Complex Fourier Transforms

Complex Fourier transform

- Consider the expression

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{in\omega_0 t}=\sum_{n=-\infty}^{\infty}F_n\cos(n\omega_0 t)+iF_n\sin(n\omega_0 t)$$

$$=\sum_{n=0}^{\infty}(F_n+F_{-n})\cos(n\omega_0 t)+i(F_n-F_{-n})\sin(n\omega_0 t)$$

- So $a_n=F_n+F_{-n}$ and $b_n=i(F_n-F_{-n})$
- Since a_n and b_n are real, we can let $F_{-n}=\overline{F_n}$

and get $a_n=2\operatorname{Re}(F_n)$ and $b_n=-2\operatorname{Im}(F_n)$

$$\operatorname{Re}(F_n)=\frac{a_n}{2}\quad\text{and}\quad\operatorname{Im}(F_n)=-\frac{b_n}{2}$$

- Thus $$F_n = \frac{1}{T}\left(\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)\,dt - i\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)\,dt \right)$$
$$= \frac{1}{T}\int_{t_0}^{t_0+T} f(t)(\cos(n\omega_0 t)\,dt - i\sin(n\omega_0 t))dt$$
$$= \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,e^{-in\omega_0 t}\,dt$$
$$= |F_n|\,e^{i\varphi_n}$$

- So you could also write $f(t) = \sum_{n=-\infty}^{\infty} |F_n| e^{i(n\omega_0 t+\varphi_n)}$

The Fourier transform G(k) and the original function g(x) are both in general complex.

$$\Im\{g(x)\} = G_r(k) + iG_i(k)$$

The Fourier transform can be written as,

$$\Im\{g(x)\} = G(k) = A(k)e^{i\Theta(k)}$$

$$A = |G| = \sqrt{G_r^2 + G_i^2}$$

$A \equiv$ amplitude spectrum, or magnitude spectrum

$\Theta \equiv$ phase spectrum

$\mathrm{A}^2 = |G|^2 = G_r^2 + G_i^2 \equiv$ power spectrum