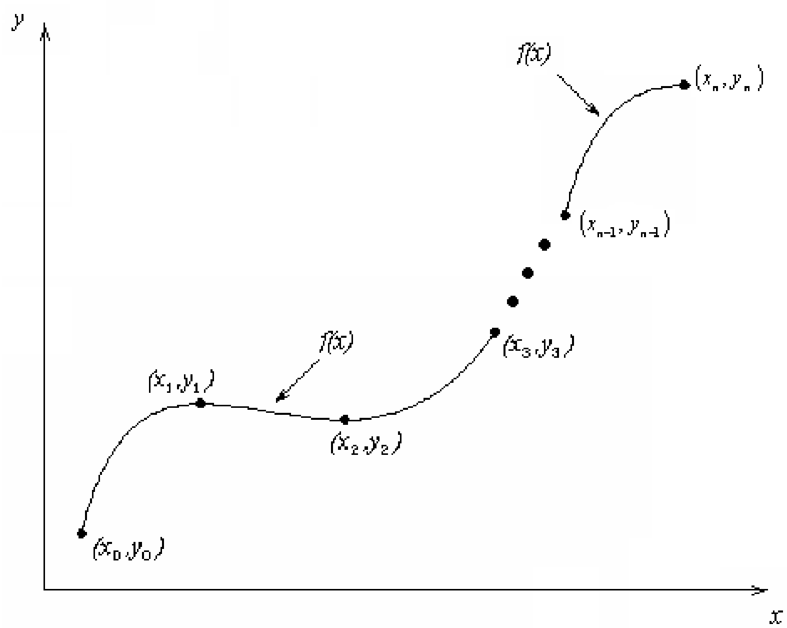
Newton's Forward and Backward Interpolation

WHAT IS INTERPOLATION?

Given (x_0,y_0), (x_1,y_1), …, (x_n,y_n), finding the value of 'y' at a value of 'x' in (x_0, x_n) is called **interpolation.**



INTERPOLANTS

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate,
- Differentiate, and
- Integrate.

NEWTONS DIVIDED DIFFERENCE

What is divided difference?

$$f[x_0,x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_1}$$

$$f[x_0, x_1, \ldots, x_{k-1}, x_k] = \frac{f[x_1, x_2 - x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}$$

for k = 3, 4, ….. n.

These I^{st}, II^{nd}... and k^{th} order differences are denoted by Δf, $\Delta^2 f$, …, $\Delta^k f$.

INTERPOLATION USING DIVIDED DIFFERENCE

The *divided difference interpolation polynomial* is:

$$P(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] + \Lambda + (x - x_0)\, \Lambda\, (x - x_{n-1})\, f[x_0, x_1, \ldots, x_n]$$

Example

For the data

x:		-1	0	2	5
f(x) :	7	10	22	235	

Find the divided difference polynomial and estimate f(1).

Solution

X	f	Δf	$\Delta^2 f$	$\Delta^3 f$
-1	7			
		3		
0	10		1	
		6		2
			13	
2	22			
		71		
5	235			

$$P(x) = f(x_0) + (x - x_0)\ f[x_0, x_1] + (x - x_0)(x - x_1)\ f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\ f[x_0, x_1, x_2, x_3]$$

$$= 7 + (x+1) \times 3 + (x+1)(x - 0) \times 1 + (x+1)(x - 0)(x - 2) \times 2$$

$$= 2x^3 - x^2 + 10$$

P (1) = 11

NEWTON FORWARD INTERPOLATION

For convenience we put $p = \frac{x - x_0}{h}$ and $f_0 = y_0$. Then we have

$$P(x_0 + ph) = y_o + pDy_0 + \frac{p(p-1)}{2!} D^2 y_0 + \frac{p(p-1)(p-2)}{3!} D^3 y_0 + \cdots +$$

$$\frac{p(p-1)(p-2)L(p-n+1)}{n!} D^n y_0$$

Example

Estimate f (3.17)from the data using Newton Forward Interpolation.

x:	3.1	3.2	3.3	3.4	3.5
f(x): 0	0.6	1.0	1.2	1.3	

Solution

First let us form the difference table

x	y	Δy	Δ²y	Δ³y	Δ⁴y
3.1	**0**				
		0.6			
3.2	0.6		- **0.2**		
		0.4		**0**	
3.3	1.0		- 0.2		**0.1**
		0.2		0.1	
3.4	1.2		-0.1		
		0.1			
3.5	1.3				

Here $x_0 = 3.1$, $x = 3.17$, $h = 0.1$.

Solution

$$P = \frac{x - x_0}{h} = \frac{0.07}{0.1} = 0.7$$

Newton forward formula is:

$$P(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$P(3.17) = 0 + 0.7 \times 0.6 + \frac{0.7(0.7-1)}{2} \times (-0.2) + \frac{0.7(0.7-1)(0.7-2)}{6} \times 0 + \frac{0.7(0.7-1)(0.7-2)(0.7-3)}{24} \times 0.1$$

$$= 0.4384$$

Thus $f(3.17) = 0.4384$.

NEWTON BACKWARD INTERPOLATION FORMULA

Taking $p = \frac{x - x_n}{h}$, we get the interpolation formula as:

$$P(x_n+ph) = y_0 + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \Lambda + \frac{p(p+1)(p+2)\ldots(p+n-1)}{n!}\nabla^n y_n$$

Example

Estimate f(42) from the following data using newton backward interpolation.

x:	20	25	30	35	40	45	
f(x):		354	332	291	260	231	204

Solution

The difference table is:

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$
20	354					
		- 22				
25	332		- 19			
		- 41		29		
30	291		10		-37	
		- 31		- 8		45
35	260		2		**8**	
		- 29		**0**		
40	231		**2**			
		- 27				
45	**204**					

Here $x_n = 45$, $h = 5$, $x = 42$

and $p = -0.6$

Solution

Newton backward formula is:

$$P(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n +$$

$$\frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n$$

$$P(42)=204+(-0.6)(-27)+\frac{(-0.6)(0.4)}{2}\times 2+\frac{(-0.6)(0.40(1.4)}{6}\times 0+\frac{(-0.6)(0.4)(1.4)(2.4)}{24}\times 8+$$

$$\frac{(-0.6)(0.4)(1.4)(2.4)(3.4)}{120}\times 45 = 219.1430$$

Thus, f(42) = 219.143

INTERPOLATION USING CENTRAL DIFFERENCES

Suppose the values of the function f (x) are known at the points a -3h, a – 2h, a – h, a, a +h, a + 2h, a + 3h, ... etc. Let these values be y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3 ..., and so on. Then we can form the central difference table as:

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
a-3h	y_{-3}						
		Δy_{-3}					
a-2h	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
a-h	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
a	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
a+h	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
a+2h	y_2		$\Delta^2 y_1$				
		Δy_2					
a+3h	y_3						

We can relate the central difference operator δ with Δ and E using the operator relation $\delta = \Delta E^{1/2}$.