## Fourier Integrals

### From Fourier Series to Fourier Integral

 Consider any periodic function f<sub>L</sub>(x) of period 2L that is represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x),$$
  $w_n = \frac{n\pi}{L}$ 

- what happens if we let  $L \rightarrow \infty$ ?
- We should expect an integral (instead of a series) involving  $\cos wx$  and  $\sin wx$  with w no longer restricted to integer multiples  $w = w_n = n\pi/L$  of  $\pi/L$  but taking *all* values.

• If we insert  $a_n$  and  $b_n$ , and denote the variable of integration by v, the Fourier series of  $f_L(x)$  becomes

$$f_L(x) = \frac{1}{2L} \int_{-L}^{L} f_L(v) \, dv + \frac{1}{L} \sum_{n=1}^{\infty} \left[ \cos w_n x \int_{-L}^{L} f_L(v) \cos w_n v \, dv + \sin w_n x \int_{-L}^{L} f_L(v) \sin w_n v \, dv \right].$$

#### We now set

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$
.

• Then  $1/L = \Delta w/\pi$ , and we may write the Fourier series in the form

(1) 
$$f_{L}(x) = \frac{1}{2L} \int_{-L}^{L} f_{L}(v) dv + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[ (\cos w_{n}x) \Delta w \int_{-L}^{L} f_{L}(v) \cos w_{n}v dv + (\sin w_{n}x) \Delta w \int_{-L}^{L} f_{L}(v) \sin w_{n}v dv \right].$$

• Let  $L \rightarrow \infty$  and assume that the resulting nonperiodic function

$$f(x) = \lim_{L \to \infty} f_L(x)$$

is **absolutely integrable** on the *x*-axis; that is, the following limits exist:

$$\lim_{a \to -\infty} \int_a^0 |f(x)| \ dx + \lim_{b \to \infty} \int_0^b |f(x)| \ dx \quad \left( \text{written } \int_{-\infty}^\infty |f(x)| \ dx \right).$$

 $1/L \rightarrow$  o, and the value of the first term on the right side of (1)  $\rightarrow$  zero. Also  $\Delta w = \pi/L \rightarrow dw$ . The infinite series in (1) becomes an integral from o to  $\infty$ , which represents f(x), namely,

(3) 
$$f(x) = \frac{1}{\pi} \int_0^\infty \left[ \cos wx \int_{-\infty}^\infty f(v) \cos wv \, dv + \sin wx \int_{-\infty}^\infty f(v) \sin wv \, dv \right] dw.$$

If we introduce the notations

(4) 
$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv, \qquad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv$$

## Fourier integral

we can write this in the form

(5) 
$$f(x) = \int_0^\infty [A(w) \cos wx + B(w) \sin wx] dw.$$

This is called a representation of f(x) by a **Fourier** integral.

### Fourier Integral

#### • THEOREM 1

#### **Fourier Integral**

If f(x) is piecewise continuous in every finite interval and has a right-hand derivative and a left-hand derivative at every point and if the integral exists, then f(x) can be represented by a Fourier integral with A and B given by (4). At a point where f(x) is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of f(x) at that point.

## Sine Integral

• The case x = o is of particular interest. If x = o, then (7) gives

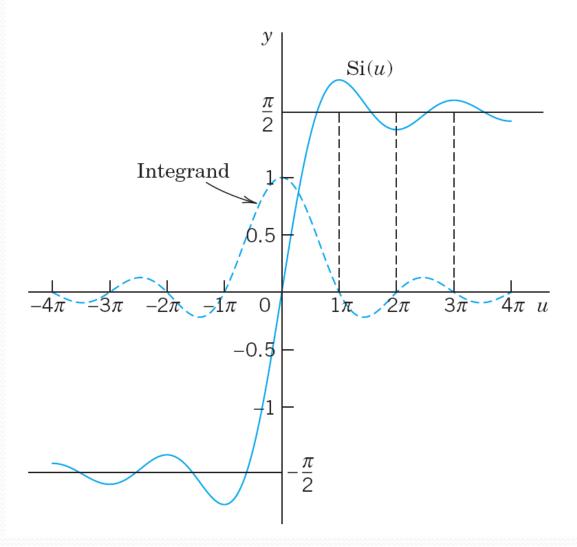
$$(8^*) \qquad \int_0^\infty \frac{\sin w}{w} \ dw = \frac{\pi}{2} \ .$$

We see that this integral is the limit of the so-called **sine integral** 

(8) 
$$\operatorname{Si}(u) = \int_0^u \frac{\sin w}{w} \ dw$$

as  $u \to \infty$ . The graphs of Si(u) and of the integrand are shown in Fig. 279.

### Fig. 279. Sine integral Si(u) and integrand



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 In the case of the Fourier integral, approximations are obtained by replacing ∞ by numbers a. Hence the integral

$$(9) \quad \frac{2}{\pi} \int_0^a \frac{\cos wx \sin w}{w} \ dw$$

which approximates f(x).

# Fourier Cosine Integral and Fourier Sine Integral

• If f(x) is an **even** function, then B(w) = 0 and

(10) 
$$A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \ dv.$$

The Fourier integral (5) then reduces to the **Fourier cosine integral** 

$$f(x) = \int_0^\infty A(w) \cos wx \, dw \qquad (f \text{ even}).$$

# Fourier Cosine Integral and Fourier Sine Integral

• If f(x) is an **odd** function, then A(w) = 0 and

(12) 
$$B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \ dv.$$

The Fourier integral (5) then reduces to the **Fourier cosine integral** 

$$f(x) = \int_0^\infty B(w) \sin wx \, dw \qquad (f \text{ odd}).$$