Solution of pde's using integral transforms

Introduction

- Integral transforms are used to:
- Simplify solutions by eliminating or reducing order of pde in a particular variable
- Offer physical insight into the problem
- Type of transform depends on BC of problem

Introduction
General form for integral transform

$$F(s) = \int_A^B K(s,t)\, f(t)\, dt$$

K(s,t) is the kernel of the transform which decides the type of transform. f(t) is transformed to F(s)

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ikx)\; f(x)\; dx$$

$$F^{-1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(+ikx)\; F(k)\, dk$$

Fourier transform (FT) and back transform

FT suitable for BC where the dependent variable vanishes at ∞

Example: Find the FT of $\exp[-ax^2]$ *Hint*: Complete the square

Fourier transforms of partial derivatives

- pde's are transformed when integral transforms are applied to the pde and BC

$$F[u_x(x,t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_x(x,t) \exp(-ikx)\, dx = ik\, F[u(x,t)]$$

$$F[u_{xx}(x,t)] = -k^2\, F[u(x,t)]$$

$$F[u_t(x,t)] = \frac{\partial}{\partial t} F[u(x,t)]$$

$$F[u_{tt}(x,t)] = \frac{\partial^2}{\partial t^2} F[u(x,t)]$$

- The first two results are obtained by integration by parts

Fourier Transform of the wave equation wrt x

$u_{tt} - c^2 u_{xx} = 0$ pde

$u(x,t) \to 0$ as $x \to \infty$ BC

$u(x,0) = \exp[-ax^2]$ IC

$F[u_{tt}] - c^2 F[u_{xx}] = 0$ FT pde

$U''(k,t) + (ck)^2 U(k,t) = 0$ ode in t

$U(k,0) = F[\exp(-ax^2)] = \frac{1}{2\sqrt{a}}\exp(-k^2/4a)$ FT IC

$U(k,t) = U(k,0)\exp[-i\omega t)]$ $\omega = ck$ ode solution

$u(x,t) = \int_{-\infty}^{\infty} U(k,0)\exp[i(kx - \omega t)]dk$ pde solution

Sine and cosine transforms
Definitions

• The choice of integral transform depends on BC. If we wish to solve a pde with boundaries at x = 0 and x → ∞ then sine and cosine transforms are appropriate.

$$F_s[u] = \;= \frac{2}{\pi}\int_0^{\infty} \sin(kx)\; u(x,t)\, dx$$

$$F_s^{-1}[u] = u(x,t) = \int_0^{\infty} \sin(kx)\; F_s[u]\, dk$$

$$F_c[u] = \;= \frac{2}{\pi}\int_0^{\infty} \cos(kx)\; u(x,t)\, dx$$

$$F_c^{-1}[u] = u(x,t) = \int_0^{\infty} \cos(kx)\; F_c[u]\, dk$$

Transforms of partial derivatives

- These results are obtained using integration by parts to eliminate derivatives from the integrand

$$F_s[u_x] = -k\,F_c[u]$$

$$F_s[u_{xx}] = -k^2\,F_s[u] + \frac{2}{\pi}\,k\,u(0,t)$$

$$F_c[u_x] = +k\,F_s[u] - \frac{2}{\pi}\,u(0,t)$$

$$F_c[u_{xx}] = -k^2\,F_c[u] - \frac{2}{\pi}\,u_x(0,t)$$

Convolution Theorem

• Convolution (or *resultant* or *Faltung*) of f and g (f*g) is defined to be

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t - \tau)\, g(\tau)\, d\tau$$

• If a function u may be written

$$u = F[f]\, F[g]$$

$$F^{-1}[u] = F^{-1}\{F[f]\, F[g]\} = f * g$$

Its inverse Fourier transform is the convolution of f and g.

• Used for solving pde's when we obtain a solution to a particular problem in terms of products of Fourier transforms

Convolution Theorem
Example

$$f(x)=x \quad g(x) = e^{-x^2}$$

$$f*g=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\xi)e^{-\xi^2}\,d\xi$$

$$\int_{-\infty}^{\infty}\xi\, e^{-\xi^2}\,d\xi=0 \text{ by symmetry}$$

$$\int_{-\infty}^{\infty} e^{-\xi^2}\,d\xi=\sqrt{\pi}$$

$$\text{Hence } \; f*g = \frac{x\sqrt{\pi}}{\sqrt{2\pi}}=\frac{x}{\sqrt{2}}$$

Application of Fourier, sine and cosine transforms

$u_{tt} = u_{xx} + \sin(\pi x)$ $\quad 0 < x < 1$

$0 < t < \infty$

$u(0,t) = 0$ $\quad 0 < t < \infty$

$u(1,t) = 0$

$u(x,0) = 1$ $\quad 0 < x < 1$

$u_t(x,0) = 1$

- We require a finite integral transform for which BC are specified as u(t) (not $u_x(t)$, *etc.*)

Application of Fourier, sine and cosine transforms
Finite sine and cosine transforms

$$S_n(t) = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) u(x,t)\,dx$$

$$u(x,t) = \sum_{n=1}^{\infty} S_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$C_n(t) = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) u(x,t)\,dx$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t) \cos\left(\frac{n\pi x}{L}\right)$$

- Note that the range of integration is [0,L] (finite)

Application of Fourier, sine and cosine transforms
Transforms of partial derivatives

$$S_n[u_x] = -\frac{n\pi}{L} C_n(t)$$

$$S_n[u_{xx}] = -\left(\frac{n\pi}{L}\right)^2 S_n(t) + \frac{2n\pi}{L^2}\left[u(0,t) + (-1)^{n+1} u(L,t)\right]$$

$$C_n[u_x] = \frac{n\pi}{L} S_n(t) + \frac{2}{L}\left[-u(0,t) + (-1)^n u(L,t)\right]$$

$$C_n[u_{xx}] = -\left(\frac{n\pi}{L}\right)^2 C_n(t) - \frac{2}{L}\left[u_x(0,t) + (-1)^{n+1} u_x(L,t)\right]$$

- Note that BC enter as u(t) in $S[u_{xx}]$ and as $u_x(t)$ in $C[u_{xx}]$. The specified BC determine which transform to choose.

Application of Fourier, sine and cosine transforms
Solving inhomogeneous BVP using finite sine transform

- BC involve $u(t)$ rather than $u_x(t)$ so ***finite sine*** transform

1. Transform pde
2. Transform IC
3. Solve resulting ode
4. Back transform

1. Transform pde

$$S[u_{tt}] = S[u_{xx}] + S[\sin(\pi x)]$$

$$S_n''(t) = -(n\pi)^2 S_n(t) + 2n\pi[u(0,t) + (-1)^{n+1}u(1,t)] + S[\sin(\pi x)]$$

$$= -(n\pi)^2 S_n(t) + S[\sin(\pi x)]$$

$$S[\sin(\pi x)] \quad = 1 \qquad n = 1$$

$$= 0 \qquad n = 2, 3, 4, \ldots$$

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2. Transform IC

$$S[u(x,0)] = S[1] = \frac{2}{1}\int_0^L \sin\left(\frac{n\pi x}{1}\right).1dx$$

$$= \frac{4}{n\pi} \quad n = 1, 3, 5, \ldots$$

$$= 0 \quad n = 2, 4, 6, \ldots$$

$$S[u_t(x,0)] = S[0] = 0 \quad n = 1, 2, 3, \ldots$$

Application of Fourier, sine and cosine transforms
Solving inhomogeneous BVP using finite sine transform

3. Solve resulting ode

$$S_n''(t) + (n\pi)^2 S_n(t) \;\begin{cases} = 1 & n = 1 \\ = 0 & n = 2, 3, 4, .. \end{cases} \qquad \text{ode}$$

$$S_n(0) = \begin{cases} \dfrac{4}{n\pi} & n = 1, 3, 5, \ldots \\ = \;0 & n = 2, 4, 6, \ldots \end{cases} \qquad \text{IC}$$

$$S_n'(t) \;= 0 \qquad n = 1, 2, 3, \ldots$$

Application of Fourier, sine and cosine transforms
Solving inhomogeneous BVP using finite sine transform

$S_1(t) \quad = A\cos(\pi t) + \pi^{-2}$

$S_n(t) \quad = 0 \qquad\qquad n = 2, 4, 6, \ldots$ solution

$S_n(t) \quad = 4\,(n\pi)^{-1}\cos(n\pi t) \qquad n = 3, 5, 7, \ldots$

$S_1(0) \quad = A + \pi^{-2} = 4\pi^{-1} \quad A = (4\pi - 1)\pi^{-2}$

$S_n(0) \quad = 0 \qquad\qquad n = 2, 4, 6, \ldots$ IC satisfied

$S_n(0) \quad = 4\,(n\pi)^{-1} \qquad n = 3, 5, 7, \ldots$

$S_1'(0) \quad = -A\pi\,\sin(\pi 0) \qquad = 0$

$S_n'(0) \quad = 0 \qquad\qquad n = 2, 4, 6, \ldots$ IC satisfied

$S_n'(0) \quad = -4\,\sin(n\pi 0) \qquad = 0 \quad n = 3, 5, 7, \ldots$

ode satisfied

$S_n''(t) + (n\pi)^2 S_n(t) \;= -A\,\pi^2\cos(\pi t) + A\,\pi^2\cos(\pi t) + 1 \quad n = 1$

$\qquad = -4n\pi\cos(n\pi t) + 4n\pi\cos(n\pi t) \qquad n = 3, 5,$

Application of Fourier, sine and cosine transforms
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4. Back transform

$$u(x,t) = \sum_{n=1}^{\infty} S_n(t) \sin\left(\frac{n\pi x}{1}\right)$$

$$= \left[A\cos(\pi t) + \pi^{-2}\right]\sin(\pi x) +$$

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1}\cos[(2n+1)\pi t]\sin\left[(2n+1)\pi x\right]$$

Summary of Integral Transforms

Kernel	**Boundary Conditions**	**Restrictions**
exp(ikx)	$u(x,t) \to 0$ as $x \to \pm\infty$	No FT exists for many functions
sin(kx)	$u(0,t) = f(t)$ $u(x,t) \to 0$ as $x \to \pm\infty$	PDE must have no 1^{st} order derivatives wrt x
cos(kx)	$u_x(0,t) = f(t)$ $u(x,t) \to 0$ as $x \to \pm\infty$	PDE must have no 1^{st} order derivatives wrt x
finite sine	$u(0,t)$, $u(L,t)$	no mixed BC
finite cosine	$u_x(0,t)$, $u_x(L,t)$	no mixed BC
exp(-st)	$u(0,t)$, $u_x(0,t)$ $u(x,t) \to 0$ as $x \to \pm\infty$ BC can be mixed	$u(x,t)$ does not grow faster than exponentially for $t>T$