Solution of System of Linear Equations

Solution of linear system of equations

- Numerical solution of differential equations (Finite Difference Method)
- Numerical solution of integral equations (Finite Element Method, Method of Moments)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
\Rightarrow
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

Consistency (Solvability)

- The linear system of equations $Ax=b$ has a solution, or said to be consistent IFF

$$\text{Rank}\{A\}=\text{Rank}\{A|b\}$$

- A system is inconsistent when

$$\text{Rank}\{A\}<\text{Rank}\{A|b\}$$

Rank{A} is the maximum number of linearly independent columns or rows of A. Rank can be found by using ERO (Elementary Row Oparations) or ECO (Elementary column operations).

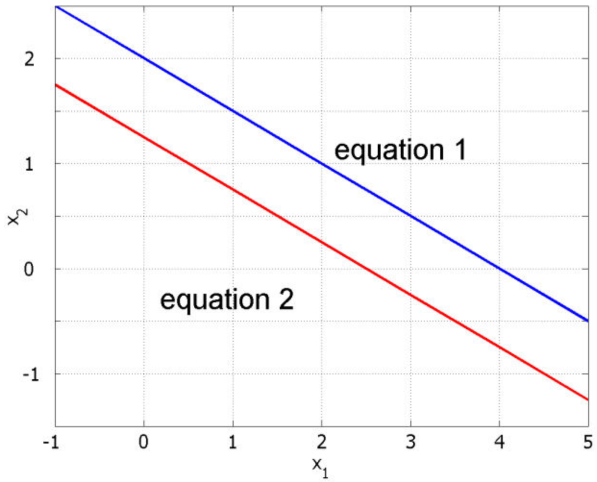
ERO$\Rightarrow$# of rows with at least one nonzero entry
ECO$\Rightarrow$# of columns with at least one nonzero entry

Elementary row operations

- The following operations applied to the augmented matrix [A|b], yield an equivalent linear system
  - Interchanges: The order of two rows can be changed
  - Scaling: Multiplying a row by a nonzero constant
  - Replacement: The row can be replaced by the sum of that row and a nonzero multiple of any other row.

An inconsistent example

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$



ERO:Multiply the first row with -2 and add to the second row

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Rank{A}=1

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -3 \end{bmatrix}$$

Rank{A|b}=2

Then this system of equations is not solvable

Uniqueness of solutions

- The system has a unique solution IFF

  Rank{A}=Rank{A|b}=n

  n is the order of the system
- Such systems are called full-rank systems

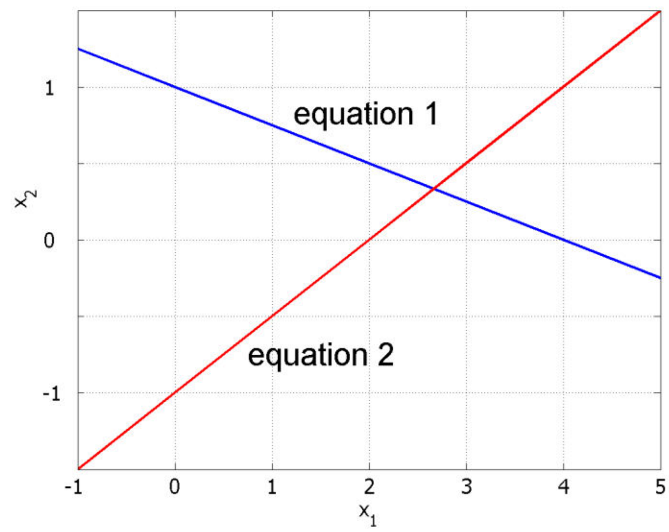
Full-rank systems

- If Rank{A}=n

  Det{A} $\neq$ 0 $\Rightarrow$ A is nonsingular so invertible

  Unique solution

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



Rank deficient matrices

- If Rank{A}=m<n

  Det{A} = 0 $\Rightarrow$ A is singular so not invertible

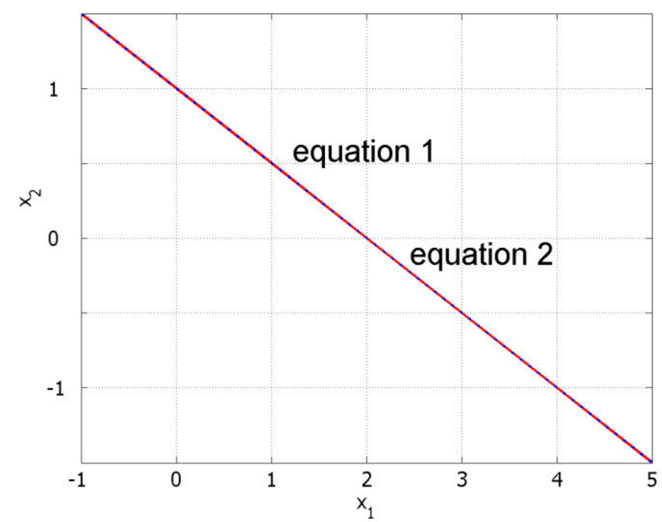
  infinite number of solutions (n-m free variables)

  under-determined system

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Rank{A}=Rank{A|b}=1

Consistent so solvable



Ill-conditioned system of equations

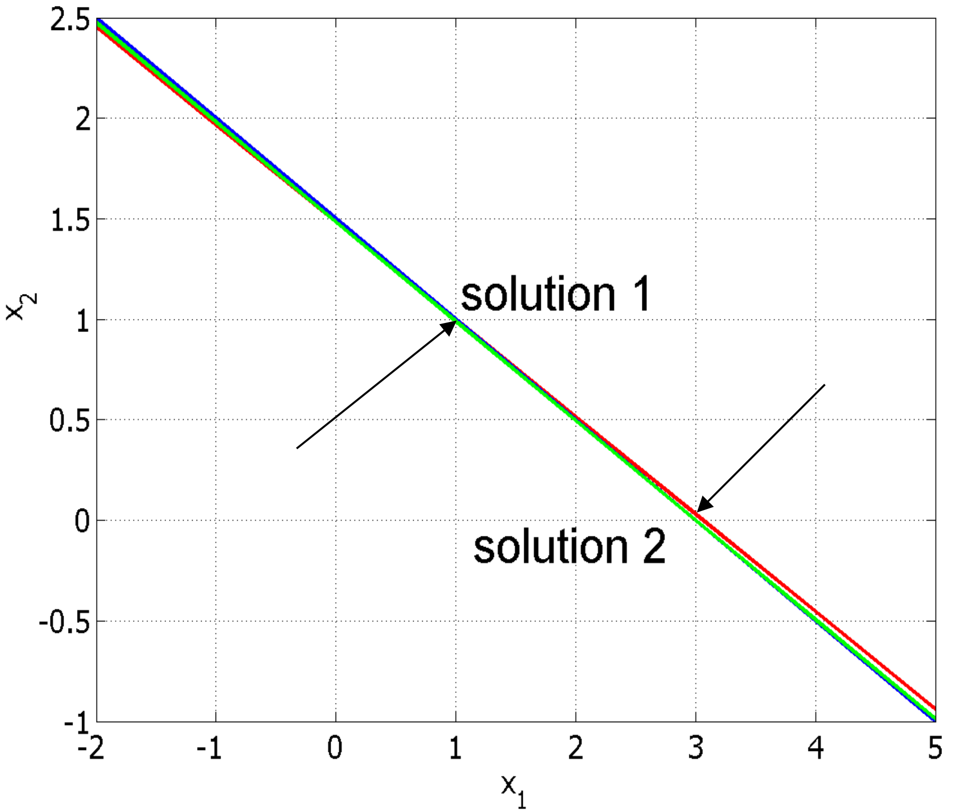
- A small deviation in the entries of A matrix, causes a large deviation in the solution.

$$\begin{bmatrix} 1 & 2 \\ 0.48 & 0.99 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1.47 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0.49 & 0.99 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1.47 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Ill-conditioned continued.....

- A linear system of equations is said to be “ill-conditioned” if the coefficient matrix tends to be singular



Types of linear system of equations

- Coefficient matrix A is square and real
- The RHS vector b is nonzero and real
- Consistent system, solvable
- Full-rank system, unique solution
- Well-conditioned system

Solution Techniques

- Direct solution methods
  - Finds a solution in a finite number of operations by transforming the system into an equivalent system that is 'easier' to solve.
  - Diagonal, upper or lower triangular systems are easier to solve
  - Number of operations is a function of system size n.
- Iterative solution methods
  - Computes succesive approximations of the solution vector for a given A and b, starting from an initial point x_0.
  - Total number of operations is uncertain, may not converge.

Direct solution Methods

- Gaussian Elimination
  - By using ERO, matrix A is transformed into an upper triangular matrix (all elements below diagonal 0)
  - Back substitution is used to solve the upper-triangular system

$$\begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} \underset{\text{ERO}}{\Rightarrow} \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \tilde{a}_{ii} & \cdots & \tilde{a}_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \tilde{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \tilde{b}_i \\ \vdots \\ \tilde{b}_n \end{bmatrix} \Big\uparrow$$

Back substitution

First step of elimination

Pivotal element

$$\begin{bmatrix} \boxed{a_{11}^{(1)}} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \\ b_3^{(1)} \\ \vdots \\ b_n^{(1)} \end{bmatrix}$$

$$\begin{array}{l} m_{2,1} = a_{21}^{(1)} / a_{11}^{(1)} \\ m_{3,1} = a_{31}^{(1)} / a_{11}^{(1)} \\ \vdots \\ m_{n,1} = a_{n1}^{(1)} / a_{11}^{(1)} \end{array} \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(2)} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(2)} \end{bmatrix}$$

Second step of elimination

Pivotal element

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & \boxed{a_{22}^{(2)}} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(2)} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(2)} \end{bmatrix}$$

$$\begin{aligned} m_{3,2} &= a_{32}^{(2)} / a_{22}^{(2)} \\ &\vdots \\ m_{n,2} &= a_{n2}^{(2)} / a_{22}^{(2)} \end{aligned} \qquad \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n3}^{(3)} & \cdots & a_{nn}^{(3)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(3)} \\ \vdots \\ b_n^{(3)} \end{bmatrix}$$

Gaussion elimination algorithm

$$m_{r,p} = a_{rp}^{(p)} / a_{pp}^{(p)}$$

$$a_{rp}^{(p)} = 0$$

$$b_r^{(p+1)} = b_r^{(p)} - m_{r,p} \times b_p^{(p)}$$

For c=p+1 to n

$$a_{rc}^{(p+1)} = a_{rc}^{(p)} - m_{r,p} \times a_{pc}^{(p)}$$

Back substitution algorithm

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{n-1n-1}^{(n)} & a_{n-1n}^{(n)} \\ 0 & 0 & 0 & 0 & a_{nn}^{(n)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(3)} \\ \vdots \\ b_{n-1}^{(n-1)} \\ b_n^{(n)} \end{bmatrix}$$

$$x_n = \frac{b_n^{(n)}}{a_{nn}^{(n)}} \qquad x_{n-1} = \frac{1}{a_{n-1n-1}^{(n-1)}}\left[b_{n-1}^{(n-1)} - a_{n-1n}^{n-1} x_n\right]$$

$$x_i = \frac{1}{a_{ii}^{(i)}}\left[b_i^{(i)} - \sum_{k=i+1}^{n} a_{ik}^{(i)} x_k\right] \qquad i = n-1, n-2, \ldots, 1$$

Operation count

- Number of arithmetic operations required by the algorithm to complete its task.
- Generally only multiplications and divisions are counted
- Elimination process $\dfrac{n^3}{3}+\dfrac{n^2}{2}-\dfrac{5n}{6}$
- Back substitution $\dfrac{n^2+n}{2}$
- Total $\dfrac{n^3}{3}+n^2-\dfrac{n}{3}$

Dominates
Not efficient for different RHS vectors