Solution of ordinary differential equations (first order, second order and simultaneous) by Euler's, Picard's and fourth-order Runge-Kutta methods

PRELIMINARIES

Consider

$$\frac{dy}{dx} = f(x, y) \text{ with an initial condition } y = y_0 \text{ at } x = x_0.$$

The function f (x, y) may be linear, nonlinear or table of values

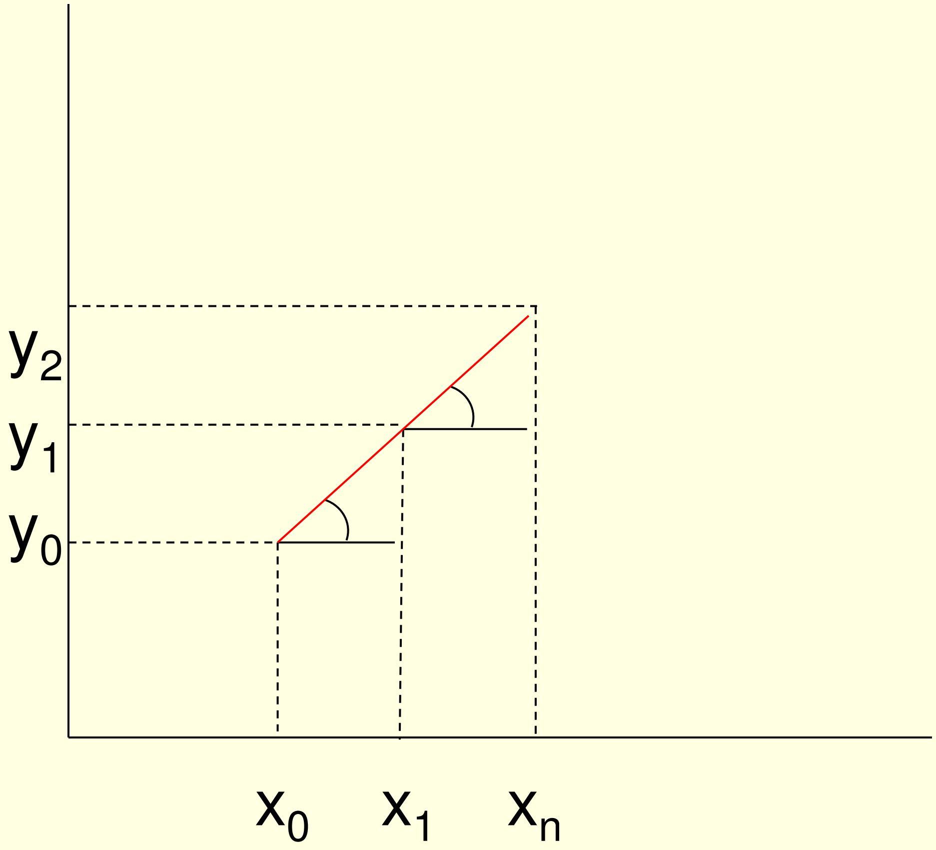
When the value of y is given at $x = x_0$ and the solution is required for $x_0 \leq x \leq x_f$ then the problem is called an ***initial value problem***. If y is given at $x = x_f$ and the solution is required for $x_f \geq x \geq x_0$ then the problem is called a ***boundary value problem***.

INITIAL VALUE PROBLEMS

- A **Solution** is a curve g (x, y) in the xy plane whose slope at each point (x, y) in the specified region is given by $\frac{dy}{dx} = f(x, y)$.

- The initial point (x_0, y_0) of the solution curve g(x, y) and the slope of the curve at this point is given. We then *extrapolate* the values of y for the required set of values in the range (x_0, x_f).

EULER'S METHOD



EULER'S METHOD

- This method uses the simplest extrapolation technique.
- The slope at (x_0, y_0) is $f(x_0, y_0)$.
- Taking a small step in the direction given by the above slope, we get

  $y_1 = y(x_0 + h) = y_0 + hf(x_0, y_0)$
- Similarly y_2 can be obtained from y_1 by taking an equal step *h* in the direction given by the slope

  $f(x_1, y_1)$.
- In general $\mathbf{y_{i+1} = y_i + h\, f(x_i, y_i)}$

Modifications

- Modified Euler Method
  - In this method the average of the slopes at (x_0, y_0) and $(x_1, y=_1^{(1)})$ is taken instead of the slope at (x_0, y_0) where $y_1^{(1)} = y_1 + h\,f\,(x_0, y_0)$.
  - In general,

$$\mathbf{y_{i+1} = y_i + \tfrac{1}{2} h\,[f\,(x_i, y_i) + f\,(x_i + h, y_i + hf\,(x_i, y_i))\,]}$$

- Improved Modified Euler Method
  - In this method points are averaged instead of slopes.

$$\mathbf{y_{i+1} = y_i + hf\left(x_i + \frac{h}{2},\; y_i + \frac{h}{2} f\,(x_i, y_i)\right)}$$

Example

- Find y (0.25) and y (0.5) given that $= 3x_2 +$ y, y(0) = 4 by

  (i) Euler Method

  (ii) Modified Euler Method

  (iii) Improved Euler Method and compare the results.

Solution

- Applying Formulae

x	y - value			
	Euler	**Modified**	**Improved**	**Exact**
0.25	5.0000	5.1484	5.1367	5.1528
0.50	6.2969	6.7194	6.6913	6.7372

TAYLOR SERIES METHOD

Consider

$\frac{dy}{dx} = f(x, y)$ with an initial condition $y = y_0$ at $x = x_0$.

The solution curve y(x) can be expressed in a Taylor series around $x = x_0$ as:

$$y(x_0 + h) = y_0 + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + \frac{h^3}{3!}\frac{d^3y}{dx^3} + \ldots$$

where $x = x_0 + h$.

Example

- Using Taylor series find y(0.1), y(0.2) and y(0.3) given that

$$\frac{dy}{dx} = x^2 - y; \; y(0) = 1$$

Solution

Applying formula

y(0.1) = 0.9052
y(0.2) = 0.8213
y(0.3) = 0.7492

PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS

- This is an iterative method.

Consider

$\frac{dy}{dx}$ = f (x, y) with an initial condition $y = y_0$ at $x = x_0$.

Integrating in $(x_0, x_0 + h)$

$$y(x_0 + h) = y(x_0) + \int_{x_0}^{x_0+h} f(x, y)\,dx$$

This integral equation is solved by successive approximations.

After *n* steps

- This process is repeated and in the n^{th} approximation, we get

$$y^{(n)} = y_0 + \int_{x_0}^{x_0+h} f(x, y^{(n-1)})\,dx$$

Example

Find y(1.1) given that $\frac{dy}{dx} = x - y$,

y(1) = 1, by Picard's Method.

Solution

- $y^{(1)}_{1.1} = 1 + \int_{1}^{1.1} (x-1)\,dx$

  $= 1.005$

Successive iterations yield 1.0045, **1.0046** , **1.0046**

Thus y (1.1) = 1.0046

Exact value is **y (1.1) = 1.0048**

RUNGE–KUTTA METHODS

- Euler Method is not very powerful in practical problems, as it requires very small step size *h* for reasonable accuracy.
- In Taylor's method, determination of higher order derivatives are involved.
- The Runge–Kutta methods give greater accuracy without the need to calculate higher derivatives.

n^{th} order R.K. Method

- This method employs the recurrence formula of the form

$$y_{i+1} = y_i + a_1 k_1 + a_2 k_2 + \Lambda + a_n k_n$$

where $k_1 = h f(x_i, y_i)$

$$k_2 = h f(x_i + p_1 h, y_i + q_{11} k_1)$$

$$k_3 = h f(x_i + p_2 h, y_i + q_{21} k_1 + q_{22} k_2)$$

……. …….. ………

$$\mathbf{k_n = h f(x_i + p_{n-1} h, y_i + q_{n-1,1} k_1 + q_{n-2,2} k_2 + \Lambda\ q_{(n-1),(n-1)} k_n)}$$

4^{th} order R.K. Method

- Most commonly used method

$$y_{n+1} = y_n + (k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(xn + \frac{h}{2},\quad y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(xn + \frac{h}{2},\ y_n + \frac{k_2}{2}\right)$$

$$k_4 = h\, f\, (x_n + h, y_3 + k_3)$$

Example

- Using R.K. Method of 4th order find y(0.1) and y(0.2).

Given that $\frac{dy}{dx} = 3x + \frac{1}{2}y$, $y(0) = 1$ taking $h = 0.1$.

Solution

$k_1 = h f(x_0, y_0) = 0.0500$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.0663$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.0667$

$k_4 = h f(x_0 + h, y_0 + k_3) = 0.0833$

$y_1 = y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.0674$

By similar procedure $y(0.2) = 1.1682$