DERIVATIVE OF ANALYTIC FUNCTION

Derivatives of Analytic Functions

$\boxed{z = x + i\,y}$

Let $f(z)$ be analytic around z, then

$$\frac{d\,f(x)}{d\,x} = g(x) \qquad \rightarrow \qquad \frac{d\,f(z)}{d\,z} = g(z)$$

Proof :

$f(z)$ analytic $\qquad \rightarrow \qquad f'(z) = \dfrac{\partial f(x+i\,y)}{\partial x} = \left.\dfrac{d\,f(x)}{d\,x}\right|_{x=z} = g(z)$

E.g. $\qquad \dfrac{d\,x^n}{d\,x} = n\,x^{n-1} \qquad \rightarrow \qquad \dfrac{d\,z^n}{d\,z} = n\,z^{n-1}$

$\therefore$ Analytic functions can be defined by Taylor series of the same coefficients as their real counterparts.

Derivative of Logarithm

CRCs $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\dfrac{y}{x}$

$$\frac{d \ln z}{dz} = \frac{1}{z}$$ for z within each branch.

Proof : $\ln z = \ln r + i\left(\theta + 2\pi n\right) = u + iv$ $\rightarrow$ $u = \ln r$, $v = \theta + 2\pi n$

$$\frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{x}{r^2} = \frac{\partial v}{\partial y} \qquad \frac{\partial \theta}{\partial y} = \left(1 + \frac{y^2}{x^2}\right)^{-1}\frac{1}{x} = \frac{x}{r^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{r}\frac{\partial r}{\partial y} = \frac{y}{r^2} = -\frac{\partial v}{\partial x} \qquad \frac{\partial \theta}{\partial x} = \left(1 + \frac{y^2}{x^2}\right)^{-1}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2}$$

$\rightarrow$ $\ln z$ is analytic within each branch.

$$\therefore \quad \frac{d \ln z}{dz} = \frac{\partial \ln z}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{r^2} - i\frac{y}{r^2} = \frac{1}{x + iy} = \frac{1}{z} \qquad r^2 = z\,z^*$$

QED

Derivatives of Functions w(t)

- Consider derivatives of complex-valued functions *w* of real variable *t*

$$w(t) = u(t) + iv(t)$$

where the function *u* and *v* are real-valued functions of *t*. The derivative

$$w'(t), or \frac{d}{dt} w(t)$$

of the function w(t) at a point *t* is defined as

$$w'(t) = u'(t) + iv'(t)$$

Derivatives of Functions w(t)

- Properties

For any complex constant $z_0=x_0+iy_0$,

$$\frac{d}{dt}[z_0 w(t)] = [(x_0+iy_0)(u+iv)]' = [\overbrace{(x_0 u \quad y_0 v)}^{u(t)} + i\overbrace{(y_0 u + x_0 v)}^{v(t)}]'$$

$$= (x_0 u \quad y_0 v)' + i(y_0 u + x_0 v)'$$

$$= (x_0 u' \quad y_0 v') + i(y_0 u' + x_0 v')$$

$$= (x_0+iy_0)(u'+iv') = z_0 w'(t)$$

Derivatives of Functions w(t)

- Properties

$$\frac{d}{dt}e^{z_0 t} = z_0 e^{z_0 t}$$

where $z_0=x_0+iy_0$. We write

$$e^{z_0 t} = e^{(x_0+iy_0)t} = \overbrace{e^{x_0 t}\cos y_0 t}^{u(t)} + i\overbrace{e^{x_0 t}\sin y_0 t}^{v(t)}$$

$$\frac{d}{dt}e^{z_0 t} = (e^{x_0 t}\cos y_0 t)' + i(e^{x_0 t}\sin y_0 t)'$$

Similar rules from calculus and some simple algebra then lead us to the expression

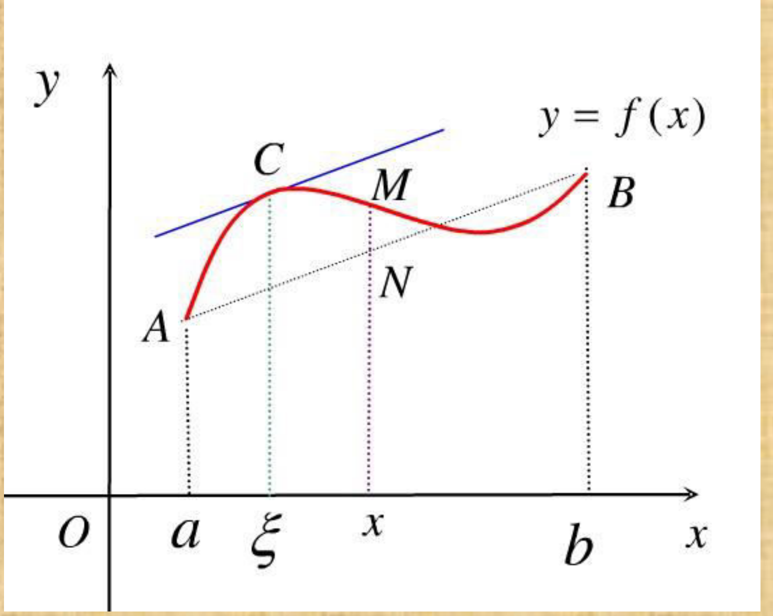
$$\frac{d}{dt}e^{z_0 t} = (x_0+iy_0)e^{(x_0+iy_0)t} = z_0 e^{z_0 t}$$

Derivatives of Functions w(t)

- Example

  Suppose that the ***real function*** f(t) is continuous on an interval a≤ t ≤b, if f'(t) exists when a<t<b, the mean value theorem for derivatives tells us that there is a number ζ in the interval a<ζ<b such that

$$f'(\ \ )=\frac{f(b)\quad f(a)}{b\quad a}$$



Derivatives of Functions w(t)

- Example (Cont')

The mean value theorem no longer applies for some ***complex functions***. For instance, the function

$$w(t) = e^{it}$$

on the interval $0 \le t \le 2\pi$.

Please note that

$$|w'(t)| = |ie^{it}| = 1$$

And this means that the derivative w'(t) is never zero, while

$$w(2\pi) - w(0) = 0 \quad \Rightarrow \quad w'(\varsigma) \neq \frac{w(2\pi) - w(0)}{2\pi - 0} = 0, \forall \varsigma \in (0, 2\pi)$$

Note: not every rules from calculus holds for complex functions