\&\; v_x = v_y = 0$$



$$\frac{du}{ds} = (grad\,u)\cdot U \qquad grad\,u = u_x i + u_y j$$

U is the unit vector along L

Example z^2 is Analytic

$z = x + i\,y$

$$f(z) = z^2 = x^2 - y^2 + 2i\,x\,y = u + i\,v$$

$$\rightarrow \quad \begin{aligned} u &= x^2 - y^2 \\ v &= 2x\,y \end{aligned} \quad \rightarrow \quad \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

$\therefore\ f'$ exists & single-valued $\forall$ finite z.

i.e., z^2 is an entire function.

Example: z^* is Not Analytic

$z = x + i\,y$

$$f(z) = z^* = x - i\,y \quad = u + i\,v$$

$$\rightarrow \quad \begin{matrix} u = x \\ v = -y \end{matrix} \quad \rightarrow \quad \frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$$

$\therefore\ f'$ doesn't exist $\forall\ z$, even though it is continuous every where.

i.e., z^2 is nowhere analytic.

Examples

- Example

  Suppose that a function $f(z)=u(x,y)+iv(x,y)$ and its conjugate $f(z)=u(x,y)-iv(x,y)$ are both analytic in a given domain D. Show that f(z) must be constant throughout D.

Proof: $f(z)=u(x,y)+iv(x,y)$ is analytic, then $u_x=v_y, u_y=-v_x$

$f(z)=u(x,y)-iv(x,y)$ is analytic, then $u_x=-v_y, u_y=v_x$

$\Rightarrow$ $u_x=0, v_x=0$ $\Rightarrow$ $f'(z)=u_x+iv_x=0$

Based on the Theorem in pp. 74, we have that f is constant throughout D

Examples

- Example

  Suppose that f is analytic throughout a given region D, and the modulus |f(z)| is constant throughout D, then the function f(z) must be constant there too.

Proof:

$$|f(z)| = c, \qquad \text{for all z in D}$$

where c is real constant.

If c=0, then f(z)=0 everywhere in D.

If $c \neq 0$, we have

$$f(z)\overline{f(z)} = c^2 \quad \Longrightarrow \quad \overline{f(z)} = \frac{c^2}{f(z)}, f(z) \neq 0 inD$$

Both *f* and it conjugate are analytic, thus *f* must be constant in D. (Refer to Ex. 3)

Uniquely Determined Analytic Function

- Lemma

  Suppose that

a) A function f is analytic throughout a domain D;

b) f(z)=0 at each point z of a domain or line segment contained in D.

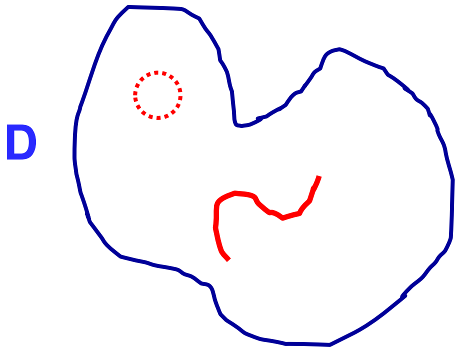
Then $f(z) \equiv 0$ in D; that is, $f(z)$ is identically equal to zero throughout D.

Refer to Chap. 6 for the proof.

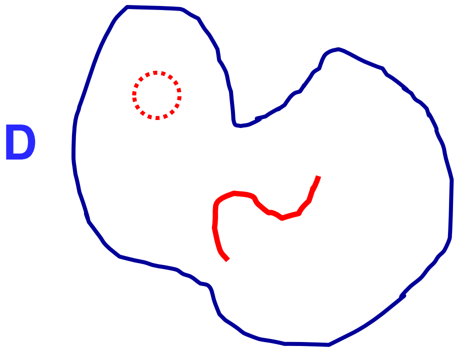
Uniquely Determined Analytic Function

- Theorem

  A function that is analytic in a domain D is uniquely determined over D by its values in a domain, or along a line segment, contained in D.



f(z)



g(z)

$$f(z) \equiv g(z)$$

Reflection Principle

- Theorem

  Suppose that a function f is analytic in some domain D which contains a segment of the x axis and whose lower half is the reflection of the upper half with respect to that axis. Then

$$\overline{f(z)} = f(\bar{z})$$

for each point z in the domain if and only if f (x) is real for each point x on the segment.