

Convolution Theorem



Convolution theorem

$$F(f * g) = F(f)F(g)$$

$$F(fg) = F(f) * F(g)$$

$$F^{-1}(F * G) = F^{-1}(F)F^{-1}(G)$$

$$F^{-1}(FG) = F^{-1}(F) * F^{-1}(G)$$

Theorem

$$F(f * g) = \int_{-\infty}^{\infty} f(t')g(t-t') \int_{-\infty}^{\infty} e^{-i\omega t} dt dt'$$

Proof (1)

$$= \int_{-\infty}^{\infty} f(t')e^{-i\omega t'} dt' \int_{-\infty}^{\infty} g(t-t')e^{-i\omega(t-t')} dt$$

$$= \int_{-\infty}^{\infty} f(t')e^{-i\omega t'} dt' \int_{-\infty}^{\infty} g(t'')e^{-i\omega t''} dt''$$

$$= F(f)F(g)$$

Convolution Theorem

Example

$$f(x) = x \quad g(x) = e^{-x^2}$$

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - \xi) e^{-\xi^2} d\xi$$

$$\int_{-\infty}^{\infty} \xi e^{-\xi^2} d\xi = 0 \text{ by symmetry}$$

$$\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi}$$

$$\text{Hence } f * g = \frac{x \sqrt{\pi}}{\sqrt{2\pi}} = \frac{x}{\sqrt{2}}$$