Singularities, Zeros And Poles

Singularities

- We have seen that the function $w = z^3$ is analytic everywhere except at $z =$ *:* whilst the function $w = z^{-1}$ is analytic everywhere except at $z = 0$.
- In fact, ***NO*** function except a ***constant*** is analytic throughout the complex plane, and every function except of a complex variable has one or more points in the z plane where it ceases to be analytic.
- These points are called "singularities".

Types of singularities

- Three types of singularities exist:
  - Poles or unessential singularities
    - “single-valued” functions
  - Essential singularities
    - “single-valued” functions
  - Branch points
    - “multivalued” functions

Poles or unessential singularities

- A pole is a point in the complex plane at which the value of a function becomes *infinite*.
- For example, $w = z^{-1}$ is infinite at $z = 0$, and we say that the function $w = z^{-1}$ has a pole at the origin.
- A pole has an "order":
  - The pole in $w = z^{-1}$ is first order.
  - The pole in $w = z^{-2}$ is second order.

The order of a pole

If $w = f(z)$ becomes infinite at the point $z = a$, we define:

$$g(z) = (z-a)^n f(z) \qquad \text{where } n \text{ is an integer.}$$

If it is possible to find a finite value of n which makes $g(z)$ analytic at $z = a$, then, the pole of $f(z)$ has been "removed" in forming $g(z)$.
The order of the pole is defined as the minimum integer value of n for whic
$g(z)$ is analytic at $z = a$.

$$w = \frac{1}{z} \qquad \text{pole, (a=0)}$$

$$(z)^n \frac{1}{z} = g(z) \qquad \text{Order} = 1$$

Essential singularities

- Certain functions of complex variables have an infinite number of terms which all approach infinity as the complex variable approaches a specific value. These could be thought of as poles of infinite order, but as the singularity cannot be removed by multiplying the function by a finite factor, they cannot be poles.
- This type of sigularity is called an *essential singularity* and is portrayed by functions which can be expanded in a descending power series of the variable.
- Example: $e^{1/z}$ has an essential sigularity at $z = 0$.

Essential singularities can be distinguished from poles by the fact that
they cannot be removed by multiplying by a factor of finite value.

Example:

$$w = e^{1/2} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \ldots + \frac{1}{n!\,z^n} + ..$$

infinite at the origin

We try to remove the singularity of the function at the origin by multiplying z^p

$$z^p w = z^p + z^{p-1} + \frac{z^{p-2}}{2!} + \ldots + \frac{z^{p-n}}{n!} + ..$$

It consists of a finite number of positive powers of z, followed by an infinite number of negative powers of z.

All terms are positive

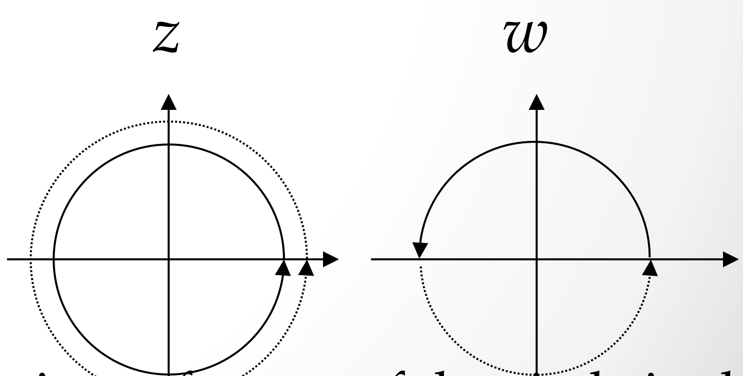
$$As \quad z \to 0, \quad z^p w \to \infty$$

It is impossible to find a finite value of *p* which will remove the singularity in $e^{1/z}$ at the origin.
The singularity is "essential".

Branch points

- The singularities described above arise from the non-analytic behaviour of single-valued functions.
- However, multi-valued functions frequently arise in the solution of engineering problems.
- For example:

$$\boxed{w = z^{\frac{1}{2}}} \xrightarrow{z = re^{i\theta}} \boxed{w = r^{\frac{1}{2}} e^{\frac{1}{2} i\theta}}$$



For any value of z represented by a point on the circumference of the circle in the z plane, there will be two corresponding values of w represented by points in the w plane.

$$\boxed{w = z^{\frac{1}{2}}} \xrightarrow{z = re^{i\theta}} \boxed{w = r^{\frac{1}{2}} e^{\frac{1}{2} i\theta}}$$

$$w = u + iv \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$u = r^{\frac{1}{2}} \cos\frac{1}{2}\theta \quad and \quad v = r^{\frac{1}{2}} \sin\frac{1}{2}\theta$$

$$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\frac{1}{2}\theta \qquad \frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin\frac{1}{2}\theta$$

$$\frac{\partial u}{\partial \theta} = -\frac{1}{2}\sqrt{r}\sin\frac{1}{2}\theta \qquad \frac{\partial v}{\partial \theta} = \frac{1}{2}\sqrt{r}\cos\frac{1}{2}\theta$$

when $0 \le \theta \le 2\pi$

A given range, where the function is single valued: the "branch"

$$\boxed{\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad and \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

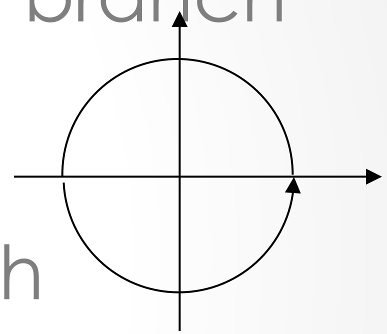
Cauchy-Riemann conditions in polar coordinates

The particular value of z at which the function becomes infinite or zero is called the "branch point".

The origin is the branch point here

Branch point

- A function is only multi-valued around closed contours which enclose the branch point.



- It is only necessary to eliminate such contours and the function will become single valued.
  - The simplest way of doing this is to erect a barrier from the branch point to infinity and not allow any curve to cross the barrier.
  - The function becomes single valued and analytic for all permitted curves.

Barrier - branch cut

- The barrier must start from the branch point but it can go to infinity in any direction in the z plane, and may be either curved or straight.
- In most normal applications, the barrier is drawn along the negative real axis.
  - The branch is termed the "principle branch".
  - The barrier is termed the "branch cut".
  - For the example given in the previous slide, the region, the barrier confines the function to the region in which the argument of z is within the range $-\pi < \theta < \pi$.

Zeros and Poles of order m

Consider a function f that is analytic at a point z_0.

(From Sec. 40). $f^{(n)}(z) \quad (n=1,2,\dots.)$ exist at z_0

$$
\begin{aligned}
If \quad & f(z_0)=0\,, \\
& f'(z_0)=0 \\
& \quad \vdots \\
& f^{(m\text{-}1)}(z_0)=0 \\
& f^{(m)}(z_0)\neq 0
\end{aligned}
$$

Then f is said to have a zero of order m at z_0.

Lemma: $f(z)=(z-z_0)^m \underline{g(z)}$

analytic and non-zero at z_0.

Example.

$$f(z) = z(e^z - 1)$$

$$= z^2\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots\ldots\right)$$

has a zero of order $m = 2$ at $z_0 = 0$

$$g(z) = \begin{cases} (e^z - 1)/z & \text{when } z \neq 0 \\ 1 & \text{when } z = 0 \end{cases}$$ is analytic at $z = 0$.

Thm. Functions p and q are analytic at z_0, and $p(z_0) \neq 0$.

If q has a zero of order m at z_0, then

$\dfrac{p(z)}{q(z)}$ has a pole of order m there.

$$q(z) = (z - z_0)^m \, g(z)$$

analytic and non zero

$$\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{(z - z_0)^m}$$

Example. $f(z)=\dfrac{1}{z(e^z-1)}$ has a pole of order 2 at $z_0=0$

Corollary: Let two functions p and q be analytic at a point z_0.

If $p(z_0)\neq 0$, $q(z_0)=0$, and $q'(z_0)\neq 0$

then z_0 is a simple pole of $\dfrac{p(z)}{q(z)}$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)}=\frac{p(z_0)}{q'(z_0)}$$

Pf:

$q(z)=(z-z_0)\ g(z)$, $g(z)$ is analytic ard non zero at z_0

$$\frac{p(z)}{q(z)}=\frac{p(z)/g(z)}{z-z_0}$$

Form Theorem in sec 56, $\operatorname*{Res}_{z=z_0} \dfrac{p(z)}{q(z)}=\dfrac{p(z_0)}{g(z_0)}$

But $g(z_0)=q'(z_0)$ $\qquad =\dfrac{p(z_0)}{q'(z_0)}$

Example.

$$f(z)=\cot z=\frac{\cos z}{\sin z}$$

$p(z)=\cos z,\ q(z)=\sin z$ both entire

The singularities of $f(z)$ occur at zeros of q, or

$$z=n\pi \quad (n=0,\ \pm 1,\ \pm 2, \ldots)$$

Since $p(n\pi)=(-1)^n \neq 0$, $q(n\pi)=0$, and $q'(n\pi)=(-1)^n \neq 0$

each singular point $z=n\pi$ of f is a simple pole,

with residue $B_n=\dfrac{p(n\pi)}{q'(n\pi)}=\dfrac{(-1)^n}{(-1)^n}=1$

$\boxed{\text{try } \tan z}$