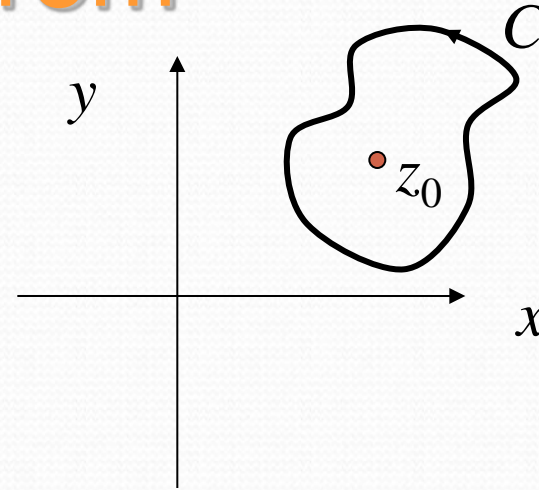


# Residue Theorem

# The Residue Theorem

Consider a line integral about a path enclosing an isolated singular point:

$$\oint_C f(z) dz$$



Expand  $f(z)$  in a Laurent series, deform the contour  $C$  to a circle of radius  $r$  centered at  $z_0$ , and evaluate the integral:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$\text{Let } z - z_0 \equiv re^{i\theta}, \quad dz = rie^{i\theta} d\theta,$$

$$\begin{aligned} \Rightarrow \oint_C f(z) dz &= i \sum_{n=-\infty}^{\infty} a_n r^{n+1} \underbrace{\int_0^{2\pi} e^{i(n+1)\theta} d\theta}_{=0, n \neq -1} \\ &= 2\pi i a_{-1} \end{aligned}$$

# The Residue Theorem (cont.)

The value  $a_{-1}$  corresponding to an isolated singular point  $z_0$  is called the “residue of  $f(z)$  at  $z_0$ ”.

$$\oint_C f(z) dz = 2\pi i \operatorname{Res} f(z_0)$$

Note path orientation is assumed counterclockwise

# The Residue Theorem (cont.)

Extend the theorem to multiple isolated singularities in the usual way:

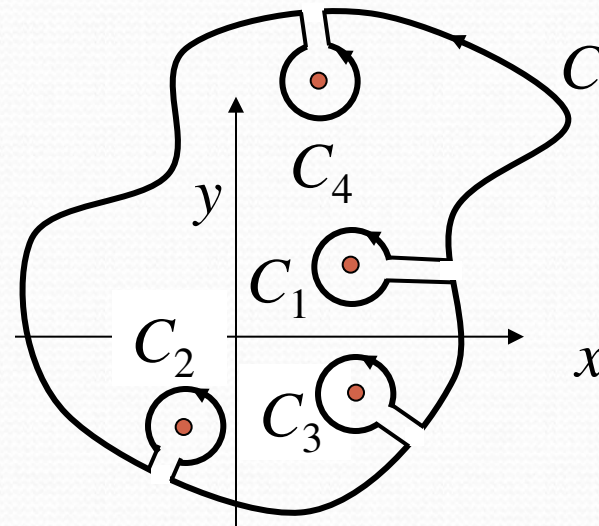
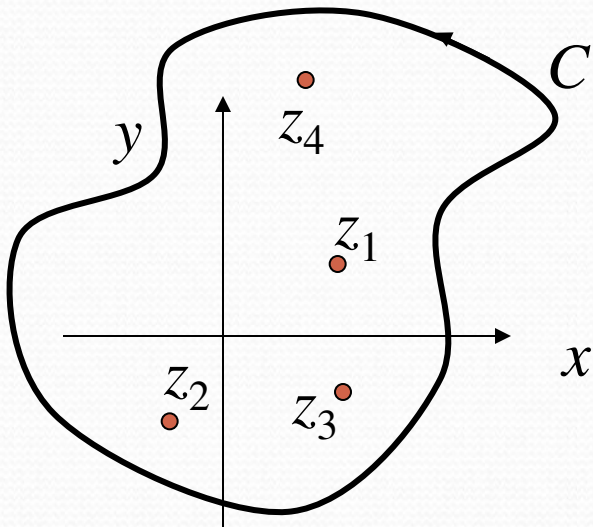
$$\oint_{C - \sum_n C_n} f(z) dz = \oint_C f(z) dz - \sum_n \oint_{C_n} f(z) dz = 0$$

$$\Rightarrow \oint_C f(z) dz = \sum_n \oint_{C_n} f(z) dz$$

$$\oint_C f(z) dz = 2\pi i \sum_n \text{Res } f(z_n)$$

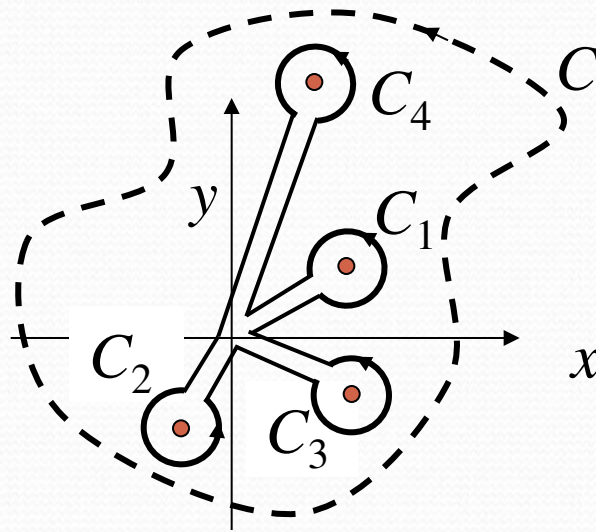
Isolated singularities at  $z_n$

=  $2\pi i$   $\times$  sum of residues enclosed by  $C$



# The Residue Theorem (cont.)

Alternatively, shrink the path to a point, leaving only the singularities encircled:



Isolated singularities at  $z_n$

$$\oint_C f(z) dz = 2\pi i \sum_n \text{Res } f(z_n)$$

=  $2\pi i$   $\times$  sum of residues enclosed by  $C$

# The Residue Theorem (cont.)

Note the residue theorem subsumes several of our earlier results and theorems, e.g. :

Cauchy's Formula:

$$\oint_C f(z) dz = 0 \text{ if } f(z) \text{ analytic (no isolated singularities) in } C$$

Cauchy Integral Formula:

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) \text{ if } f(z) \text{ analytic}$$

(no isolated singularities) in  $C$

# Evaluating Residues

- Construct Laurent series about each singularity  $z_n$ , identify coefficient  $a_{-1,n} = \text{Res } f(z_n)$ . Sometimes this can be a tedious approach.
- For a *simple* pole at  $z = z_0$ ,

$$\begin{aligned}\text{Res } f(z_0) &= \lim_{z \rightarrow z_0} (z - z_0) f(z) \\ &= \lim_{z \rightarrow z_0} (z - z_0) \left( \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots \right) \\ &= a_{-1}\end{aligned}$$

$$\text{Res } f(z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

# Evaluating Residues (cont.)

Example:

- $f(z) = \frac{2z}{(z+2)(z^2+1)}$  has simple poles at  $z = -2, +i, -i$

$\Rightarrow$

$$\text{Res } f(-2) = \lim_{z \rightarrow -2} \cancel{(z+2)} \frac{2z}{\cancel{(z+2)}(z^2+1)} = -\frac{4}{5}$$

$$\text{Res } f(i) = \lim_{z \rightarrow i} \cancel{(z-i)} \frac{2z}{(z+2)(z+i)\cancel{(z-i)}} = \frac{\cancel{2i}}{(2+i)\cancel{2i}} = \frac{2-i}{5}$$

$$\text{Res } f(-i) = \lim_{z \rightarrow -i} \cancel{(z+i)} \frac{2z}{(z+2)(z-i)\cancel{(z+i)}} = \frac{\cancel{(-2i)}}{(2-i)\cancel{(-2i)}} = \frac{2+i}{5}$$



# Evaluating Residues (cont.)

Example:

$$f(z) = \frac{1}{\sin z} \text{ has simple poles at } z = n\pi, \quad n = 0, \pm 1, \pm 2, \dots$$

$\Rightarrow$

$$\text{Res } f(n\pi) = \lim_{z \rightarrow n\pi} (z - n\pi) \times \frac{1}{\sin z} \stackrel{\text{L'Hospital's rule}}{=} \lim_{z \rightarrow n\pi} \frac{1}{\cos z} = (-1)^n$$

Alternatively,

$$\begin{aligned} \text{Res } f(n\pi) &= \lim_{z \rightarrow n\pi} (z - n\pi) \frac{1}{\sin z} = \lim_{z \rightarrow n\pi} (z - n\pi) \frac{1}{\sin(z - n\pi + n\pi)} \\ &= \lim_{z \rightarrow n\pi} (z - n\pi) \frac{1}{\sin(z - n\pi) \cos n\pi + \cos(z - n\pi) \sin n\pi} \\ &= \lim_{z \rightarrow n\pi} \frac{(z - n\pi)}{\sin(z - n\pi) \cos n\pi} = (-1)^n \text{ since we already know } \lim_{z \rightarrow 0} \frac{z}{\sin z} \rightarrow 1 \end{aligned}$$

# Evaluating Residues (cont.)

- The above rule can be specialized to functions of the form  $g(z)/f(z)$  where  $g(z)$  is analytic, non-vanishing, and  $f(z)$  has a simple zero, all at  $z_0$ :

$$\begin{aligned}\operatorname{Res} \frac{g(z_0)}{f(z_0)} &= \lim_{z \rightarrow z_0} (z - z_0) \frac{g(z)}{f(z)} = \lim_{z \rightarrow z_0} (z - z_0) \frac{g(z)}{f(z) - f(z_0)} \\ &= \frac{g(z_0)}{\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{(z - z_0)}} = \frac{g(z_0)}{f'(z_0)}\end{aligned}$$

$$\Rightarrow \operatorname{Res} \frac{g(z_0)}{f(z_0)} = \frac{g(z_0)}{f'(z_0)}$$

Example:

$$\tan z = \frac{\sin z}{\cos z} \text{ has simple poles at } z = \frac{(2n+1)\pi}{2},$$

for  $n = 0, \pm 1, \pm 2, \dots$

$$\Rightarrow \operatorname{Res} \tan \frac{(2n+1)\pi}{2} = \frac{\cancel{\sin \frac{(2n+1)\pi}{2}}}{-\cancel{\sin \frac{(2n+1)\pi}{2}}} = -1$$

# Evaluating Residues (cont.)

- For a pole of finite order  $m$  at  $z = z_0$ :

At the pole, the Laurent series is

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

Hence the following series is a Taylor series about the pole,

$$(z - z_0)^m f(z) = a_{-m} + a_{-m+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{m-1} + a_0(z - z_0)^m + \cdots$$

for which the formula for the coefficient of  $(z - z_0)^{m-1}$  is

$$a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[ (z - z_0)^m f(z) \right] \Big|_{z=z_0}$$

$$\text{Res } f(z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[ (z - z_0)^m f(z) \right] \Big|_{z=z_0} \quad (m \text{ finite})$$

# Evaluating Residues (cont.)

$$\text{Res } f(z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[ (z - z_0)^m f(z) \right] \Big|_{z=z_0} \quad (m \text{ finite})$$

Example:

$$f(z) = \frac{2z^2 + z}{(z+2)^3} \quad \text{has a pole of order 3 at } z = -2$$

$$\begin{aligned} \Rightarrow \text{Res } f(-2) &= \frac{1}{2!} \frac{d^2}{dz^2} \left[ \cancel{(z+2)^3} \frac{2z^2 + z}{\cancel{(z+2)^3}} \right] \Big|_{z=-2} \\ &= \frac{1}{2!} \frac{d^2}{dz^2} [2z^2 + z] \Big|_{z=-2} = \frac{4}{2} = 2 \end{aligned}$$

# Evaluating Residues (cont.)

**Example:**  $f(z) = \frac{1}{\sin^2 z}$  has poles of order 2 at  $z = n\pi$ ,  $n = 0, \pm 1, \pm 2, \dots$

$$\Rightarrow \text{Res } f(n\pi) = \left. \frac{d}{dz} \left[ \frac{(z - n\pi)^2}{\sin^2 z} \right] \right|_{z = n\pi} = \left. \frac{d}{dz} \left[ \frac{(z - n\pi)^2}{\sin^2(z - n\pi + n\pi)} \right] \right|_{z = n\pi}$$

$$= \left. \frac{d}{dz} \left[ \frac{(z - n\pi)^2}{[\sin(z - n\pi) \cos n\pi + \cos(z - n\pi) \sin n\pi]^2} \right] \right|_{z = n\pi} = \left. \frac{d}{dz} \left[ \frac{(z - n\pi)^2}{\sin^2(z - n\pi)} \right] \right|_{z = n\pi}$$

Let  $u = z - n\pi$

$$= \left. \frac{d}{du} \left[ \frac{u^2}{\sin^2 u} \right] \right|_{u=0} = \left. \frac{2u(\sin^2 u - u \cancel{\sin u} \cos u)}{\sin^4 u} \right|_{u=0}$$

L'hospital's rule

$$= \left. \frac{2(\sin u - u \cos u) + 2u^2 \sin u}{3 \sin^2 u \cos u} \right|_{u=0}$$

L'hospital's rule

$$= \left. \frac{6u \sin u + 2u^2 \cos u}{3 \sin u (2 \cos^2 u - \sin^2 u)} \right|_{u=0}$$

L'hospital's rule

$$= \left. \frac{6 \sin u + 10u \cos u - 2u^2 \sin u}{3 \cos u (2 \cos^2 u - \sin^2 u) - 18 \sin^2 u \cos u} \right|_{u=0} = 0$$

# Evaluating Residues (cont.)

The previous result might be “more easily” obtained as follows:

$$\begin{aligned}
 f(z) &= \frac{1}{\sin^2 z} = \frac{1}{\sin^2(z - n\pi + n\pi)} = \frac{1}{\sin^2(z - n\pi)} = \frac{1}{\sin^2 u} \Big|_{u = z - n\pi \rightarrow 0} \\
 &= \frac{1}{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \dots\right)^2} = \frac{1}{u^2 \left(1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \dots\right)^2}
 \end{aligned}$$

$$\stackrel{\text{Geometric Series}}{=} \frac{1}{u^2} \left[ 1 + \left(\frac{u^2}{3!} - \frac{u^4}{5!} + \frac{u^6}{7!} \dots\right) + \left(\frac{u^2}{3!} - \frac{u^4}{5!} + \frac{u^6}{7!} \dots\right)^2 + \dots \right]^2$$

$$= \frac{1}{u^2} \left[ 1 + \frac{u^2}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)u^4 + \dots \right]^2 = \frac{1}{u^2} + \frac{2}{3!} + \dots$$

missing  $\frac{a_{-1}}{u}$  term  $\Rightarrow \text{Res } f(n\pi) = 0$

# Evaluating Residues (cont.)

Example:

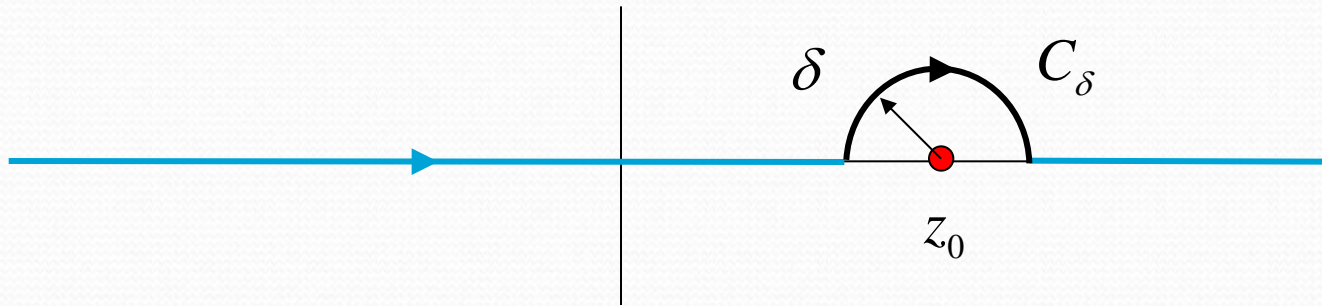
$$f(z) = e^{\frac{1}{z}} \quad (\text{isolated essential singularity at } z = 0)$$

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!} \left(\frac{1}{z}\right)^2 + \frac{1}{3!} \left(\frac{1}{z}\right)^3 + \dots$$

Residue:  $a_{-1} = 1$

# Evaluating Residues (cont.)

Extension for **simple poles** (going halfway around)



As  $\delta \rightarrow 0$ :

$$\int_{C_\delta} f(z) dz = -\pi i \operatorname{Res} f(z_0)$$

**Proof:**

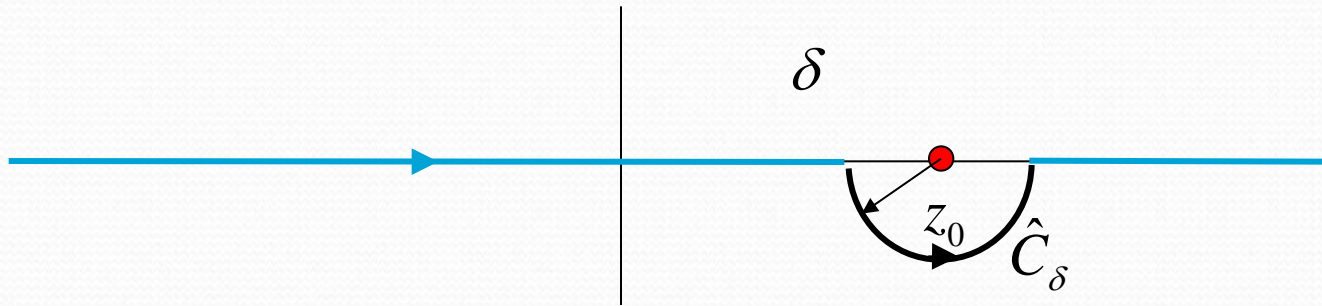
$$z - z_0 = \delta e^{i\theta}$$

$$\int_{C_\delta} \frac{a_{-1}}{z - z_0} dz = a_{-1} \int_{C_\delta} \frac{dz}{z - z_0} = a_{-1} \int_{\pi}^0 \frac{i\delta e^{i\theta}}{\delta e^{i\theta}} d\theta = a_{-1} (-\pi i)$$



# Evaluating Residues (cont.)

What is the result if the contour goes around the simple pole on the opposite side?



Hint: The contour  $\hat{C}_\delta - C_\delta$  encircles the pole in the counterclockwise direction so that

$$\oint_{\hat{C}_\delta - C_\delta} f(z) dz = 2\pi i \operatorname{Res} f(z_0)$$

$$\Rightarrow \int_{\hat{C}_\delta} f(z) dz = \pi i \operatorname{Res} f(z_0)$$

# Evaluation of residues at multiple poles

If  $f(z)$  has a pole of order  $n$  at  $z = a$  and no other singularity,  $f(z)$  is:

$$f(z) = \frac{F(z)}{(z-a)^n}$$

where  $n$  is a finite integer, and  $F(z)$  is analytic at  $z = a$ .

$F(z)$  can be expanded by the Taylor series:

$$F(z) = F(a) + (z-a)F'(a) + \frac{(z-a)^2}{2!} F''(a) + \dots + \frac{(z-a)^n}{n!} F^n(a) + \dots$$

Dividing throughout by  $(z-a)^n$

$$f(z) = \frac{F(a)}{(z-a)^n} + \frac{F'(a)}{(z-a)^{n-1}} + \dots + \frac{F^{n-1}(a)}{(n-1)!(z-a)} + \dots$$

The residue at  $z = a$  is the coefficient of  $(z-a)^{-1}$

The residue at a pole of order  $n$  situated at  $z = a$  is:

$$B_1 = \frac{F^{n-1}(a)}{(n-1)!} \Big|_{z=a} = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[ (z-a)^n f(z) \right]_{z=a}$$

### EXAMPLE:

Evaluate  $\int_C \frac{\cos 2z}{(z-a)^3} dz$  around a circle of radius  $|z| > |a|$ .

$\frac{\cos 2z}{(z-a)^3}$  has a pole of order 3 at  $z = a$ , and the residue is:

$$B_1 = \frac{F^{n-1}(a)}{(n-1)!} \Big|_{z=a} = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[ (z-a)^n f(z) \right]_{z=a}$$
$$= \frac{1}{2!} \frac{d^2}{dz^2} \left[ (z-a)^3 \frac{\cos 2z}{(z-a)^3} \right]_{z=a} = -2 \cos 2a$$

$$\therefore \int_C \frac{\cos 2z}{(z-a)^3} dz = 2\pi i (-2 \cos 2a)$$