



Jacobi Method

Basic Idea on Jacobi method

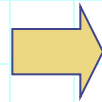
Convert the system: $Ax = B$

into the equivalent system: $x = Cx + d$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$



$$x_1 = -\frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 + \frac{b_1}{a_{11}}$$

$$x_2 = -\frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}}{a_{22}}x_3 + \frac{b_2}{a_{22}}$$

$$x_3 = -\frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 + \frac{b_3}{a_{33}}$$

- Generate a sequence of approximation

$$x^{(1)}, x^{(2)}, \dots$$

$$x^{(k)} = Cx^{(k-1)} + d$$

Jacobi iteration method

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

$$x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

$$\begin{aligned} x_1^1 &= \frac{1}{a_{11}} (b_1 - a_{12}x_2^0 - \cdots - a_{1n}x_n^0) & x_i^{k+1} &= \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^n a_{ij}x_j^k \right] \\ x_2^1 &= \frac{1}{a_{22}} (b_2 - a_{21}x_1^0 - a_{23}x_3^0 - \cdots - a_{2n}x_n^0) \\ x_n^1 &= \frac{1}{a_{nn}} (b_n - a_{n1}x_1^0 - a_{n2}x_2^0 - \cdots - a_{nn-1}x_{n-1}^0) \end{aligned}$$

$x^{k+1} = Ex^k + f$ iteration for Jacobi method

A can be written as $A=L+D+U$ (*not decomposition*)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

$$Ax=b \Rightarrow (L+D+U)x=b$$

$$Dx^{k+1} = -(L+U)x^k + b$$

$$x_i^{k+1} = \frac{1}{a_{ii}} \left[b_i - \underbrace{\sum_{j=1}^{i-1} a_{ij} x_j^k}_{Lx^k} - \underbrace{\sum_{j=i+1}^n a_{ij} x_j^k}_{Ux^k} \right]$$

Dx^{k+1}

$$x^{k+1} = -D^{-1}(L+U)x^k + D^{-1}b$$

$$E = -D^{-1}(L+U)$$

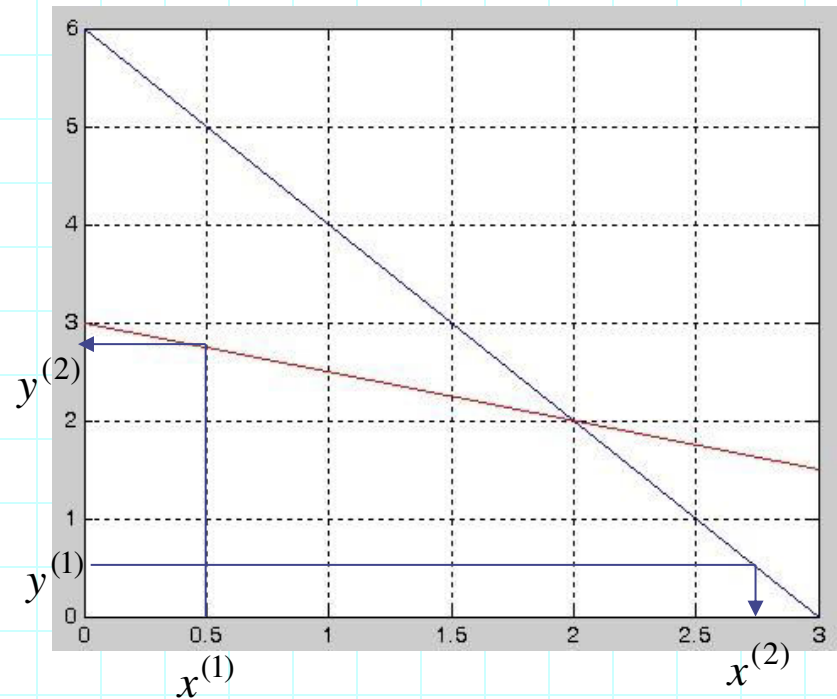
$$f = D^{-1}b$$

Jacobi Method with examples

- Consider the two-by-two system
- Start with $x^{(1)} = y^{(1)} = 1/2$
- *Simultaneous updating*
 - New values of the variables are not used until a new iteration step is begun

$$\begin{array}{l} 2x + y = 6 \\ x + 2y = 6 \end{array} \Rightarrow \begin{array}{l} x = -\frac{1}{2}y + 3 \\ y = -\frac{1}{2}x + 3 \end{array}$$

$$x^{(2)} = -\frac{1}{2}y^{(1)} + 3 = -\frac{1}{4} + 3 = \frac{11}{4}$$
$$y^{(2)} = -\frac{1}{2}x^{(1)} + 3 = -\frac{1}{4} + 3 = \frac{11}{4}$$



Jacobi Method

- Con't

$$x^{(3)} = -\frac{1}{2}y^{(2)} + 3 = -\frac{11}{8} + 3 = \frac{13}{8}$$

$$y^{(3)} = -\frac{1}{2}x^{(2)} + 3 = -\frac{11}{8} + 3 = \frac{13}{8}$$

>> Jacobi_f(A,b,x0,0.001,50)

1.0000	2.7500	2.7500
2.0000	1.6250	1.6250
3.0000	2.1875	2.1875
4.0000	1.9063	1.9063
5.0000	2.0469	2.0469
6.0000	1.9766	1.9766
7.0000	2.0117	2.0117
8.0000	1.9941	1.9941
9.0000	2.0029	2.0029
10.0000	1.9985	1.9985
11.0000	2.0007	2.0007
12.0000	1.9996	1.9996
Jacobi method converged		
13.0000	2.0002	2.0002

Jacobi Method

- Consider the three-by-three system

$$2x_1 - x_2 + x_3 = -1$$

$$x_1 + 2x_2 - x_3 = 6$$

$$x_1 - x_2 + 2x_3 = -3$$

$$x_1 = +0.5x_2 - 0.5x_3 - 0.5$$

$$x_2 = -0.5x_1 + 0.5x_3 + 3.0$$

$$x_3 = -0.5x_1 + 0.5x_2 - 1.5$$

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} 0.0 & 0.5 & -0.5 \\ -0.5 & 0.0 & 0.5 \\ -0.5 & 0.5 & 0.0 \end{bmatrix} \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \begin{bmatrix} -0.5 \\ 3.0 \\ -1.5 \end{bmatrix}$$

- Start with $x^{(0)} = (0,0,0)$

```

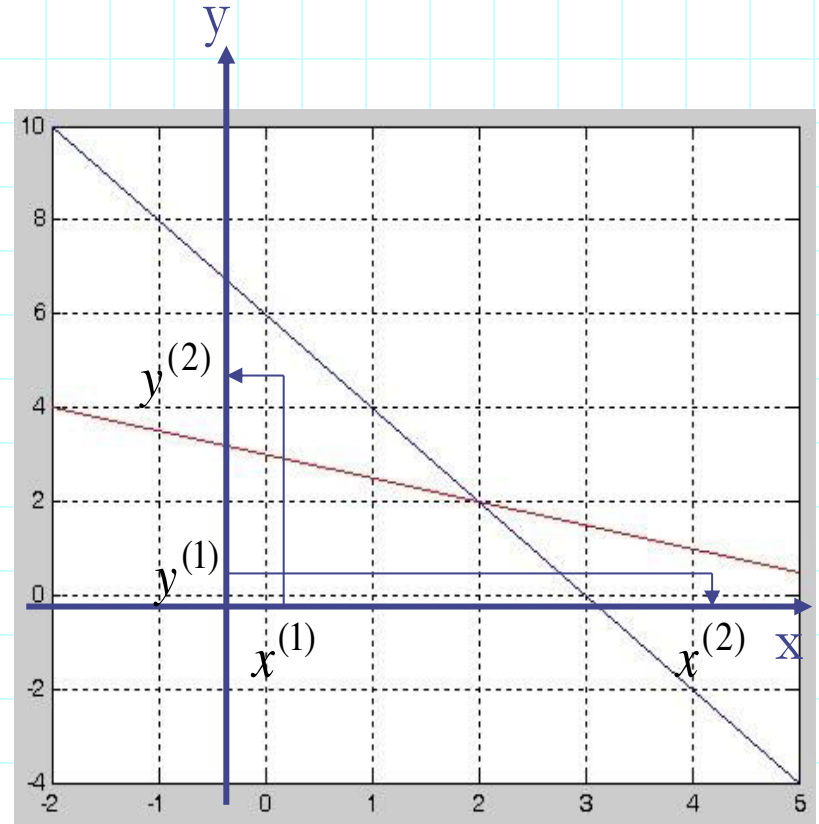
>> Jacobi_f(A,b,x0,0.001,50)
1.0000 -0.5000 3.0000 -1.5000
2.0000 1.7500 2.5000 0.2500
3.0000 0.6250 2.2500 -1.1250
4.0000 1.1875 2.1250 -0.6875
5.0000 0.9063 2.0625 -1.0313
6.0000 1.0469 2.0313 -0.9219
7.0000 0.9766 2.0156 -1.0078
8.0000 1.0117 2.0078 -0.9805
9.0000 0.9941 2.0039 -1.0020
10.0000 1.0029 2.0020 -0.9951
11.0000 0.9985 2.0010 -1.0005
12.0000 1.0007 2.0005 -0.9988
13.0000 0.9996 2.0002 -1.0001
Jacobi method converged
14.0000 1.0002 2.0001 -0.9997
    
```

Jacobi Method

- Discussion

$$\begin{array}{l} x + 2y = 6 \\ 2x + y = 6 \end{array} \quad \Rightarrow \quad \begin{array}{l} x = -2y + 6 \\ y = -2x + 6 \end{array}$$

- A necessary and sufficient condition for the convergence of the Jacobi method
 - The magnitude of the largest eigenvalue of the iteration matrix C be less than 1



Jacobi method : Example 1

Consider a circuit shown in figure here; currents i_1 , i_2 , and i_3 are given by

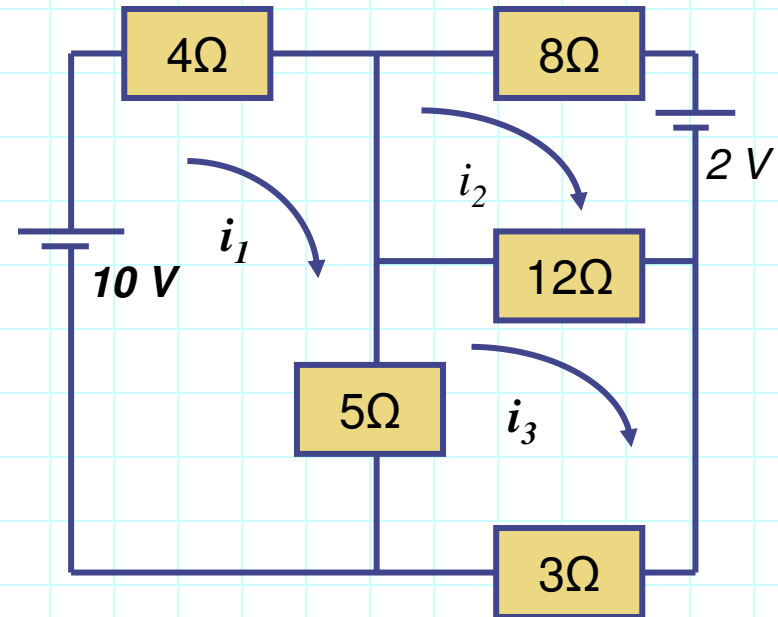
$$9i_1 + 0i_2 - 5i_3 = 10$$

$$0i_1 + 20i_2 - 12i_3 = -2$$

$$-5i_1 - 12i_2 + 20i_3 = 0$$

The matrix form is:

$$\begin{bmatrix} 9 & 0 & -5 \\ 0 & 20 & -12 \\ -5 & -12 & 20 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 0 \end{bmatrix}$$



Notice that magnitude of any diagonal element is greater than the sum of the magnitudes of other elements in that row

A matrix with this property is said to be **Diagonally dominant**.

Jacobi method : Example 1

The set of equations:

$$9i_1 + 0i_2 - 5i_3 = 10$$

$$0i_1 + 20i_2 - 12i_3 = -2$$

$$-5i_1 - 12i_2 + 20i_3 = 0$$

Let us write for i_1 , i_2 and i_3 as

$$i_1 = (10 + 5i_3)/9 = 1.1111 + 0.5556i_3 \quad (1)$$

$$i_2 = (-2 + 12i_3)/20 = -0.1000 + 0.6000i_3 \quad (2)$$

$$i_3 = (5i_1 + 12i_2)/20 = 0.2500i_1 + 0.6000i_2 \quad (3)$$

Let us make an initial guess as $i_1 = 0.0$; $i_2 = 0.0$ and $i_3 = 0.0$

First iteration results: $i_1 = 1.1111$; $i_2 = -0.1000$ and $i_3 = 0.0$

Jacobi method : Example 1

$$i_1 = (10 + 5i_3)/9 = 1.1111 + 0.5556i_3 \quad (1)$$

$$i_2 = (-2 + 12i_3)/20 = -0.1000 + 0.6000i_3 \quad (2)$$

$$i_3 = (5i_1 + 12i_2)/20 = 0.2500i_1 + 0.6000i_2 \quad (3)$$

First iteration results: $i_1 = 1.1111$; $i_2 = -0.1000$ and $i_3 = 0.0$

2nd iteration results: $i_1 = 1.1111$; $i_2 = -0.1000$ and $i_3 = 0.22$

3rd iteration results: $i_1 = 1.23$; $i_2 = 0.03$ and $i_3 = 0.22$

4th iteration results: $i_1 = 1.23$; $i_2 = 0.03$ and $i_3 = 0.33$

5th iteration results: $i_1 = 1.29$; $i_2 = 0.1$ and $i_3 = 0.33$

6th iteration results: $i_1 = 1.29$; $i_2 = 0.1$ and $i_3 = 0.38$

Jacobi method : Example 2

$$4X_1 + 2X_2 = 2$$

$$2X_1 + 10X_2 + 4X_3 = 6$$

$$4X_2 + 5X_3 = 5$$

Solution: $(X_1, X_2, X_3) = (0.41379, 0.17241, 0.86206)$

Jacobi method : Example 2

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 10 & 4 \\ 0 & 4 & 5 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 6 \\ 5 \end{Bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} + \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 6 \\ 5 \end{Bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 6 \\ 5 \end{Bmatrix} - \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

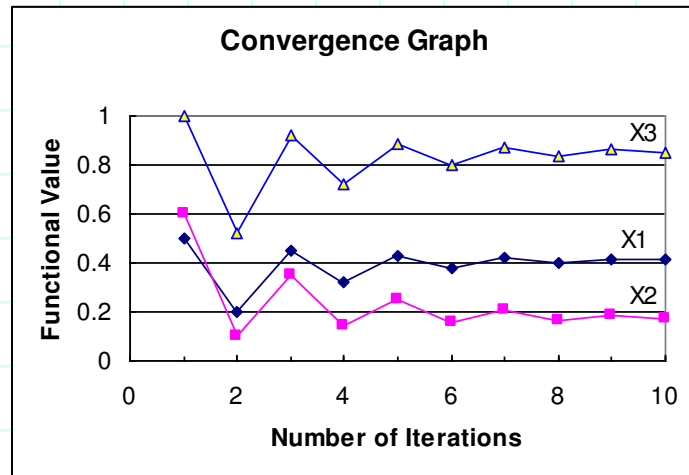
Jacobi method : Example 2

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 6 \\ 5 \end{Bmatrix} - \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$\begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \begin{Bmatrix} 2 \\ 6 \\ 5 \end{Bmatrix} - \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

Jacobi method : Example 2



The actual Solution:

$$(X_1, X_2, X_3) =$$

$$(0.41379, 0.17241, 0.86206)$$

Iteration	1	2	3	4	5	6	7
X_1	0.5	0.2	0.45	0.324	0.429	0.376	0.42
X_2	0.6	0.1	0.352	0.142	0.248	0.16	0.204
X_3	1	0.52	0.92	0.718	0.886	0.802	0.872

Jacobi method : Example 3

Consider the following set of equations.

$$10x_1 - x_2 + 2x_3 = 6$$

$$-x_1 + 11x_2 - x_3 + 3x_4 = 25$$

$$2x_1 - x_2 + 10x_3 - x_4 = -11$$

$$3x_2 - x_3 + 8x_4 = 15$$

Convert the set $Ax = b$ in the form of $x = Tx + c$.

$$x_1 = \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5}$$

$$x_2 = \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11}$$

$$x_3 = -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10}$$

$$x_4 = -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}$$

Jacobi method : Example 3

$$\begin{aligned}x_1^{(1)} &= \frac{1}{10}x_2^{(0)} - \frac{1}{5}x_3^{(0)} + \frac{3}{5} \\x_2^{(1)} &= \frac{1}{11}x_1^{(0)} + \frac{1}{11}x_3^{(0)} - \frac{3}{11}x_4^{(0)} + \frac{25}{11} \\x_3^{(1)} &= -\frac{1}{5}x_1^{(0)} + \frac{1}{10}x_2^{(0)} + \frac{1}{10}x_4^{(0)} - \frac{11}{10} \\x_4^{(1)} &= -\frac{3}{8}x_2^{(0)} + \frac{1}{8}x_3^{(0)} + \frac{15}{8}\end{aligned}$$

$$x_1^{(0)} = 0, x_2^{(0)} = 0, x_3^{(0)} = 0 \text{ and } x_4^{(0)} = 0.$$

$$\begin{aligned}x_1^{(1)} &= \frac{1}{10}(0) - \frac{1}{5}(0) + \frac{3}{5} \\x_2^{(1)} &= \frac{1}{11}(0) + \frac{1}{11}(0) - \frac{3}{11}(0) + \frac{25}{11} \\x_3^{(1)} &= -\frac{1}{5}(0) + \frac{1}{10}(0) + \frac{1}{10}(0) - \frac{11}{10} \\x_4^{(1)} &= -\frac{3}{8}(0) + \frac{1}{8}(0) + \frac{15}{8}\end{aligned}$$

$$\begin{aligned}x_1^{(1)} &= 0.6000, \\x_2^{(1)} &= 2.2727, \\x_3^{(1)} &= -1.1000 \\x_4^{(1)} &= 1.8750\end{aligned}$$

Jacobi method : Example 3

$$\begin{aligned}x_1^{(2)} &= \frac{1}{10}x_2^{(1)} - \frac{1}{5}x_3^{(1)} + \frac{3}{5} \\x_2^{(2)} &= \frac{1}{11}x_1^{(1)} + \frac{1}{11}x_3^{(1)} - \frac{3}{11}x_4^{(1)} + \frac{25}{11} \\x_3^{(2)} &= -\frac{1}{5}x_1^{(1)} + \frac{1}{10}x_2^{(1)} + \frac{1}{10}x_4^{(1)} - \frac{11}{10} \\x_4^{(2)} &= -\frac{3}{8}x_2^{(1)} + \frac{1}{8}x_3^{(1)} + \frac{15}{8}\end{aligned}$$

$$\begin{aligned}x_1^{(k)} &= \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} + \frac{3}{5} \\x_2^{(k)} &= \frac{1}{11}x_1^{(k-1)} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11} \\x_3^{(k)} &= -\frac{1}{5}x_1^{(k-1)} + \frac{1}{10}x_2^{(k-1)} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10} \\x_4^{(k)} &= -\frac{3}{8}x_2^{(k-1)} + \frac{1}{8}x_3^{(k-1)} + \frac{15}{8}\end{aligned}$$

Jacobi method : Example 3

Results:

<i>iteration</i>	0	1	2	3
$x_1^{(k)}$	0.0000	0.6000	1.0473	0.9326
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.0530
$x_3^{(k)}$	0.0000	-1.1000	-0.8052	-1.0493
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309

Jacobi method : Example 4

A diverging case study:

$$\begin{bmatrix} -2 & 1 & 5 \\ 4 & -8 & 1 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ -21 \\ 7 \end{bmatrix} \quad x^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \|b - Ax^0\|_2 = 26.7395$$

The matrix is not diagonally dominant

$$x_1^1 = \frac{-15 + x_2^0 + 5x_3^0}{2} = \frac{-15}{2} = -7.5$$

$$x_2^1 = \frac{21 + 4x_1^0 + x_3^0}{8} = \frac{21}{8} = 2.625$$

$$x_3^1 = 7 - 4x_1^0 + x_2^0 = 7.0$$

$$\|b - Ax^1\|_2 = 54.8546$$

Jacobi method : Example 4

$$x_1^1 = \frac{-15 + 2.625 + 5 \times 7}{2} = 11.3125$$

$$x_2^1 = \frac{21 - 4 \times 7.5 + 7}{8} = -0.25$$

$$x_3^1 = 7 + 4 \times 7.5 + 2.625 = 39.625$$

$$\|b - Ax^2\|_2 = 208.3761$$

The residual term is increasing at each iteration, so the iterations are *diverging*.

Note that the matrix is not diagonally dominant

The rules

We set up the Jacobi iteration but did not ask the question “when will the Jacobi iteration converge and how fast?”

Definition:

A matrix \mathbf{A} is diagonally dominant if
$$\sum_{j=1, j \neq i}^N |a_{ij}| < |a_{ii}|$$

Theorem:

If the matrix \mathbf{A} is diagonally dominant then $\mathbf{Ax}=\mathbf{b}$ has a unique solution \mathbf{x} and the Jacobi iteration produces a sequence which converges to \mathbf{x} for any initial guess

Informally:

The “**more diagonally dominant**” a matrix is the faster it will converge... this holds some of the time.