Structural Analysis - II

Indeterminate structures; Force method of analysis

Module I

Statically and kinematically indeterminate structures

- Degree of static indeterminacy, Degree of kinematic indeterminacy, Force and displacement method of analysis

Force method of analysis

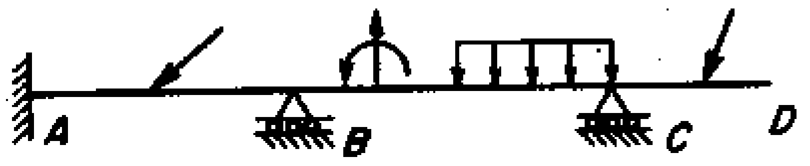
- Method of **consistent deformation**-Analysis of fixed and continuous beams

- Clapeyron's **theorem of three moments**-Analysis of fixed and continuous beams

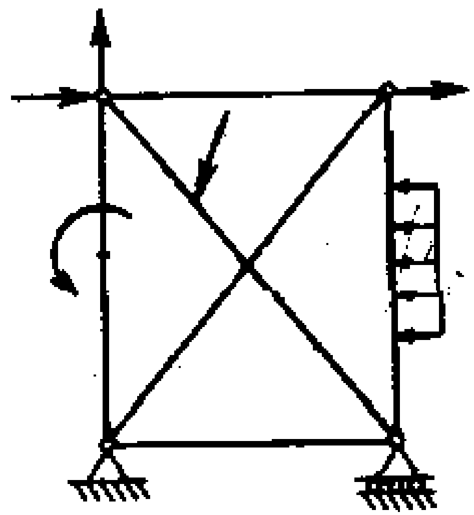
- **Principle of minimum strain energy**-Castigliano's second theorem-Analysis of beams, plane trusses and plane frames.

Types of Framed Structures

- a. Beams: may support bending moment, shear force and axial force



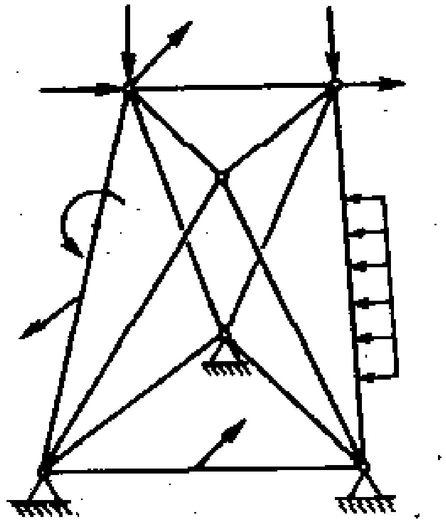
(a)



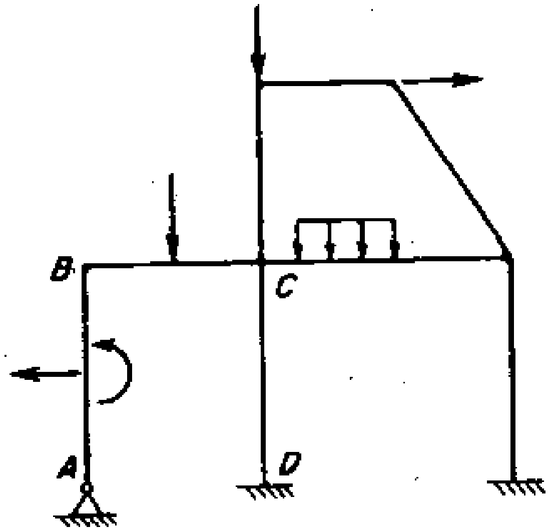
(b)

- b. Plane trusses: hinge joints; In addition to axial forces, a member CAN have bending moments and shear forces if it has loads directly acting on them, in addition to joint loads

•c. Space trusses: hinge joints; any couple acting on a member should have moment vector perpendicular to the axis of the member, since a truss member is incapable of supporting a twisting moment



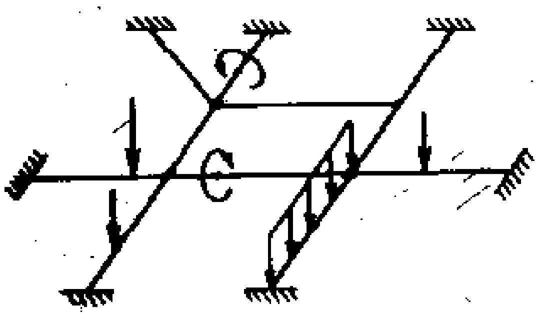
(c)



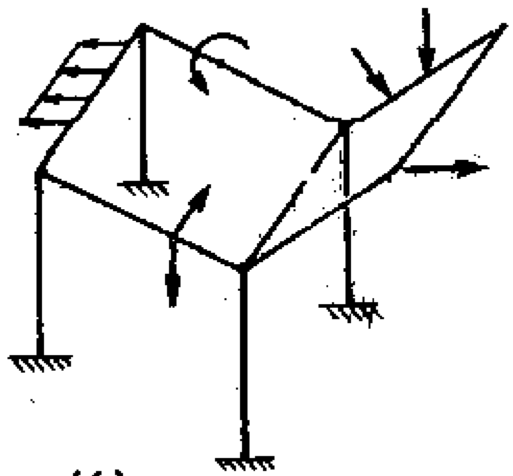
(d)

•d. Plane frames: Joints are rigid; all forces in the plane of the frame, all couples normal to the plane of the frame

•e. Grids: all forces normal to the plane of the grid, all couples in the plane of the grid (includes bending and torsion)



(e)



(f)

•f. Space frames: most general framed structure; may support bending moment, shear force, axial force and torsion

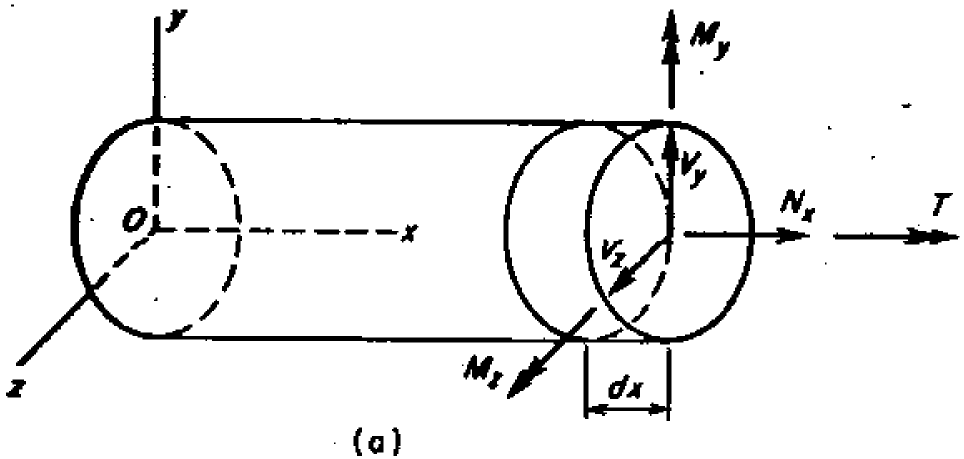
Deformations in Framed Structures

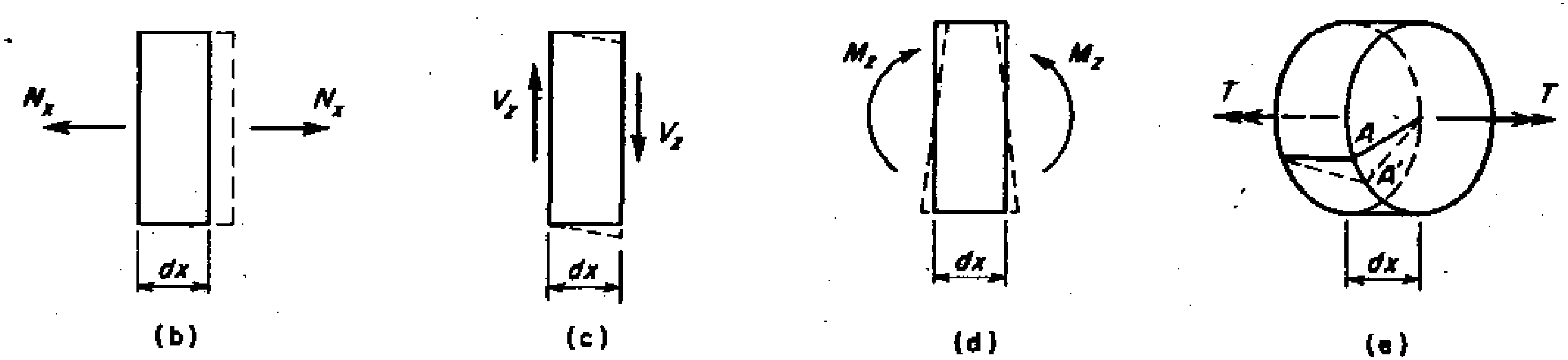
Three forces: N_x, V_y, V_z

Three couples: T_x, M_y, M_z

- Significant deformations in framed structures

Structure	Significant deformations
Beams	flexural
Plane trusses	axial
Space trusses	axial
Plane frames	flexural and axial
Grids	flexural and torsional
Space frames	axial, flexural and torsional





Types of deformations in framed structures
b) axial c) shearing d) flexural e) torsional

Equilibrium

- Resultant of all actions (a force, a couple or both) must vanish for static equilibrium

- Resultant *force* vector must be zero; resultant *moment* vector must be zero

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$

$$\sum M_x = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 0$$

- For 2-dimensional problems (forces are in one plane and couples have vectors normal to the plane),

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = 0$$

Compatibility

- Compatibility conditions: Conditions of continuity of displacements throughout the structure

- Eg: at a rigid connection between two members, the displacements (translations and rotations) of both members must be the same

Indeterminate Structures

Force method and Displacement method

- **Force method** (Flexibility method)
  - Actions are the primary unknowns
  - Static indeterminacy: excess of unknown actions than the available number of equations of static equilibrium
- **Displacement method** (Stiffness method)
  - Displacements of the joints are the primary unknowns
  - Kinematic indeterminacy: number of independent translations and rotations

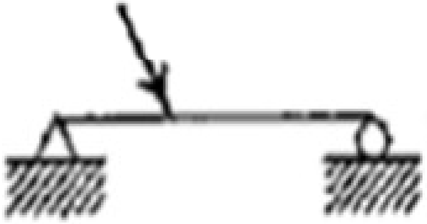
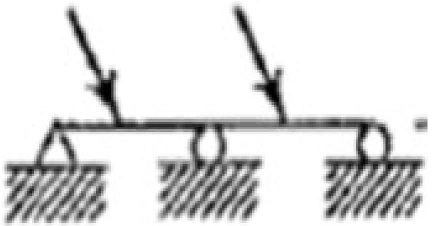
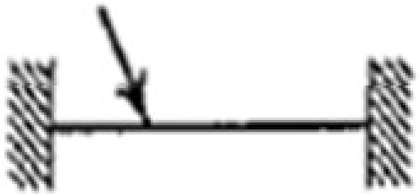
Static indeterminacy

- Beam:

  •Static indeterminacy = Reaction components - number of eqns available $E = R - 3$

  - Examples:

    •Single span beam with both ends hinged with inclined loads
    - Continuous beam
    - Propped cantilever
    - Fixed beam

Structure	Degree of Static Indeterminacy
(a) 	**0**
(b) 	**1**
(c) 	**3**

Structure	Degree of Static Indeterminacy
	**0**
	**1**
	**5**
	**2**

- Rigid frame (Plane):

  - External indeterminacy = Reaction components - number of eqns available $E = R - 3$

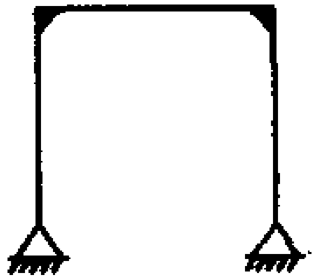
  - Internal indeterminacy = 3 × closed frames $I = 3a$

- Total indeterminacy = External indeterminacy + Internal indeterminacy

$$T = E + I = (R - 3) + 3a$$

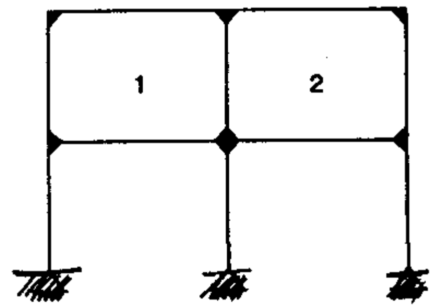
  - Note: An internal hinge will provide an additional eqn

Example 1



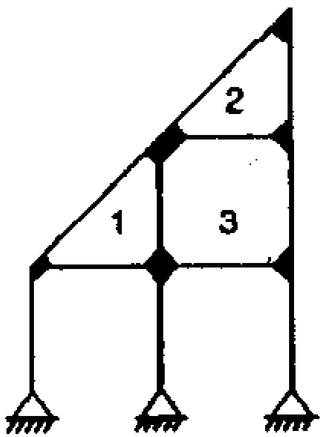
$$T = E + I = (R - 3) + 3a$$
$$= (2 \times 2 - 3) + 3 \times 0 = 1$$

Example 2



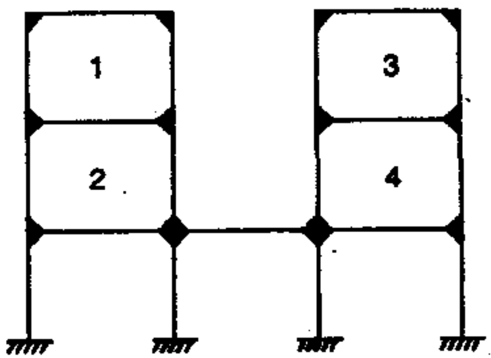
$$T = E + I = (R - 3) + 3a$$
$$= (3 \times 3 - 3) + 3 \times 2 = 12$$

Example 3

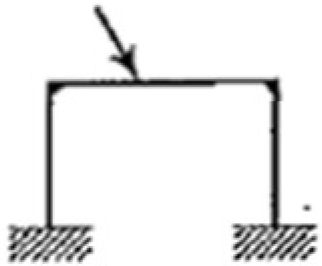
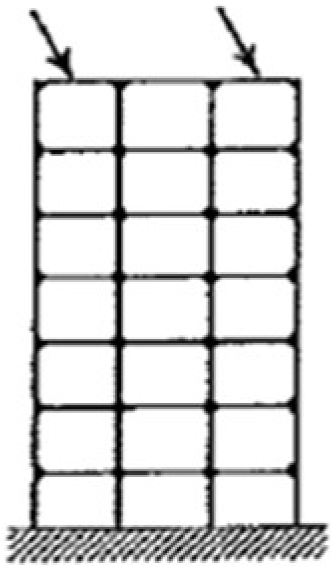


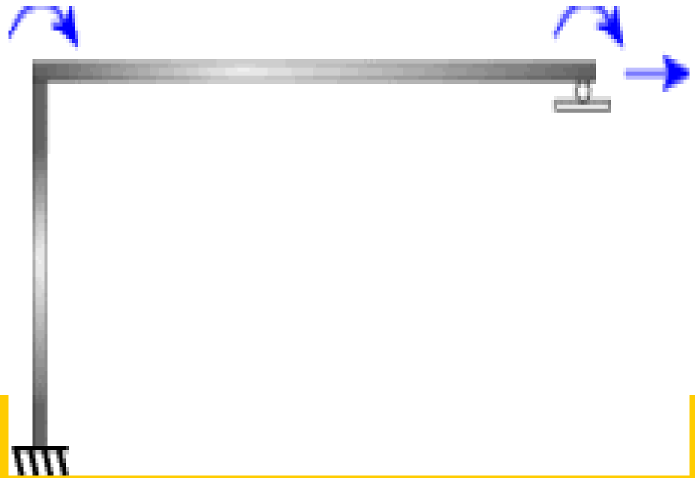
$$T = E + I = (R - 3) + 3a$$
$$= (3 \times 2 - 3) + 3 \times 3 = 12$$

Example 4



$$T = E + I = (R - 3) + 3a$$
$$= (4 \times 3 - 3) + 3 \times 4 = 21$$

Structure	Degree of Static Indeterminacy
	3
	63

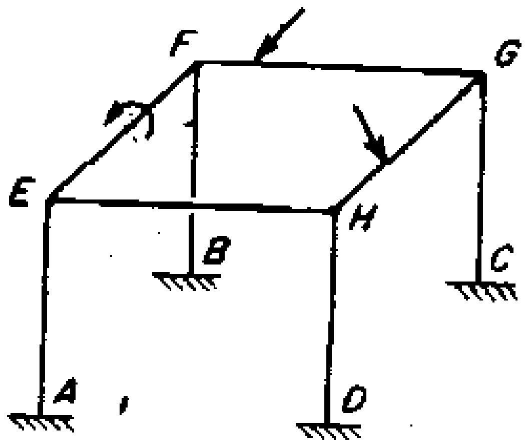
Structure	Degree of Static Indeterminacy
	**2**
	**1**

• Rigid frame (Space):

- External indeterminacy = Reaction components - number of eqns available $E = R - 6$

- Internal indeterminacy = $6 \times$ closed frames

Example 1



$$T = E + I = (R - 6) + 6a$$
$$= (4 \times 6 - 6) + 6 \times 1 = 24$$

If axial deformations are neglected, static indeterminacy is not affected since the same number of actions still exist in the structure

- Plane truss (general):

- External indeterminacy = Reaction components - number of eqns available

$$E = R - 3$$

- Minimum 3 members and 3 joints.
- Any additional joint requires 2 additional members.
- Hence, number of members for stability, $m = 3 + 2(j - 3) = 2j - 3$
- Hence, internal indeterminacy, $I = m - (2j - 3)$
- Total (Internal and external) indeterminacy

$$T = E + I = R - 3 + m - (2j - 3)$$

$$= m + R - 2j$$

  - $\boldsymbol{m}$: number of members
  - $\boldsymbol{R}$: number of reaction components
  - $\boldsymbol{j}$: number of joints

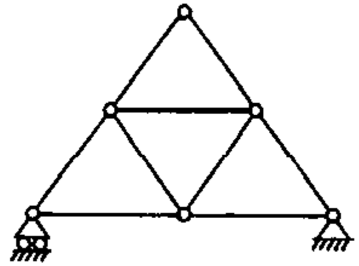
- Note: Internal hinge will provide additional eqn

Example 1

$$T = m + R - 2j = 9 + 3 - 2 \times 6 = 0$$

$$E = R - 3 = 3 - 3 = 0$$

$$I = T - E = 0$$

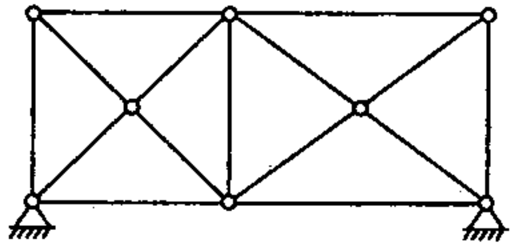


Example 2

$$T = m + R - 2j = 15 + 4 - 2 \times 8 = 3$$

$$E = R - 3 = 4 - 3 = 1$$

$$I = T - E = 2$$



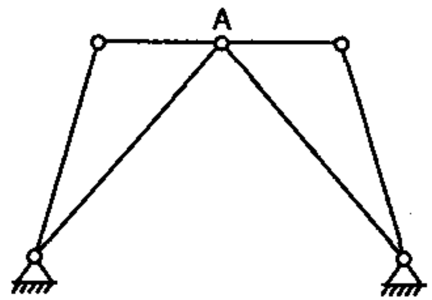
Example 3

$$T = m + R - 2j = 6 + 4 - 2 \times 5 = 0$$

$$E = R - (3 + 1) = 4 - 4 = 0$$

Hinge at A

$$I = T - E = 0$$

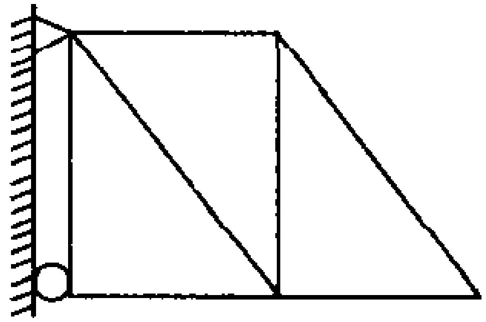


Example 4

$$T = m + R - 2j = 7 + 3 - 2\times5 = 0$$

$$E = R - 3 = 3 - 3 = 0$$

$$I = T - E = 0$$

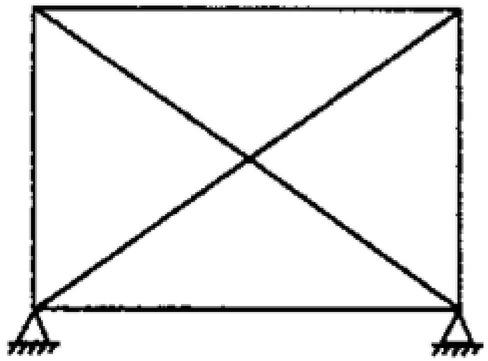


Example 5

$$T = m + R - 2j = 6 + 4 - 2\times4 = 2$$

$$E = R - 3 = 4 - 3 = 1$$

$$I = T - E = 1$$

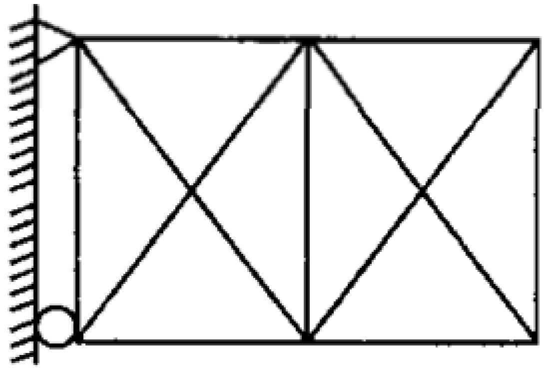


Example 6

$$T = m + R - 2j = 11 + 3 - 2\times6 = 2$$

$$E = R - 3 = 3 - 3 = 0$$

$$I = T - E = 2$$

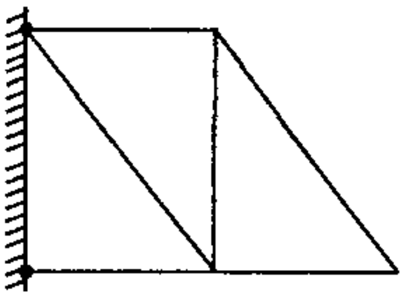


• Wall or roof attached pin jointed plane truss (Exception to the above general case):

- Internal indeterminacy $I = m - 2j$

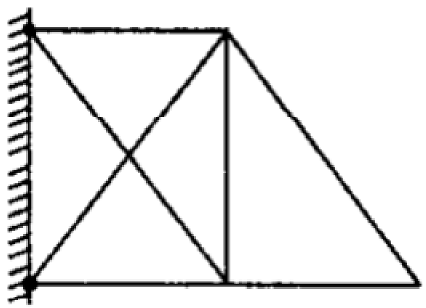
- External indeterminacy = 0 (Since, once the member forces are determined, reactions are determinable)

Example 1



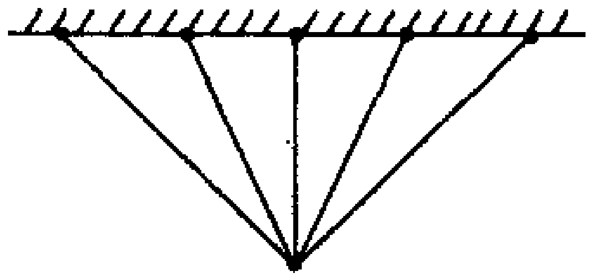
$$T = I = m - 2j$$
$$= 6 - 2 \times 3 = 0$$

Example 2



$$T = I = m - 2j$$
$$= 7 - 2 \times 3 = 1$$

Example 3



$$T = I = m - 2j$$
$$= 5 - 2 \times 1 = 3$$

- Space Truss:

•External indeterminacy = Reaction components - number of eqns available

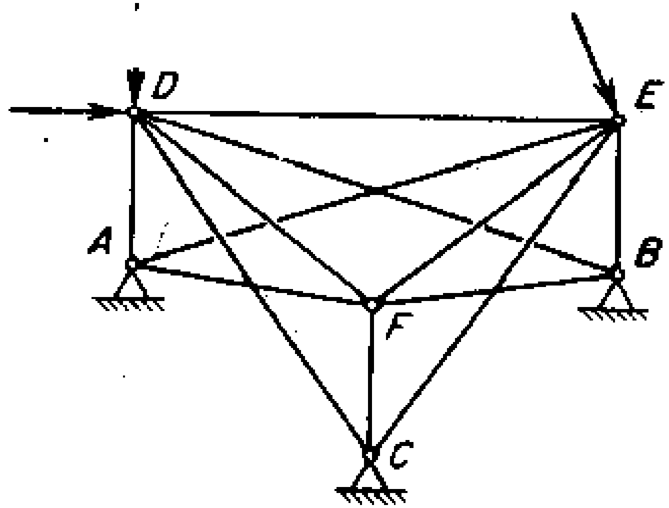
$$E = R - 6$$

- Minimum 6 members and 4 joints.
- Any additional joint requires 3 additional members.
- Hence, number of members for stability, $m = 6 + 3(j - 4) = 3j - 6$
- Hence, internal indeterminacy, $I = m - (3j - 6)$
- Total (Internal and external) indeterminacy

$$T = E + I = R - 6 + m - (3j - 6)$$

$$= m + R - 3j$$

Example



- Total (Internal and external) indeterminacy $T = m + R - 3j$

$$\therefore T = 12 + 9 - 3 \times 6 = 3$$

$$E = R - 6 = 9 - 6 = 3$$

Kinematic indeterminacy

- joints: where members meet, supports, free ends
- joints undergo translations or rotations

•in some cases joint displacements will be known, from the restraint conditions

•the unknown joint displacements are the kinematically indeterminate quantities

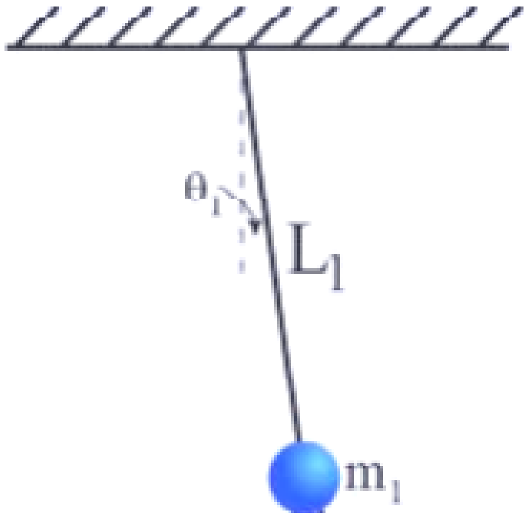
  ○degree of kinematic indeterminacy: number of degrees of freedom

Two types of DOF

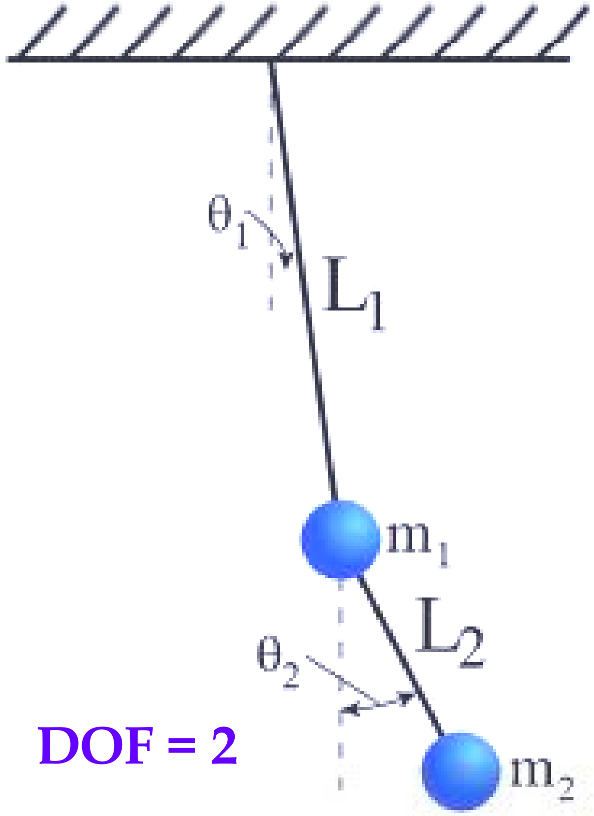
- Nodal type DOF
- Joint type DOF

•degree of **kinematic indeterminacy** (degrees of freedom) is defined as:

•the number of **independent translations and rotations** in a structure.



DOF = 1

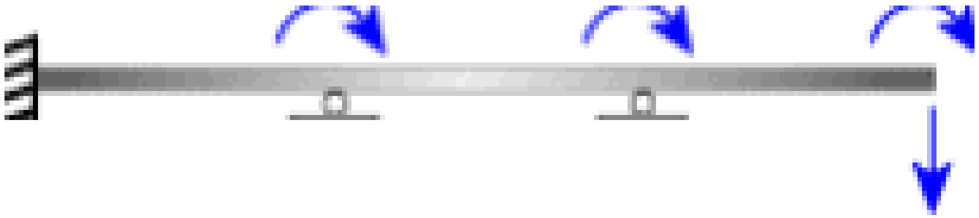


DOF = 2

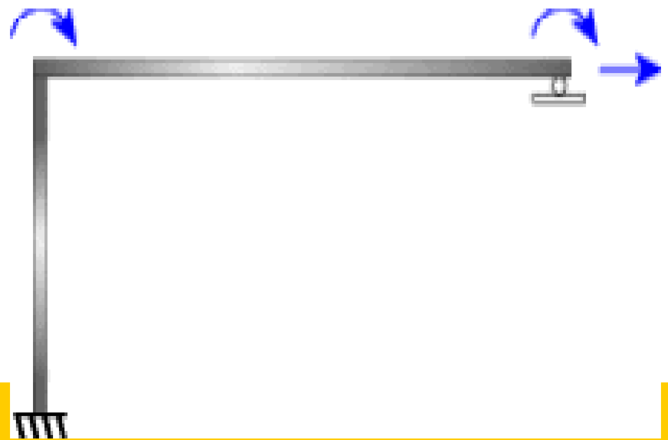
- in a truss, the joint rotation is not regarded as a degree of freedom. joint rotations do not have any physical significance as they have no effects in the members of the truss

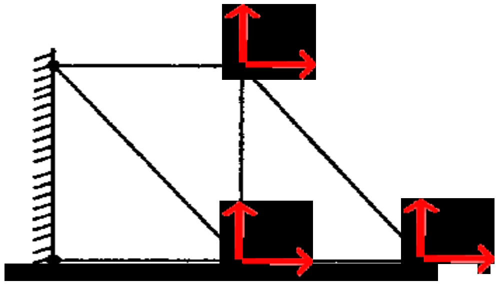
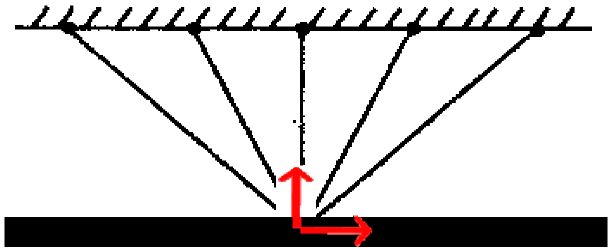
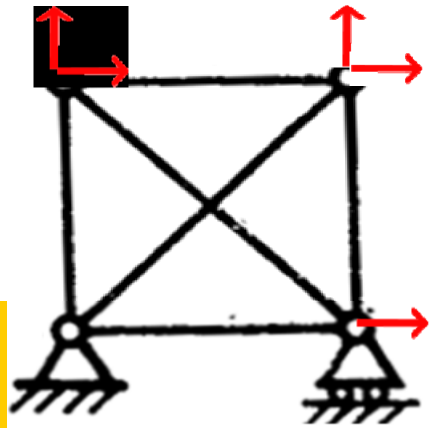
- in a frame, degrees of freedom due to axial deformations can be neglected

Structure	Kinematic Degree of Freedom
	**2**
	**1**
	**2**

Structure	Kinematic Degree of Freedom
	**4**

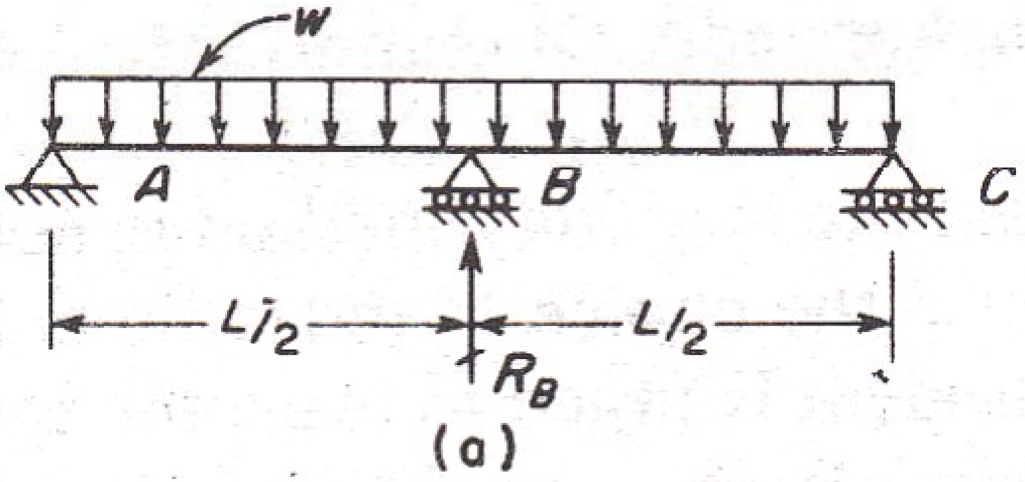
If the effect of the cantilever portion is considered as a joint load at the roller support on the far right, kinematic indeterminacy can be taken as 2.

Structure	Kinematic Degree of Freedom
	**2**
	**3**

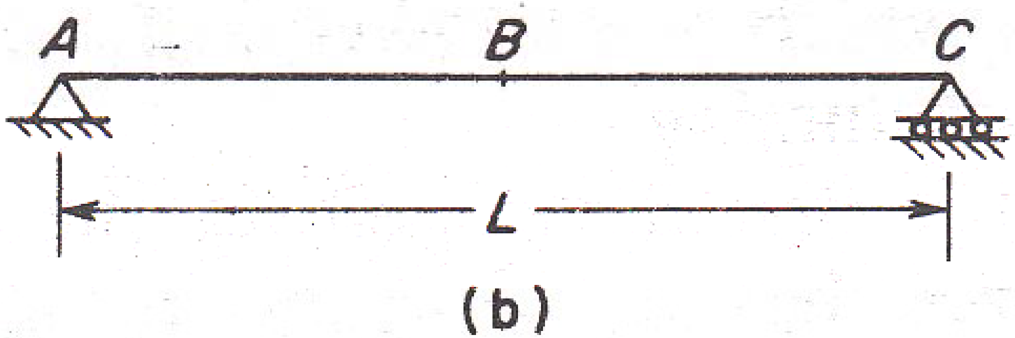
Structure	Kinematic Degree of Freedom
	**6**
	**2**
	**5**

Method of Consistent Deformation

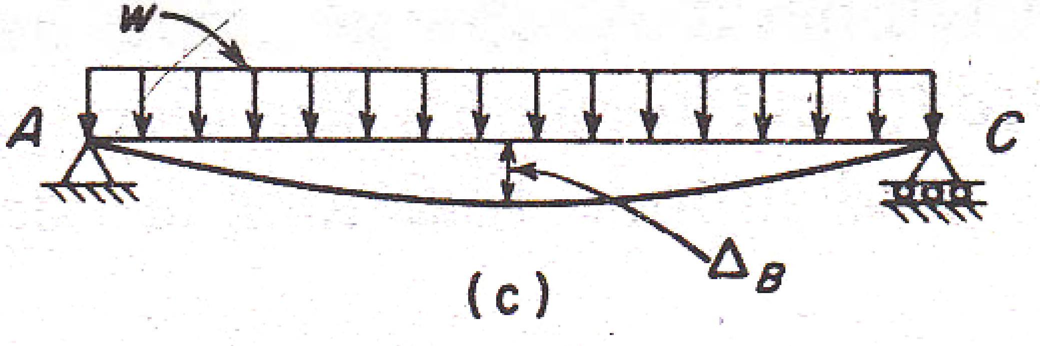
Illustration of the method



Problem

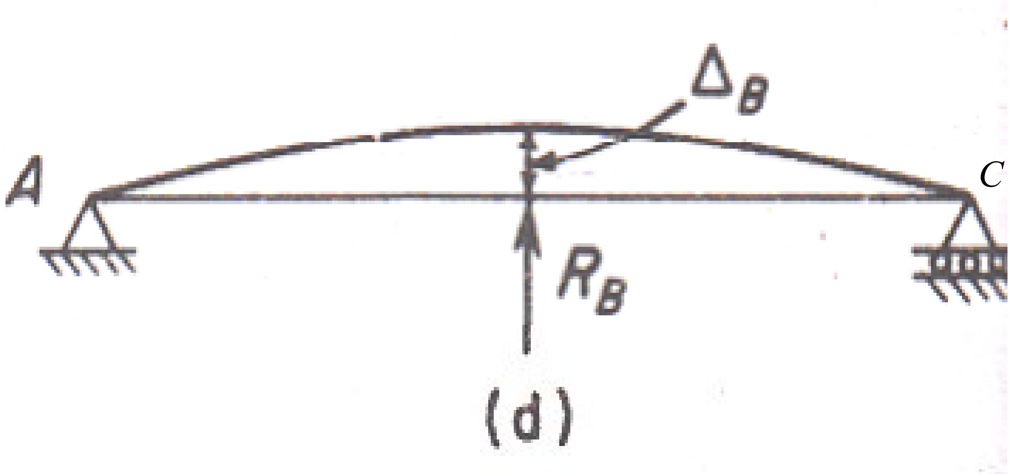


Released structure



Deflection of released structure due to actual loads

$$\Delta_B = \frac{5wL^4}{384EI}$$



Deflection of released structure due to redundant applied as a load

$$\text{Deflection due to } R_B = \frac{R_B L^3}{48EI}$$

$$\Delta_B = \frac{R_B L^3}{48EI}$$

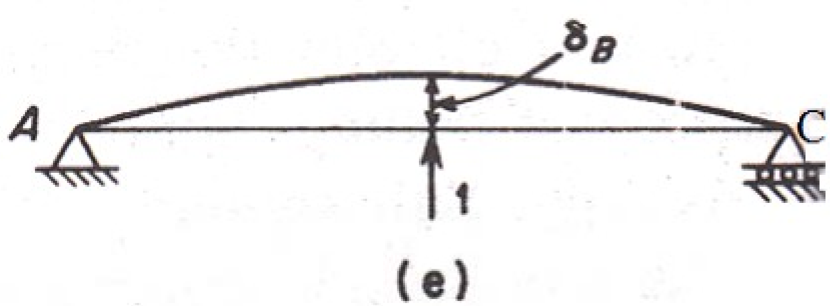
$$\frac{5wL^4}{384EI} = \frac{R_B L^3}{48EI}$$

Compatibility condition (or equation of superposition or equation of geometry)

[redacted]

- A general approach (applying consistent sign convention for loads and displacements):

  - Apply unit load corresponding to R_B



Let the displacement due to unit load $\dfrac{L^3}{48EI}$

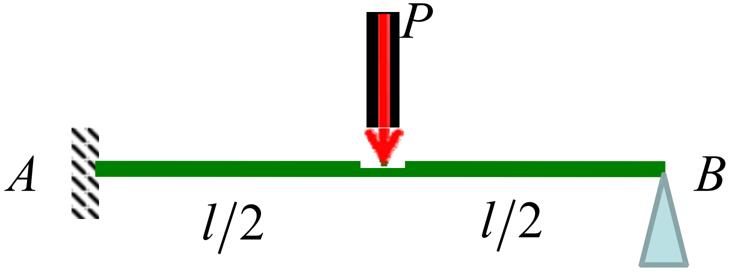
b Displacement due to R_B is $R_B \delta_B$

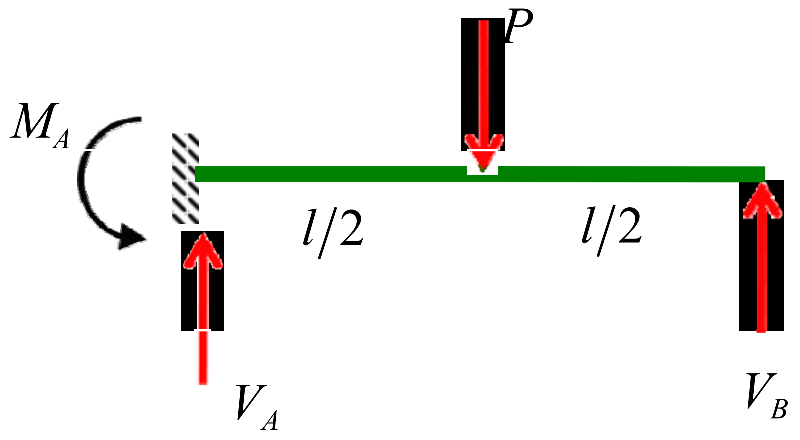
$$\Delta_B = -\frac{5wL^4}{384EI}$$ (Negative, since deflection is downward)

$$\Delta_B + R_B\delta_B = 0$$ (Compatibility condition)

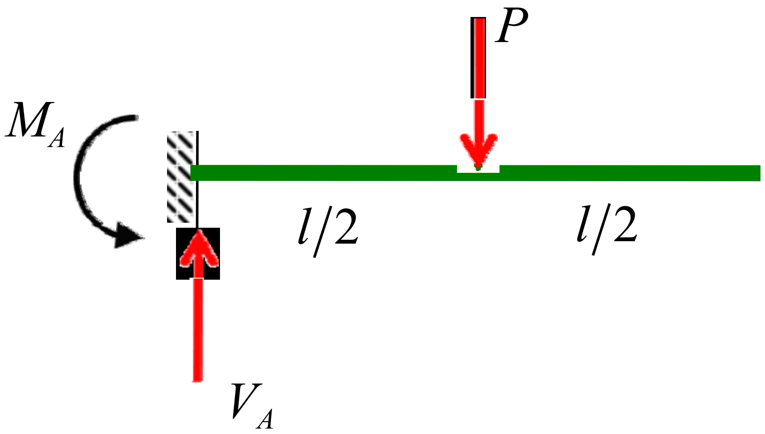
$$R_B = -\frac{\Delta_B}{\delta_B}$$ [redacted]

Example 1: Propped cantilever



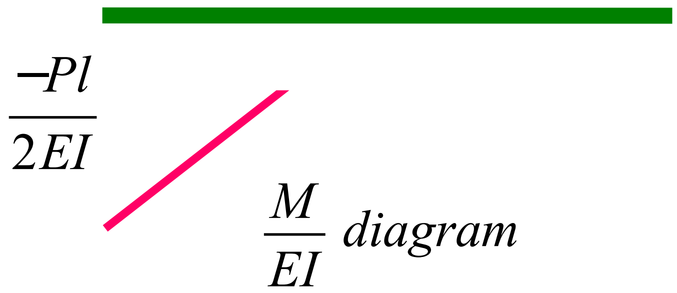


Choose V_B as the redundant



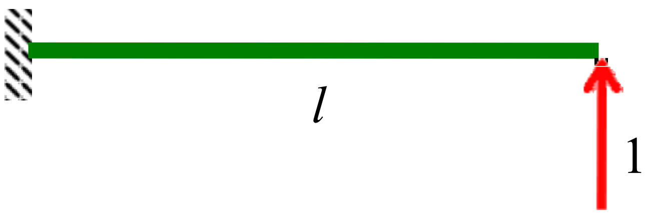
Released structure

Find deflection at ***B*** of the released structure



$$\Delta_B = \frac{1}{2} \cdot \frac{-PL}{2EI} \cdot \frac{l}{2} \left(\frac{l}{2} + \frac{2}{3} \cdot \frac{l}{2} \right) = \frac{-5Pl^3}{48EI}$$

Apply unit load on released structure corresponding to $\boldsymbol{V_B}$ and find deflection at ***B***



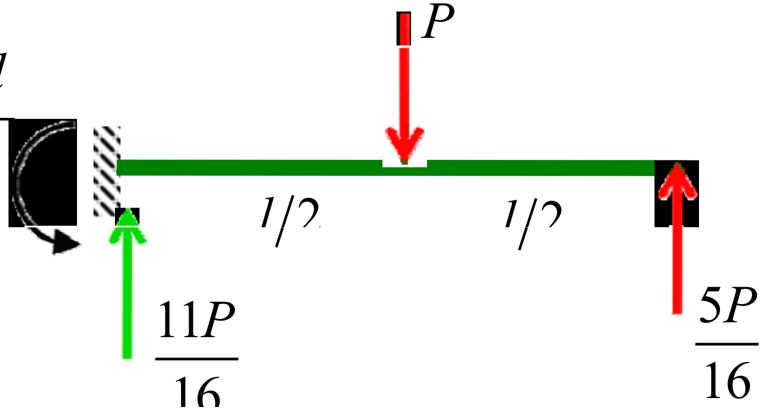
$$\delta_B = \frac{l^3}{3EI}$$

$$\Delta_B + R_B \delta_B = 0 \qquad R_B = \frac{-\Delta_B}{\delta_B} = \frac{5Pl^3}{48EI} \cdot \frac{3EI}{l^3} = \frac{5P}{16}$$

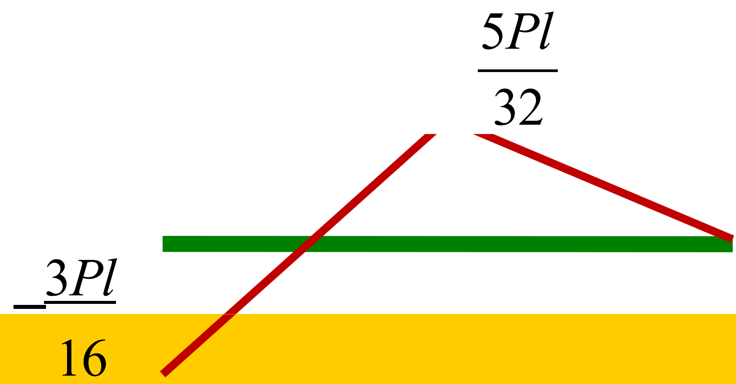
+ve sign indicates that V_B is in the same direction of the unit load.
i.e., in the upward direction.

Other reactions

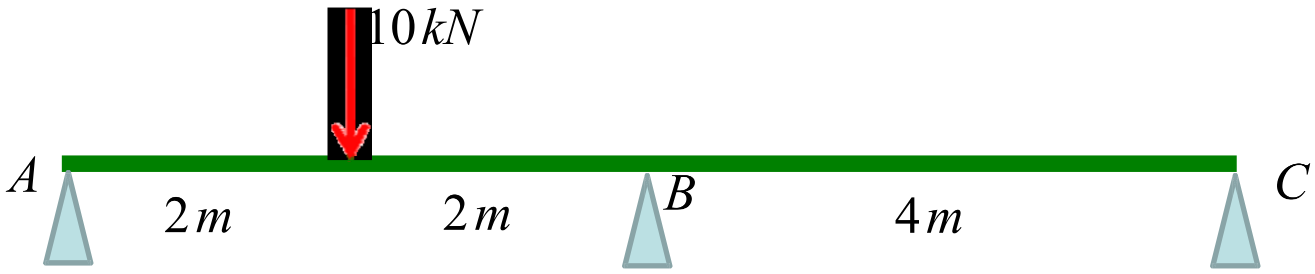
$$M_A = \frac{Pl}{2} - \frac{5Pl}{16} = \frac{3Pl}{16}$$

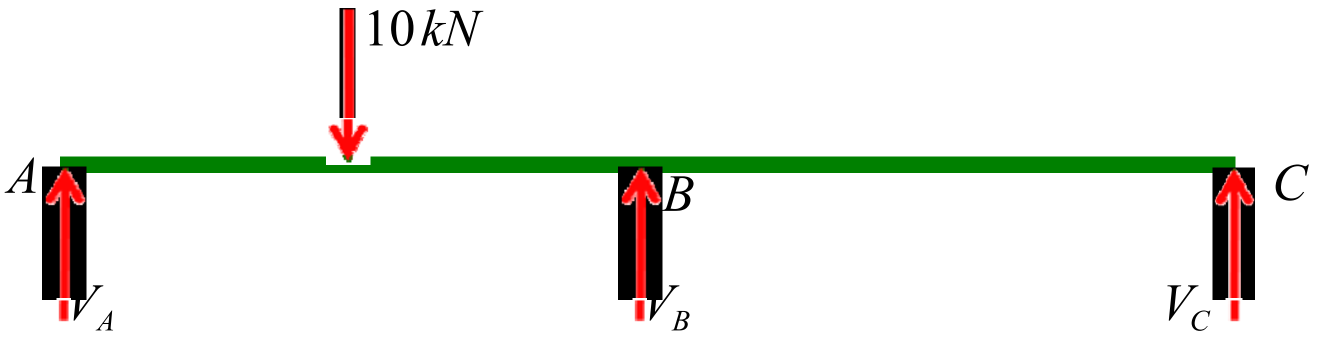


Bending moment diagram

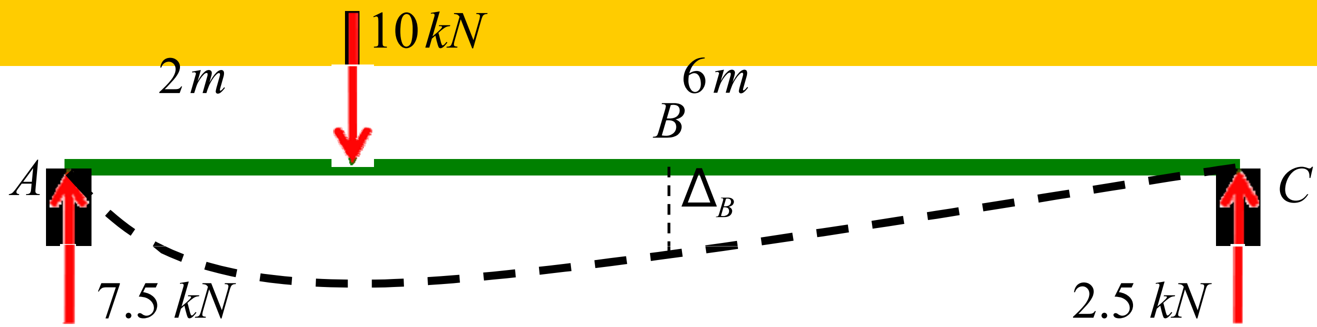


Example 2: Continuous beam

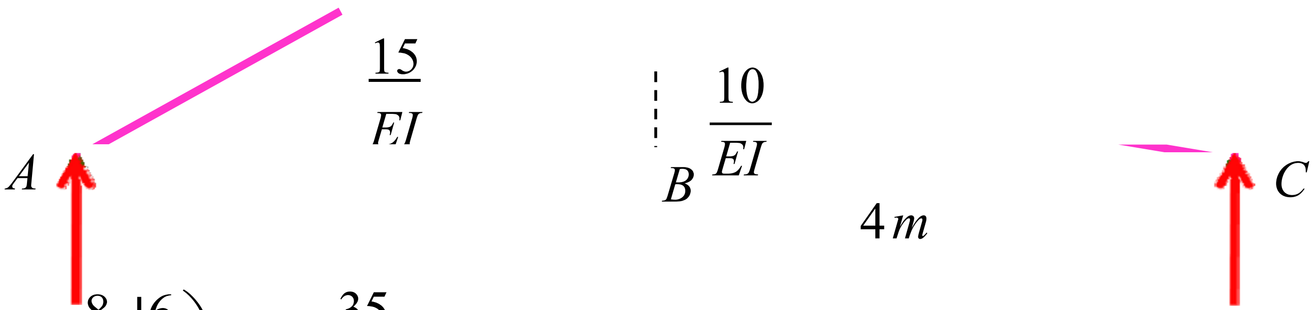




Choose V_B as the redundant



Released structure

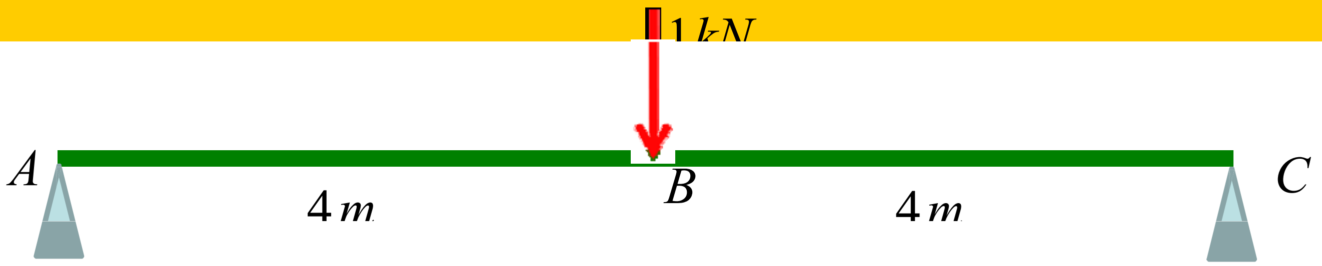


$$\frac{1}{EI}\left(0.5\times15\times8\times\frac{8+6}{3}\right)\div 8=\frac{35}{EI}$$

$$\frac{1}{EI}\left(0.5\times15\times8-35\right)=\frac{25}{EI}$$

Conjugate beam of the released structure (to find Δ_B)

$$\therefore \Delta_B=\frac{1}{EI}\left(25\times4-0.5\times10\times4\times\frac{4}{3}\right)=\frac{73.333}{EI}$$ (numerically)



Released structure with unit load corresponding to $\boldsymbol{V_B}$ (to find $\boldsymbol{\delta_B}$)

$$\delta_B = \frac{l^3}{48EI} = \frac{8^3}{48EI} = \frac{10.667}{EI}$$

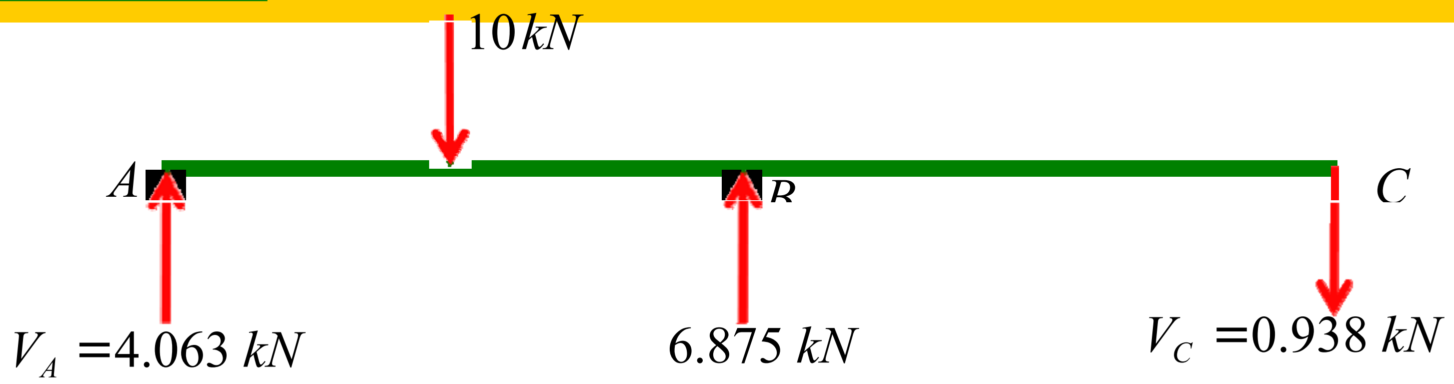
(numerically)
(direction is same as Δ_B i.e., downwards)

$$\Delta_B + R_B \delta_B = 0$$

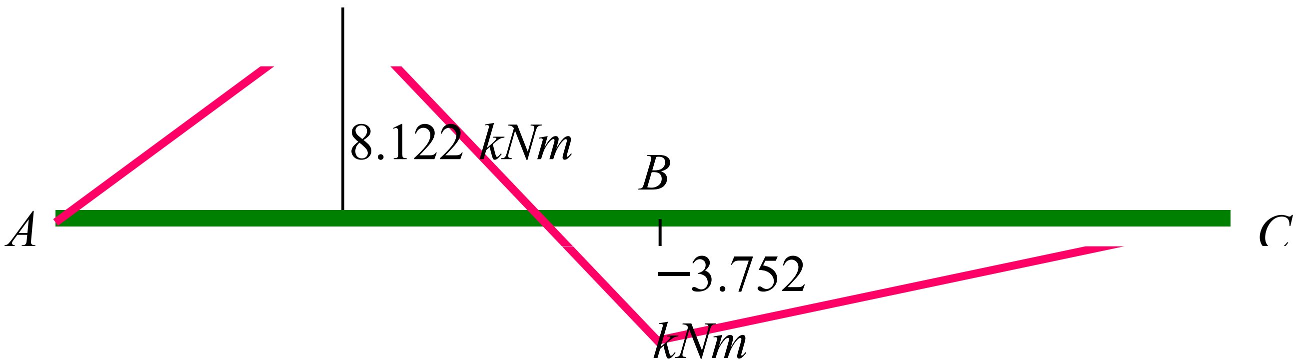
$$R_B = \frac{-\Delta_B}{\delta_B} = \frac{-73.333}{EI} \cdot \frac{EI}{10.667} = -6.875 \; kNm$$

-ve sign indicates that $\boldsymbol{V_B}$ is in the <u>opposite direction of the unit load</u>.
i.e., in the upward direction.

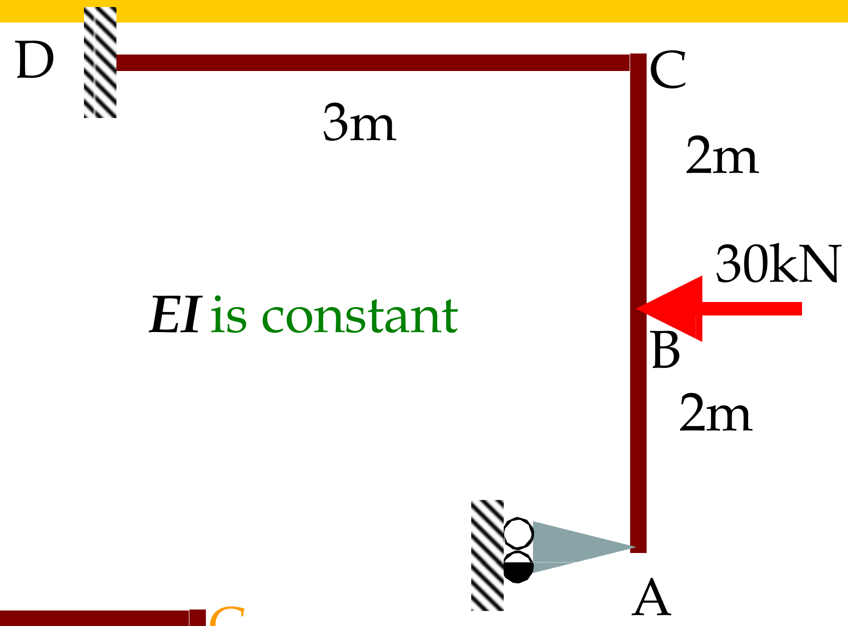
Other reactions



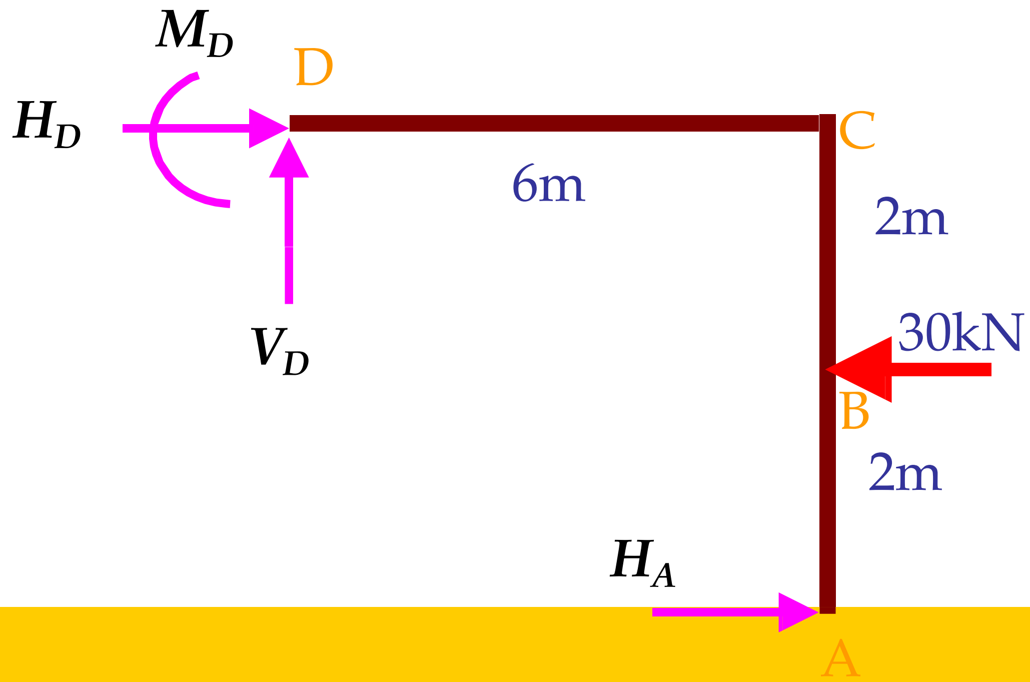
Bending moment diagram



Example 3A:

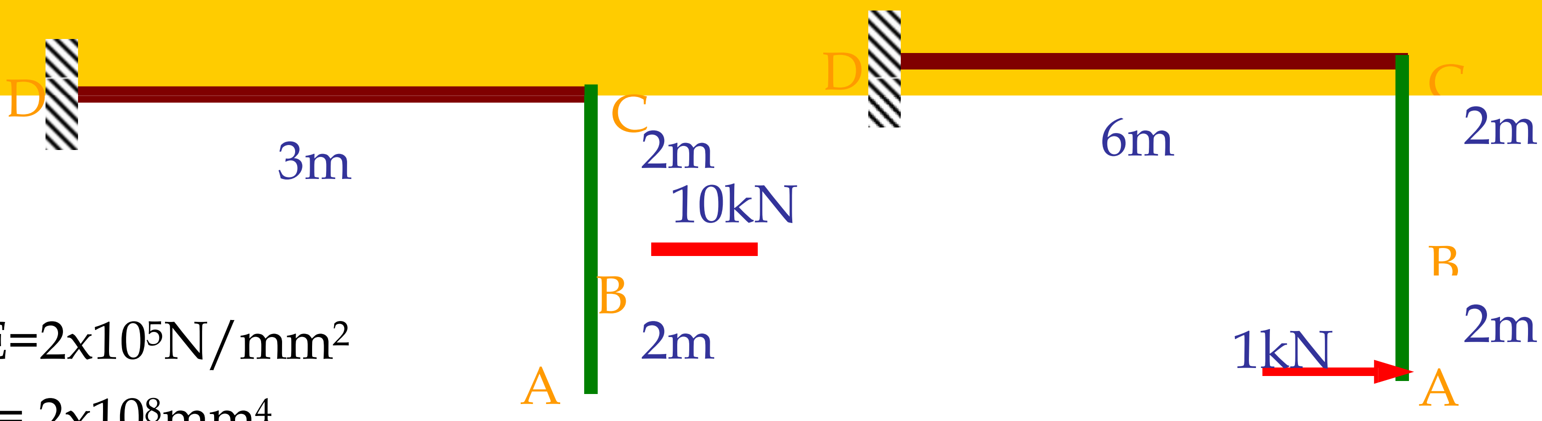


***EI* is constant**



Choose H_A as the redundant

To find Δ_{HA} and δ_{HA} (using unit load method)



$E = 2 \times 10^5 N/mm^2$

$I = 2 \times 10^8 mm^4$

Portion	Origin	Limits	M	m	Mm	m^2
AB	A	0-2	0	x	0	x^2
BC	A	2-4	-10(x-2)	x	$-10x^2+20x$	x^2
CD	A	0-3	-20	4	-80	16

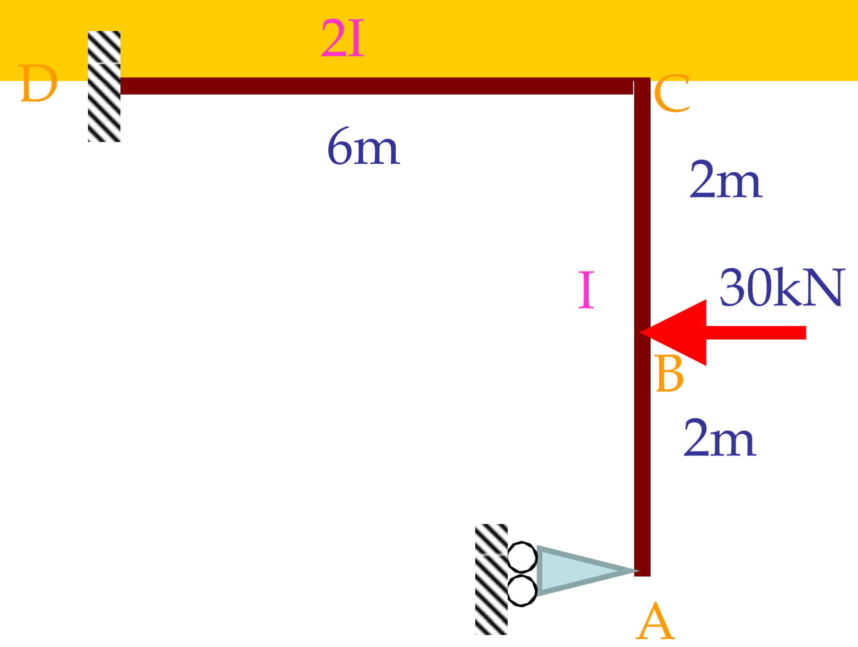
$$\Delta_{HA} = \int \frac{Mm\,dx}{EI} = \int_2^4 \frac{1}{EI}\left(-10x^2 + 20x\right)dx + \int_0^3 \frac{1}{EI}(-80)\,dx$$

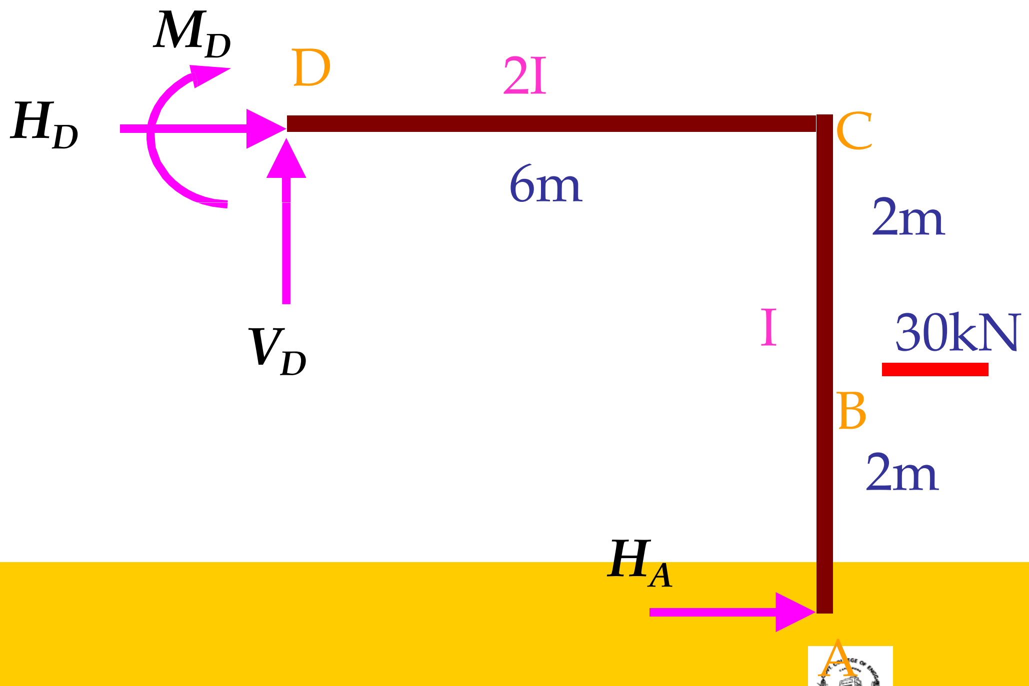
$$\Delta_{HA} = \frac{1}{EI}\left[-10\frac{x^3}{3} + 10x^2\right]_2^4 + \frac{1}{EI}\left[-80x\right]_0^3 = \frac{-306.67}{EI}$$

$$\delta_{HA} = \int \frac{m^2 dx}{EI} = \int_0^4 \frac{x^2}{EI} dx + \int_0^3 \frac{16}{EI} dx = \left[\frac{x^3}{3EI}\right]_0^4 + \left[\frac{16x}{EI}\right]_0^3 = \frac{69.33}{EI}$$

$$\Delta_{HA} + H_A \delta_{HA} = 0 \qquad H_A = \frac{-\Delta_B}{\delta_B} = \frac{306.67}{EI} \cdot \frac{EI}{69.33} = 4.423\ kN$$

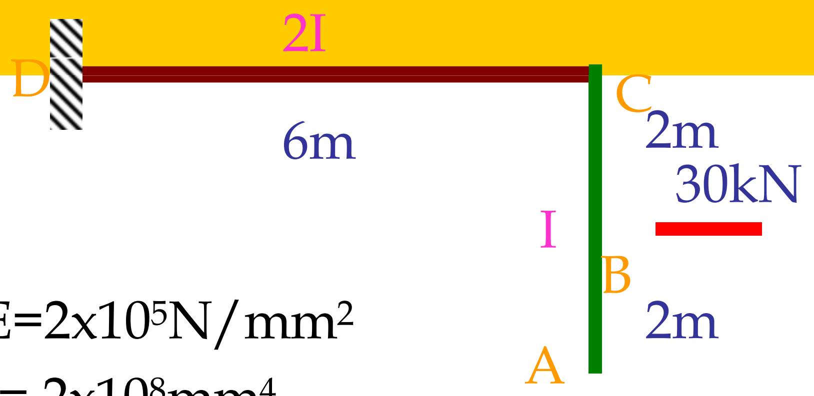
Example 3B:

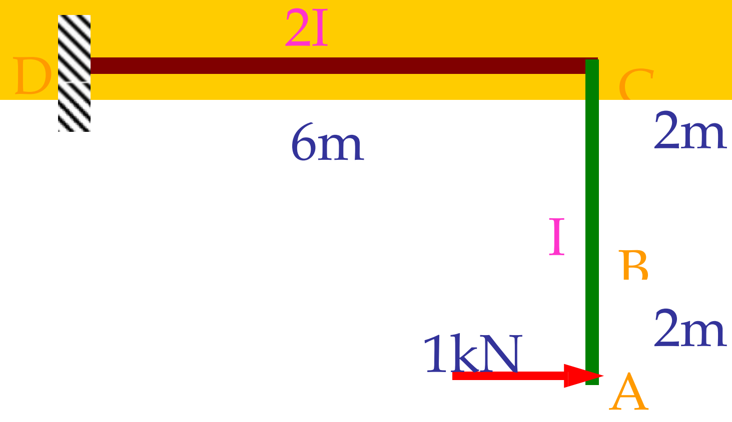




Choose H_A as the redundant

To find Δ_{HA} and δ_{HA} (using unit load method)





$E = 2 \times 10^5 N/mm^2$

$I = 2 \times 10^8 mm^4$

Portion	Origin	Limits	M	m	Mm	m^2
AB	A	0-2	0	x	0	x^2
BC	A	2-4	-30(x-2)	x	$-30x^2+60x$	x^2
CD	A	0-6	-60	4	-240	16

$$\Delta_{HA} = \int \frac{Mmdx}{EI} = \int_2^4 \frac{1}{EI}\left(-30x^2 + 60x\right)dx + \int_0^6 \frac{1}{2EI}\left(-240\right)dx$$

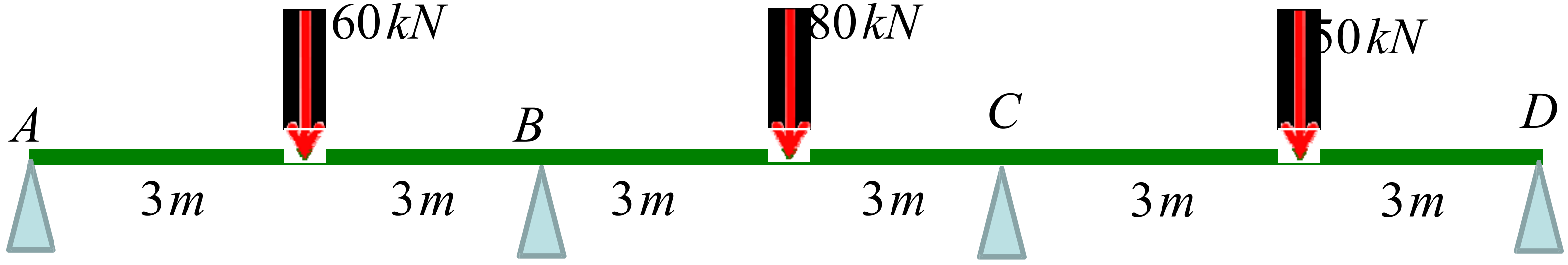
$$\Delta_{HA} = \frac{1}{EI}\left[-10x^3 + 30x^2\right]_2^4 + \frac{1}{2EI}\left[-240x\right]_0^6 = \frac{-920}{EI}$$

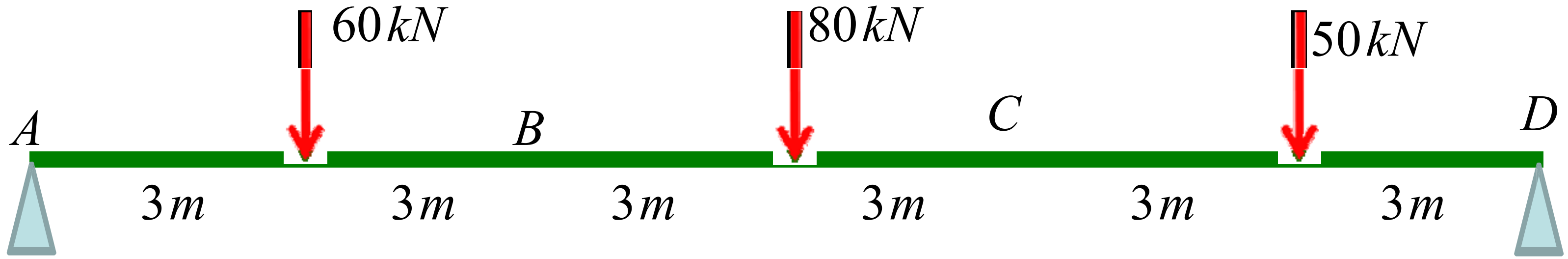
$$\delta_{HA} = \int \frac{m^2 dx}{EI} = \int_0^4 \frac{x^2}{EI} dx + \int_0^6 \frac{16}{2EI} dx = \left[\frac{x^3}{3EI}\right]_0^4 + \left[\frac{8x}{EI}\right]_0^6 = \frac{69.333}{EI}$$

$$\Delta_{HA} + H_A \; {}_{HA} = 0$$

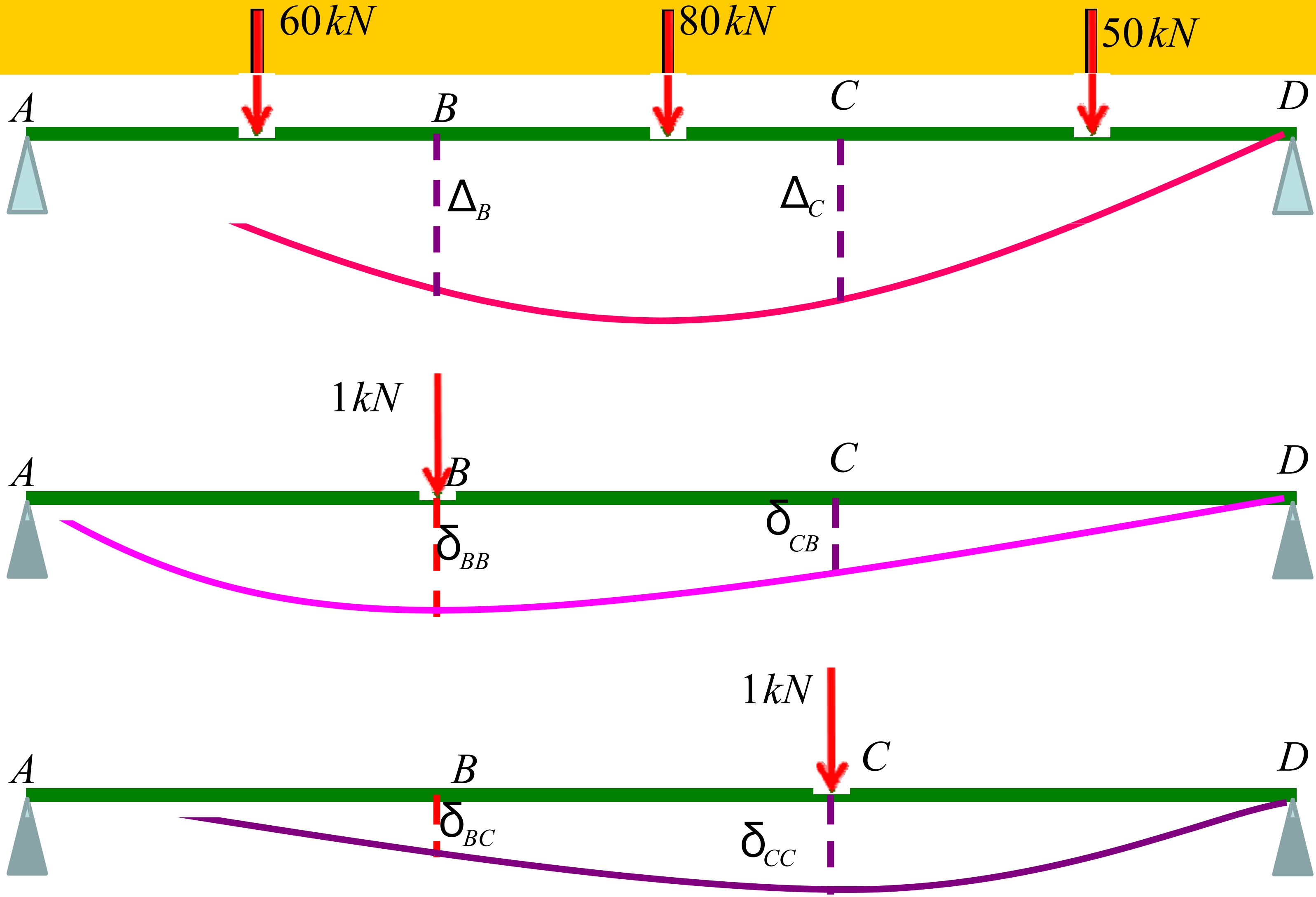
$$H_A = \frac{-\Delta_{HA}}{\delta_{HA}} = \frac{920}{EI} \cdot \frac{EI}{69.333} = 13.269\ kN$$

Example 4: Two or more redundants

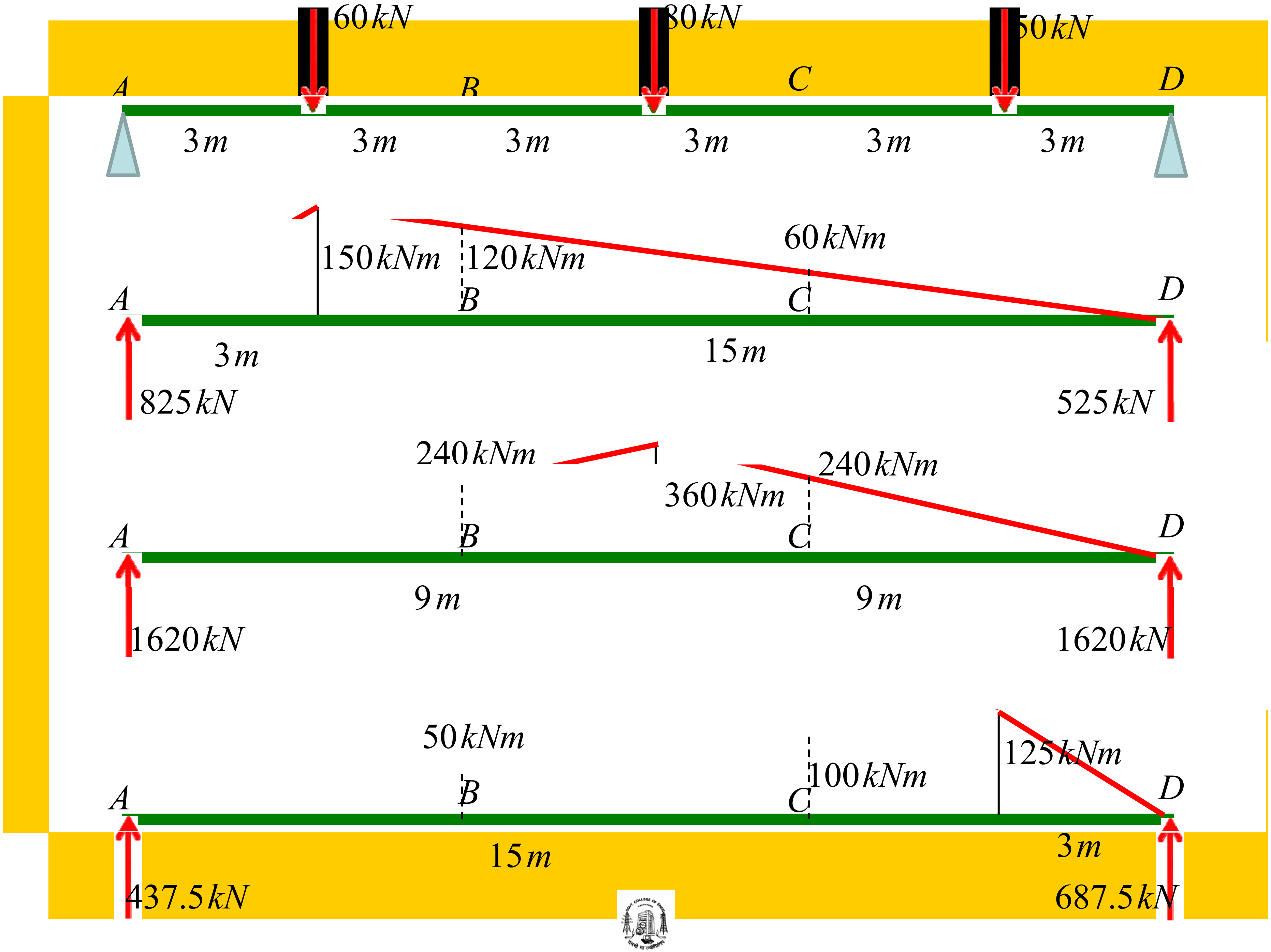




Choose V_B and V_c as the redundant



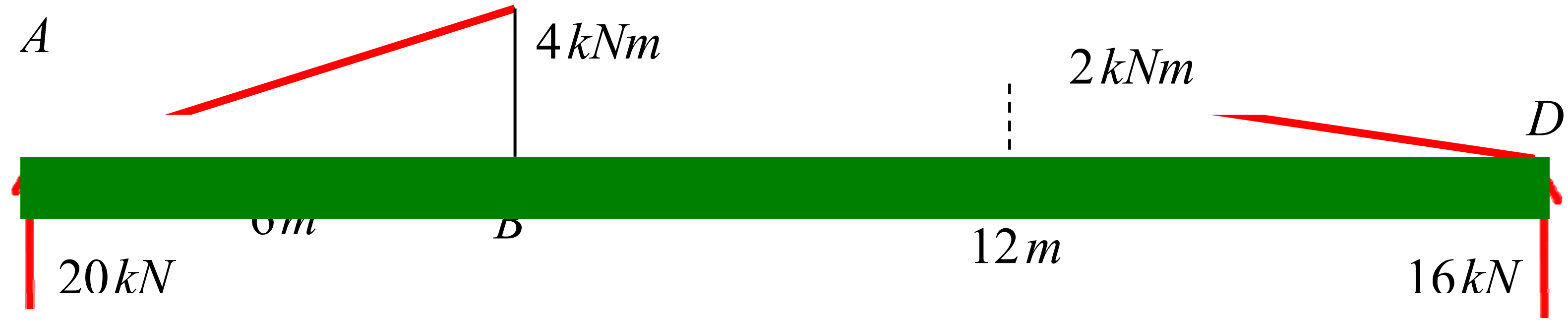
$$\Delta_B + V_B \delta_{BB} + V_C \delta_{BC} = 0 \qquad \Delta_C + V_B \delta_{CB} + V_C \delta_{CC} = 0$$



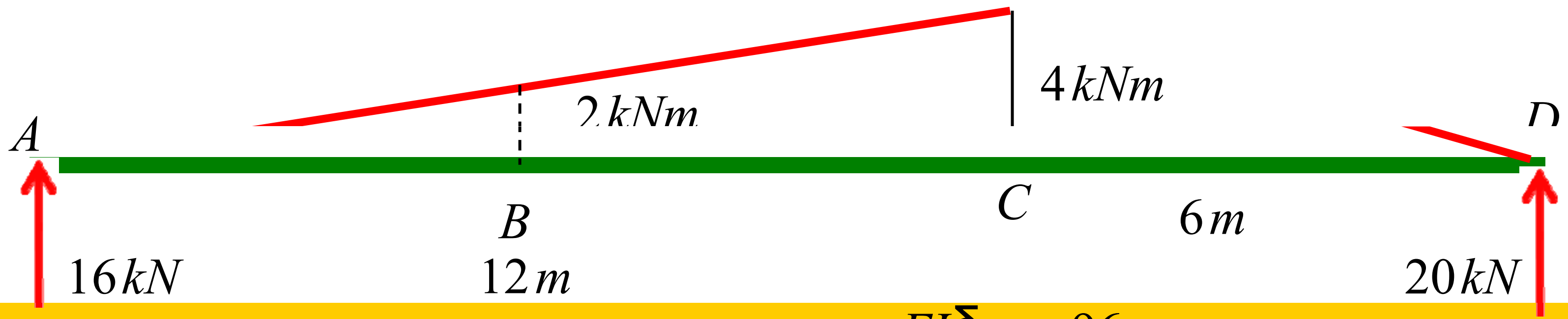
60 kN
80 kN
50 kN
A
B
C
D
3 m
3 m
3 m
3 m
3 m
3 m
150 kNm
120 kNm
60 kNm
A
B
C
D
3 m
15 m
825 kN
525 kN
240 kNm
360 kNm
240 kNm
A
B
C
D
9 m
9 m
1620 kN
1620 kN
50 kNm
100 kNm
125 kNm
A
B
C
D
15 m
3 m
437.5 kN
687.5 kN

$$EI\Delta_B = 3420 + 8280 + 2325 = 14025$$

$$EI\Delta_C = 2790 + 8280 + 2850 = 13920$$



$$EI\delta_{BB} = 96 \qquad EI\delta_{CB} = 84$$



$$EI\delta_{BC} = 84 \qquad EI\delta_{CC} = 96$$

$$\Delta_B + V_B \quad {}_{BB} + V_C\delta_{BC} = 0 \Rightarrow 14025 + 96V_B + 84V_C = 0$$

$$\Delta_C + V_B\delta_{CB} + V_C\delta_{CC} = 0 \Rightarrow 13920 + 84V_B + 96V_C = 0$$

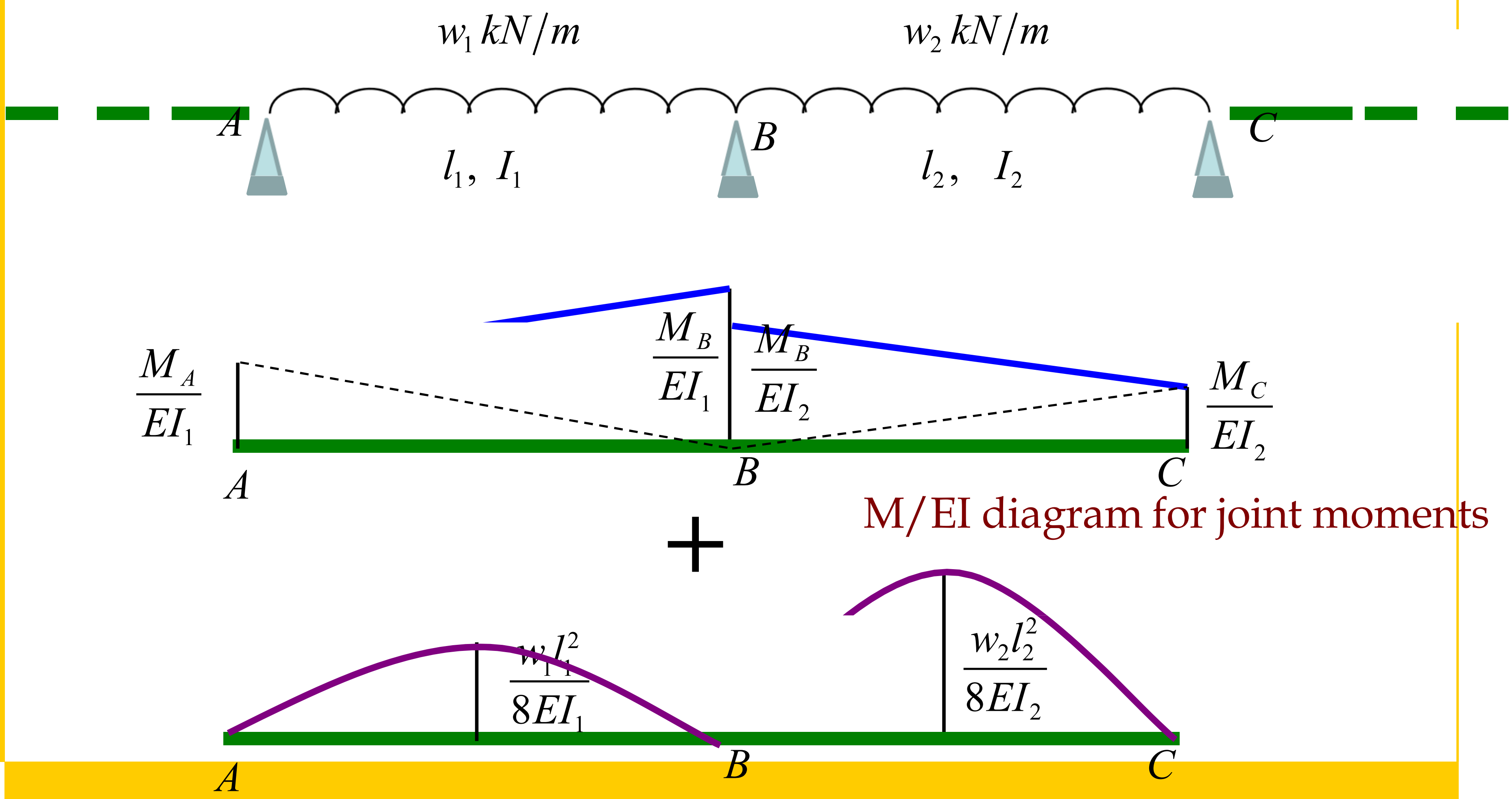
$V_B = -82$ kN $\quad$ $V_C = -73.25$ kN

$V_B = 82\ kN\,(\uparrow)$ $\quad$ $V_C = 73.25\ kN\,(\uparrow)$

$V_A = 19.25\ kN\,(\uparrow)$ $\quad$ $V_D = 15.5\ kN\,(\uparrow)$

Clapeyron's **theorem of three moments**

1. Uniform loading



M/EI diagram for joint moments

M/EI diagram for simple beam moments

To find slopes at B using Conjugate Beam Method:

From span AB:

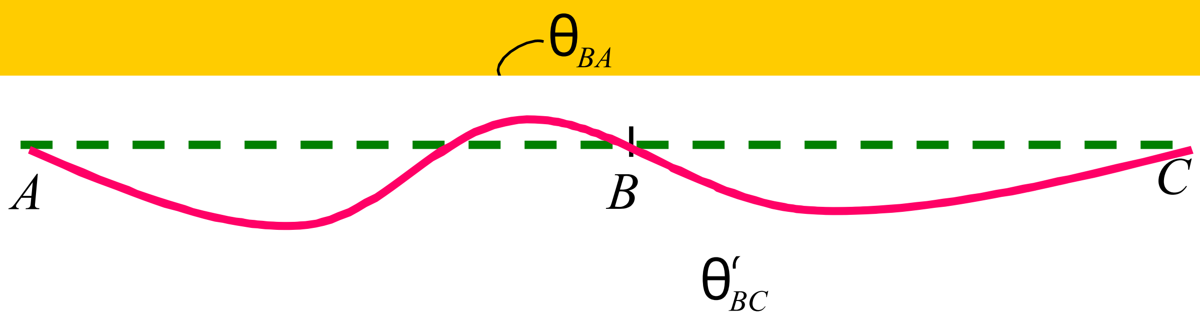
$$V_{B1} l_1 = \left[\frac{1}{2}\cdot\frac{M_A l_1}{EI_1}\cdot\frac{l_1}{3} + \frac{1}{2}\cdot\frac{M_B l_1}{EI_1}\cdot\frac{2l_1}{3}\right] + \left[\frac{2}{3}\cdot l_1\cdot\frac{w_1 l_1^2}{8EI_1}\cdot\frac{l_1}{2}\right]$$

$$\Rightarrow V_{B1} = \frac{M_A l_1}{6EI_1} + \frac{M_B l_1}{3EI_1} + \frac{w_1 l_1^3}{24EI_1} = \theta_{BA}$$

From span BC:

$$V_{B2} l_2 = \left[\frac{1}{2}\cdot\frac{M_B l_2}{EI_2}\cdot\frac{l_2}{3} + \frac{1}{2}\cdot\frac{M_C l_2}{EI_2}\cdot\frac{2l_2}{3}\right] + \left[\frac{2}{3}\cdot l_2\cdot\frac{w_2 l_2^2}{8EI_2}\cdot\frac{l_2}{2}\right]$$

$$\Rightarrow V_{B2} = \frac{M_B l_2}{3EI_2} + \frac{M_C l_2}{6EI_2} + \frac{w_2 l_2^3}{24EI_2} = \theta_{BC}$$



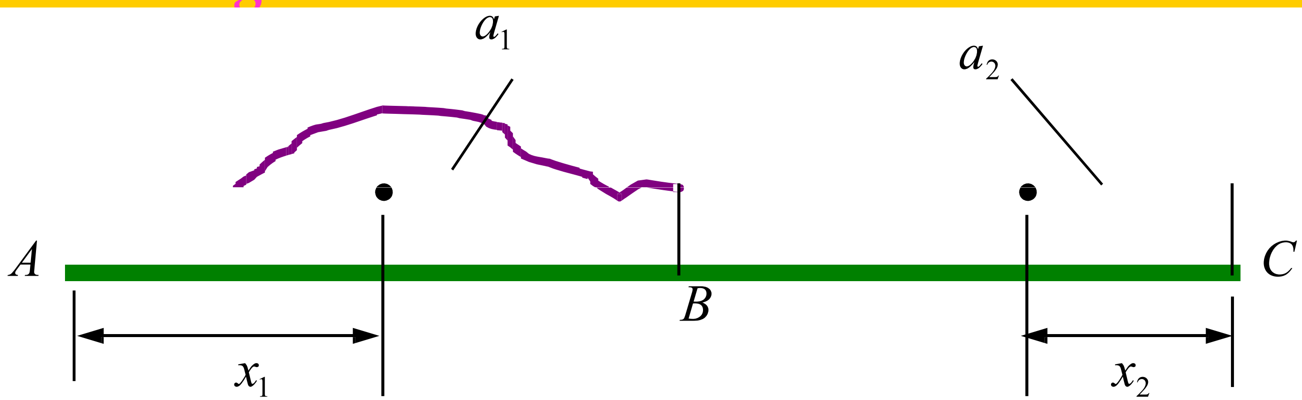
Deflected shape

$$\theta_{BA} + \theta_{BC} = 0 \Rightarrow \theta_{BA} = -\theta_{BC}$$

$$\Rightarrow \frac{M_A l_1}{6EI_1} + \frac{M_B l_1}{3EI_1} + \frac{w_1 l_1^3}{24EI_1} \; l_2 = -\left(\frac{M_B l_2}{3EI_2} + \frac{M_C}{6EI_2} + \frac{w_2 l_2^3}{24EI_2}\right)$$

$\Rightarrow$ [illegible]

2. General loading

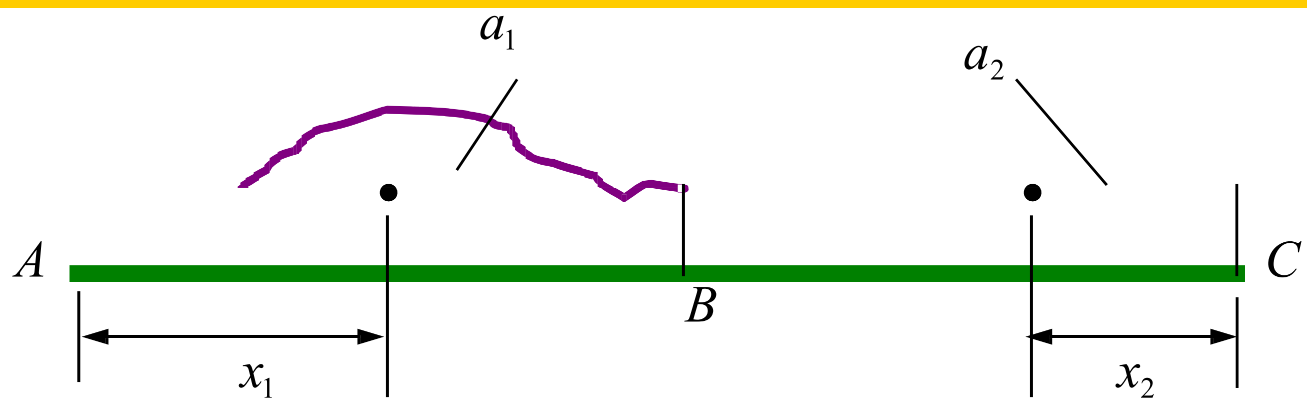


$$\theta_{BA} = \frac{M_A l_1}{6EI_1} + \frac{M_B l_1}{3EI_1} + \frac{a_1 x_1}{EI_1 l_1}$$

$$\theta_{BC} = \frac{M_B l_2}{3EI_2} + \frac{M_C l_2}{6EI_2} + \frac{a_2 x_2}{EI_2 l_2}$$

$\theta_{BA} = -\theta_{BC}$ =

2. General loading with support settlement



$\delta_A > \delta_B;$

$\delta_B < \delta_C$

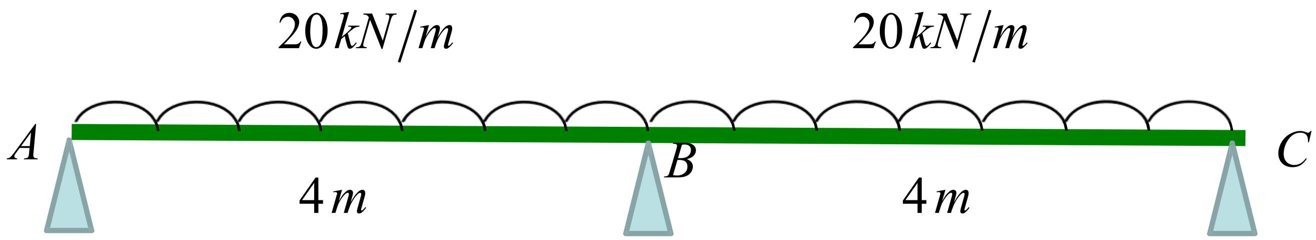
$$\theta_{BA} = \frac{M_A l_1}{6EI_1} + \frac{M_B l_1}{3EI_1} + \frac{a_1 x_1}{EI_1 l_1} + \frac{\delta_A - \delta_B}{l_1}$$

$$\theta_{BC} = \frac{M_B l_2}{3EI_2} + \frac{M_C l_2}{6EI_2} + \frac{a_2 x_2}{EI_2 l_2} + \frac{\delta_C - \delta_B}{l_2}$$

$$\theta_{BA} = -\theta_{BC}$$

$\Rightarrow$

Example 1:



EI is constant

$$\frac{M_A l_1}{EI_1} + 2M_B\left(\frac{l_1}{EI_1} + \frac{l_2}{EI_2}\right) + \frac{M_C l_2}{EI_2} = -\frac{w_1 l_1^3}{4EI_1} - \frac{w_2 l_2^3}{4EI_2}$$

$$l_1 = l_2$$

$$w_1 = w_2$$

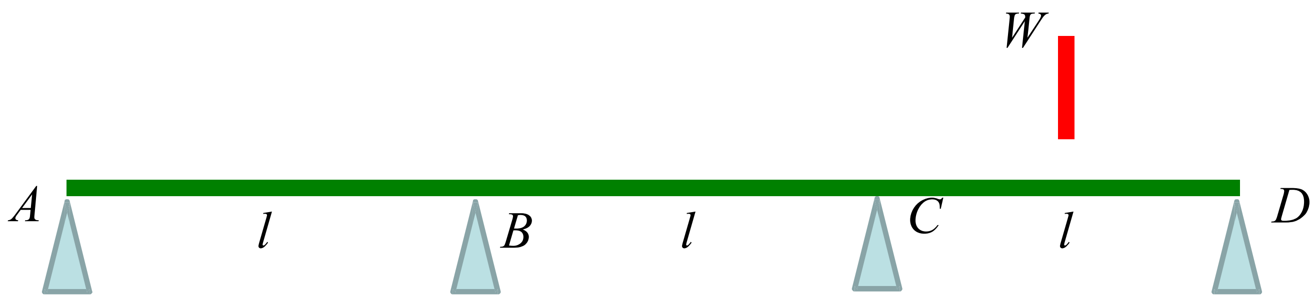
$$4M_B = \frac{2wl^2}{4}$$

$$-$$

$$M_A = M_C = 0$$

$$\therefore M_B = \frac{wl^2}{8} = \frac{20 \times 4^2}{8} = -40$$

Example 2:



EI is constant

Span ABC

$$l_1 = l_2 \qquad EI_1 = EI_2 \qquad M_A = 0$$

$$\frac{M_A l_1}{EI_1} + 2M_B\left(\frac{l_1}{EI_1} + \frac{l_2}{EI_2}\right) + \frac{M_C l_2}{EI_2} = -\frac{w_1 l_1^3}{4EI_1} - \frac{w_2 l_2^3}{4EI_2}$$

$$\Rightarrow M_A + 4M_B + M_C = 0$$

$$\Rightarrow 4M_B + M_C = 0 \qquad \text{———————} \qquad (1)$$

$$a_2 = \frac{1}{2}\cdot\frac{Wl}{4}\cdot l$$

$$x_2 = \frac{l}{2}$$

Span BCD

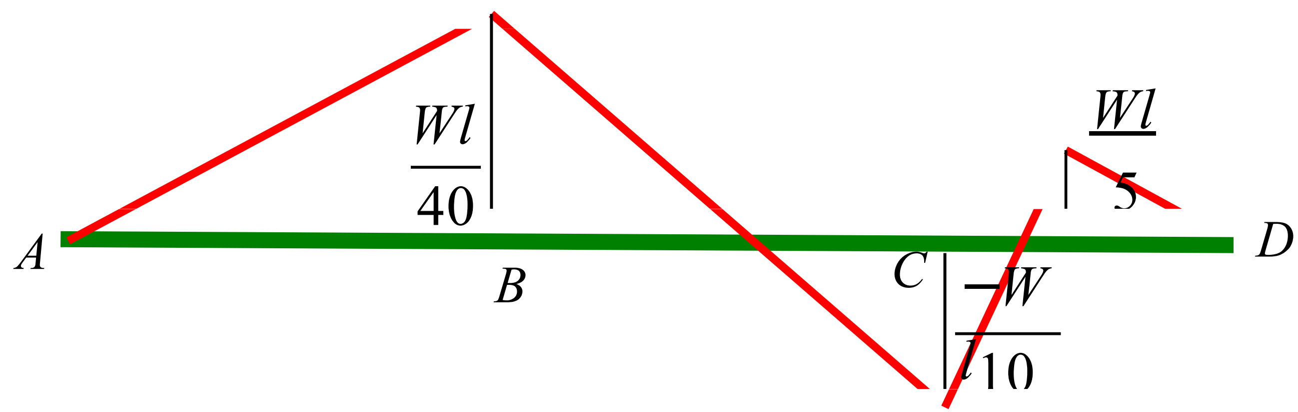
$$M_D = 0$$

$$l\left(M_B + 4M_C + M_D\right) = -\frac{6a_1}{l}x_1 - \frac{6a_2}{l}x_2$$

$$l^2\left(M_B + 4M_C\right) = -6\left(\frac{1}{2}\cdot\frac{Wl}{4}\cdot l\right)\frac{\;}{2}$$

$$M_B + 4M_C = -\frac{3Wl}{8} \qquad (2)$$

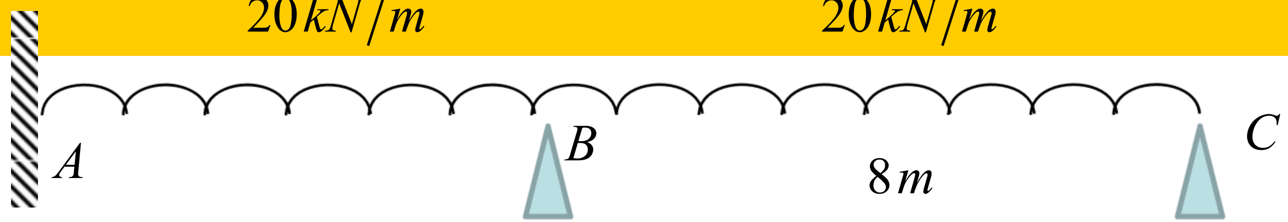
$(1)\,\&\,(2) \longrightarrow \quad M_C = \frac{-W}{10} \qquad M_B = \frac{Wl}{40}$



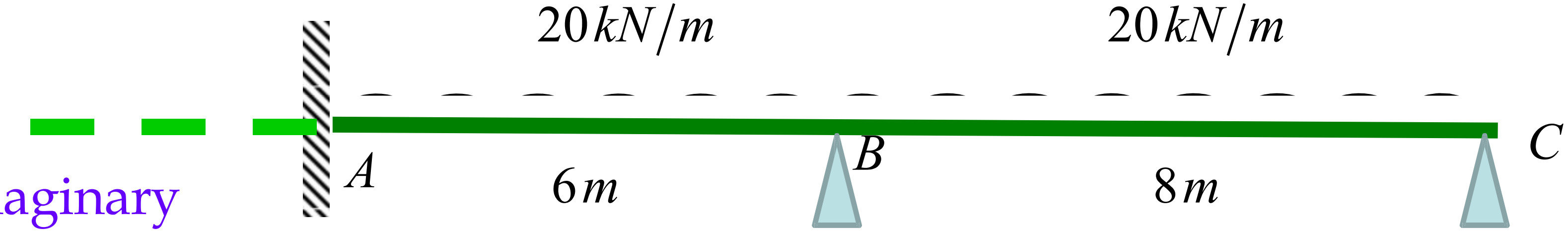
Wl
40
A
B
C
D
Wl
5
−W
l10

Example 3:

EI is constant



Imaginary span $A'A$



Span $A'AB$

$$w_1 = 0$$

$$w_2 = 20\ kN/m$$

$$M_{A'} = 0$$

$$EI_1 = EI_2$$

$$\frac{M_{A'} l_1}{EI_1} + 2M_A\left(\frac{l_1}{EI_1} + \frac{l_2}{EI_2}\right) + \frac{M_B l_2}{EI_2} = -\frac{w_1 l_1^3}{4EI_1} - \frac{\quad^{3}}{4EI_2}$$

$$2M_A \times 6 + M_B \times 6 = -\frac{20 \times 6^3}{4}$$

$$2M_A + M_B = -180 \qquad \text{———————— (1)}$$

Span *ABC*

$$\frac{M_A l_1}{EI_1} + 2M_B\left(\frac{l_1}{EI_1} + \frac{l_2}{EI_2}\right) + \frac{M_C l_2}{EI_2} = -\frac{w_1 l_1^3}{4EI_1} - \frac{w_2 l_2^3}{4EI_2}$$

$w_1 = w_2 = 20\ kN/m$

$M_C = 0$

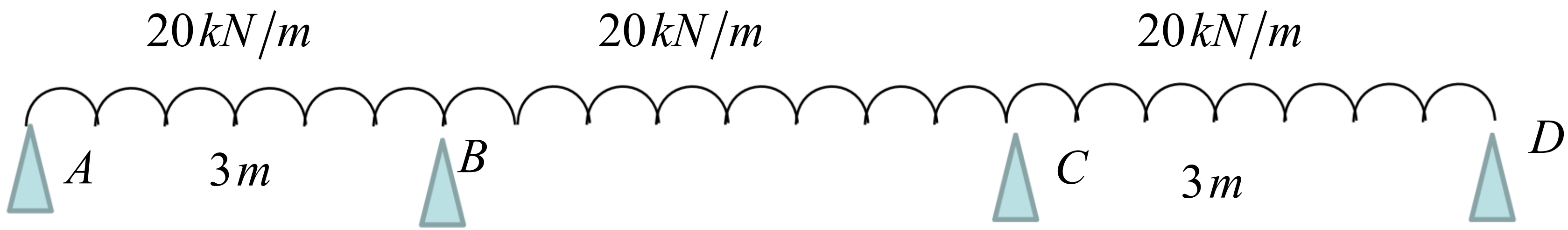
$$6M_A + 2M_B \times 14 = \frac{20 \times 6^3}{4} - \frac{20 \times 8^3}{4}$$

$$3M_A + 14M_B = -1820 \qquad \text{———————— (2)}$$

(1) & (2) ⟶ $M_A = -28\ kNm$ $\qquad$ $M_B = -124\ kNm$

Example 4 (Support settlement):

EI is constant $\qquad EI = 26320\ kNm^2 \qquad \delta_B = \delta_C = 87\ mm$



Span ***ABC***

$$M_A = 0 \qquad EI_1 = EI_2 = EI$$

$$\frac{M_A l_1}{EI_1} + 2M_B\left(\frac{l_1}{EI_1} + \frac{l_2}{EI_2}\right) + \frac{M_C l_2}{EI_2} = -\frac{w_1 l_1^3}{4EI_1} - \frac{w_2 l_2^3}{4EI_2} - \frac{6(\delta_A - \delta_B)}{l_1} - \frac{6(\delta_C - \delta_B)}{l_2}$$

$$\frac{1}{26320}(18M_B + 6M_C) = -\frac{20 \times 3^3}{4 \times 26320} - \frac{20 \times 6^3}{4 \times 26320} - \frac{6(-0.087)}{3} - \frac{6 \times 0}{6}$$

$$3M_B + M_C = 560.78 \qquad \text{———} \qquad (1)$$

Span *BCD*

$$M_D = 0$$

$$\frac{M_B l_1}{EI_1} + 2M_C\left(\frac{l_1}{EI_1} + \frac{l_2}{EI_2}\right) + \frac{M_D l_2}{EI_2} = \frac{w_1 l_1^3}{4EI_1} - \frac{w_2 l_2^3}{4EI_2} - \frac{6(\delta_B - \delta_C)}{l_1} - \frac{6(\delta_D - \delta_C)}{l_2}$$

$$\frac{1}{26320}\, 6M_B + 18M_C = \frac{20\times 6^3}{4\times 26320} - \frac{20\times 3^3}{4\times 26320} - \frac{6\times 0}{6} - \frac{6(-0.087}{3}$$

$$-$$

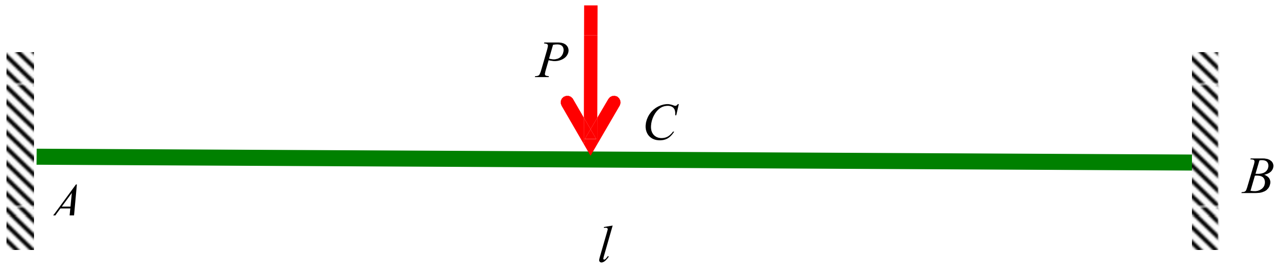
$$M_B + 3M_C = 560.78 \qquad \cdots \qquad (2)$$

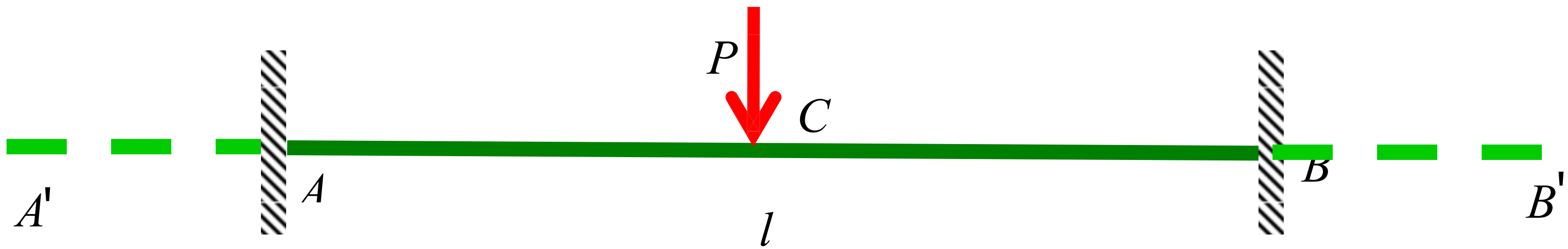
$$M_B = 140.195\ kNm \qquad M_C = 140.195\ kNm$$

Alternatively, from symmetry, $M_B = M_C$

$$3M_B + M_C = 560.78 \Rightarrow 4M_B = 560.78 \Rightarrow M_B = 140.195\ kNm$$

Examples 5 (Fixed Beam)





Imaginary span ***A′A***

Imaginary span ***BB′***

$$\frac{M_A l_1}{EI_1} + 2M_B\left(\frac{l_1}{EI_1} + \frac{l_2}{EI_2}\right) + \frac{M_C l_2}{EI_2} = -\frac{6a_1 x_1}{EI_1 l_1} - \frac{6a_2 x_2}{EI_2 l_2}$$

Span *A′AB*

$$M_{A'} = 0$$

$$EI_1 = EI_2$$

$$a_2 = \frac{1}{2}.\frac{Pl}{4}.l$$

$$x_2 = \frac{l}{2}$$

$$\Rightarrow 2M_A + M_B = -\frac{6Pl}{16} \quad \text{———————} \quad (1)$$

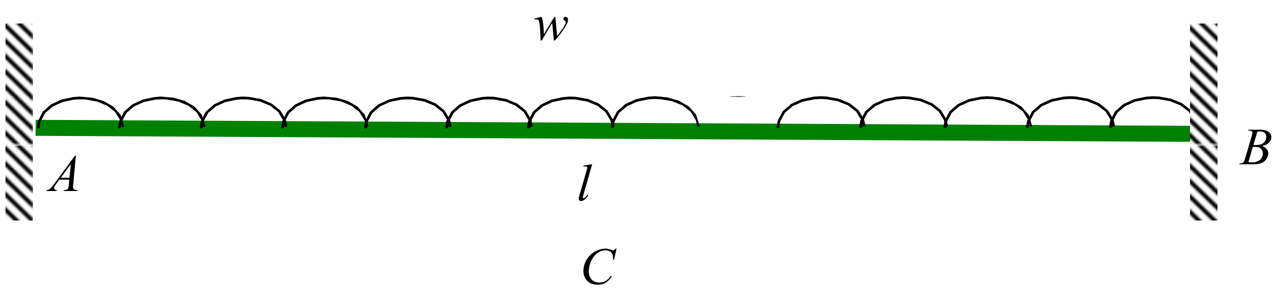
Span *ABB′*

$$M_{B'} = 0 \qquad EI_1 = EI_2 \qquad a_1 = \frac{Pl^2}{8}, \quad x_1 = \frac{l}{2}$$

$$\Rightarrow M_A + 2M_B = -\frac{6Pl}{16} \quad \text{———————} \quad (2)$$

$$(1)\&(2) \quad \text{————} \quad M_A = M_B = -\frac{Pl}{8}$$

Examples 6 (Fixed Beam)



ENERGY PRINCIPLES BASED ON DISPLACEMENT FIELD

[redacted] Energy (PMTPE)

Castigliano's Theorem (Part I)

alternative forms of

[redacted]

Principle of Stationary Total Potential Energy (PSTPE)

When the displacement field in a loaded elastic structure is given a small and arbitrary perturbation, maintaining compatibility and without disturbing the associated force field, then the first variation of the total potential energy is equal to zero, if the forces are in a state of static equilibrium.

Alternative form of Principle of Stationary Total Potential Energy (PSTPE)

The total potential energy π in a loaded elastic structure expressed as a function of n independent displacements D_1, D_2, … D_n in a compatible displacement field must be rendered stationary, with the partial derivative of π with respect to every D_j being equal to zero, if the associated force field is to be in a state of static equilibrium.

Principle of Minimum Total Potential Energy (PMTPE)

When the displacement field in a loaded linear elastic structure is given a small and arbitrary perturbation, maintaining compatibility and without disturbing the associated force field, then the first variation of the total potential energy is equal to zero, if the forces are in a state of static equilibrium.

Castigliano's Theorem (Part I)

If the strain energy, U, in an elastic structure, subject to a system of external forces in static equilibrium, can be expressed as a function of n independent displacements $D_1, D_2, \ldots D_n$ satisfying compatibility, then the partial derivative of U with respect to every D_j will be equal to the value of the conjugate force, F_j.

ENERGY PRINCIPLES BASED ON FORCE FIELD

Principle of Minimum Total Complementary Potential

Castigliano's Theorem (Part II)

Work

Principle of Stationary Total Complementary Potential Energy (PSTCPE)

When the force field in a loaded elastic structure is given a small and arbitrary perturbation, maintaining equilibrium compatibility and without disturbing the associated displacement field, then the first variation of the total complementary potential energy is equal to zero, if the displacements satisfy compatibility.

Alternative form of Principle of Stationary Total Complementary Potential Energy (PSTCPE)

The total complementary potential energy π^* in a loaded elastic structure expressed as a function of n independent forces F_1, F_2, … F_n in a statically admissible force field must be rendered stationary, with the partial derivative of π^* with respect to every F_j being equal to zero, if the associated displacement field is to satisfy compatibility.

Principle of Minimum Total Complementary Potential Energy (PMTCPE)

When the force field in a loaded linear elastic structure is given a small and arbitrary perturbation, maintaining equilibrium compatibility and without disturbing the associated displacement field, then the first variation of the total complementary potential energy is equal to zero, if the displacement satisfy compatibility.

Castigliano's Theorem (Part II)

If the complementary strain energy, U^*, in an elastic structure, with a kinematically admissible displacement field, is expressed as a function of n independent external forces F_1, F_2, … F_n satisfying equilibrium, then the partial derivative of U^* with respect to every F_j will be equal to the value of the conjugate displacement, D_j.

If the behaviour is linear elastic, U^* can be replaced by U.

Thus, Castigliano's Theorem (Part II) can otherwise be stated as:

"If U is the total strain energy in a linear elastic structure due to application of external forces F_1, F_2, F_3, … F_n at points A_1, A_2, A_3,…, A_n respectively in the directions AX_1, AX_2, AX_3,…, AX_n then the displacements at points A_1, A_2, A_3,…, A_n respectively in the directions AX_1, AX_2, AX_3,…, AX_n are $\partial U/\partial F_1$, $\partial U/\partial F_2$, $\partial U/\partial F_3$,…, $\partial U/\partial F_n$ respectively."

Proof for Castigliano's theorem (Part II)

Let $x_1, x_2, x_3, \ldots x_n$ be deflections at points $A_1, A_2, A_3, \ldots A_n$ due to $F_1, F_2, F_3, \ldots F_n$

Total strain energy, $$U = \frac{1}{2}F_1x_1 + \frac{1}{2}F_2x_2 + \frac{1}{2}F_3x_3 + \ldots + \frac{1}{2}F_nx_n \longrightarrow (1)$$

Let the load F_1 alone be increased by δF_1

Let $\delta x_1, \delta x_2, \delta x_3 \ldots, \delta x_n$ be the additional deflections at points $A_1, A_2, A_3, \ldots A_n$

Increase in strain energy,

$$\delta U = \left(F_1 + \frac{1}{2}\delta F_1\right)\delta x_1 + F_2\delta x_2 + F_3\delta x_3 + \ldots + F_n\delta x_n$$

$$\delta U = F_1\delta x_1 + F_2\delta x_2 + F_3\delta x_3 + \ldots + F_n\delta x_n$$, neglecting small quantities.

$\longrightarrow$ (2)

Let $(F_1 + \delta F_1), F_2, F_3 \ldots, F_n$ are acting on the original structure

Total strain energy,

$$U + \delta U = \frac{1}{2}(F_1 + \delta F_1)(x_1 + \delta x_1) + \frac{1}{2}F_2(x_2 + \delta x_2) + \frac{1}{2}F_3(x_3 + \delta x_3) + \ldots + \frac{1}{2}F_n(x_n + \delta x_n) \longrightarrow (3)$$

$$(1) \longrightarrow \quad U = \frac{1}{2}F_1 x_1 + \frac{1}{2}F_2 x_2 + \frac{1}{2}F_3 x_3 + \ldots + \frac{1}{2}F_n x_n$$

$$(3)\text{-}(1) \longrightarrow \quad \delta U = \frac{1}{2} x_1 \delta F_1 + \left(\frac{1}{2} F_1 \delta x_1 + \frac{1}{2} F_2 \delta x_2 + \ldots + \frac{1}{2} F_n \delta x_n\right)$$

$$2\ U = x_1 \delta F_1 + F_1 \delta x_1 + F_2 \delta x_2 + \ldots + F_n \delta x_n \longrightarrow (4)$$

$$(2) \longrightarrow \quad \delta U = F_1 \delta x_1 + F_2 \delta x_2 + F_3 \delta x_3 + \ldots + F_n \delta x_n$$

$$(4)\text{-}(2) \longrightarrow \quad \delta U = x_1 \delta F_1$$

$$\frac{\delta U}{\delta F_1} = x_1 \qquad When\ \delta F_1 \to 0, \frac{\partial U}{\partial F_1} = x_1$$

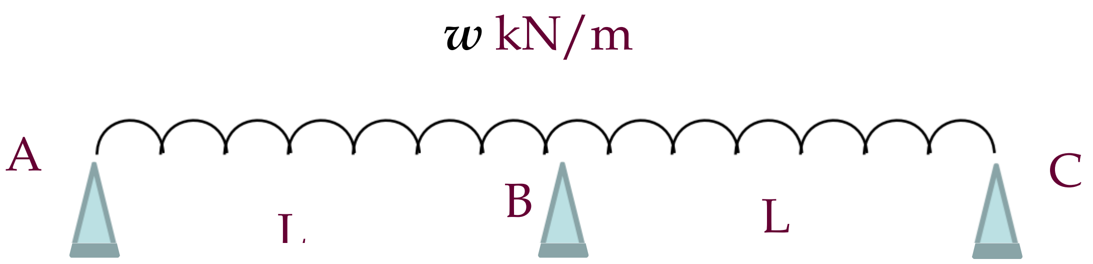
$$Similarly,\ x_2 = \frac{\partial U}{\partial F_2}, x_2 = \frac{\partial U}{\partial F_3}, \ldots x_n = \frac{\partial U}{\partial F_n}$$

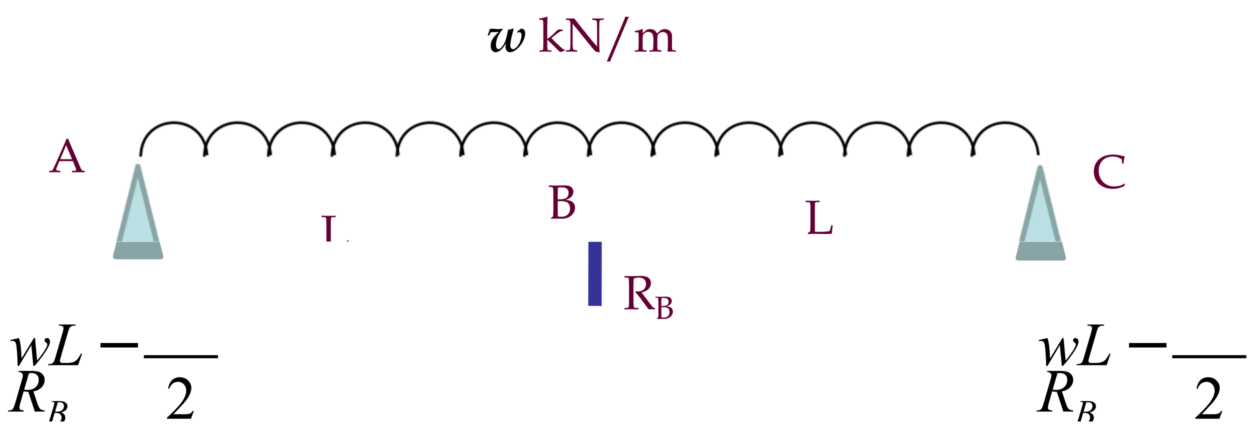
For example, in the case of bending,

$$U = \int \left(\frac{M^2}{2EI} \right) dx$$

$$\therefore \delta = \frac{\partial U}{\partial F} = \int \left(\frac{M}{EI} \right) \frac{\partial M}{\partial F} dx$$

Example 1: Using Castigliano's Theorem, analyse the continuous beam shown in figure.





Let R_B be the redundant.

From ***A*** to ***B***,
$$M_x = \left(wL - \frac{R_B}{2}\right)x - \frac{wx^2}{2}$$

$$\delta_B = 0 \qquad \Rightarrow \frac{\partial U}{\partial R_B} = \int \left(\frac{M}{EI}\right)\frac{\partial M}{\partial R_B}\, dx = 0$$

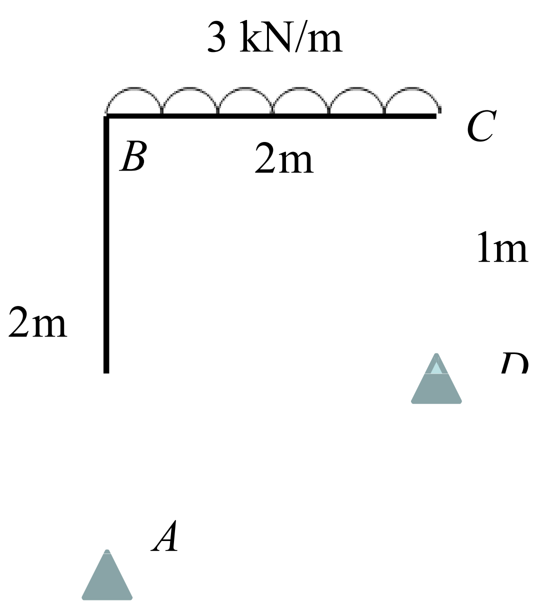
$$\frac{\partial M}{\partial R_B} = -\frac{x}{2}$$

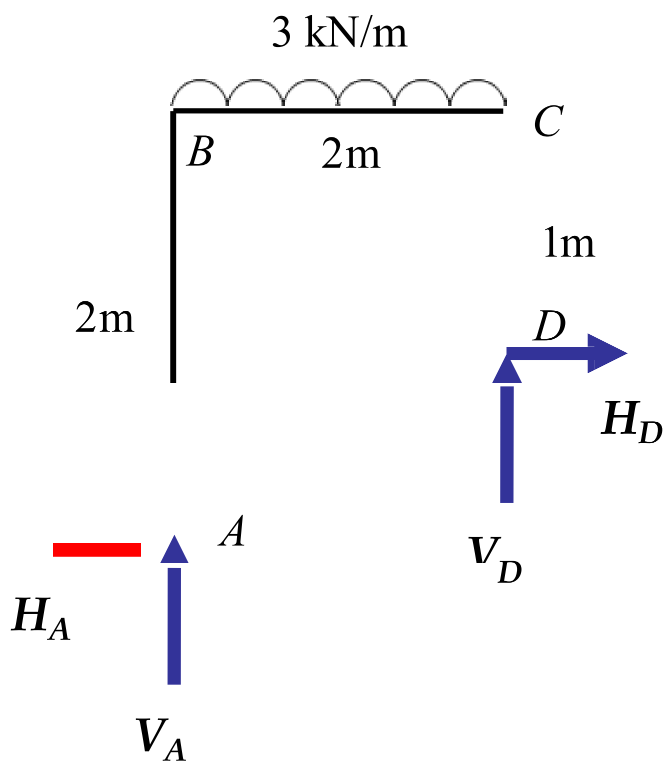
For the entire span ***AC***

$$\therefore \int \left(\frac{M}{EI}\right)\frac{\partial M}{\partial R_B}\, dx = 0 \Rightarrow \frac{2}{EI}\int \left(wLx - \frac{R_B x}{2} - \frac{wx^2}{2}\right)\left(-\frac{x}{2}\right) dx = 0$$

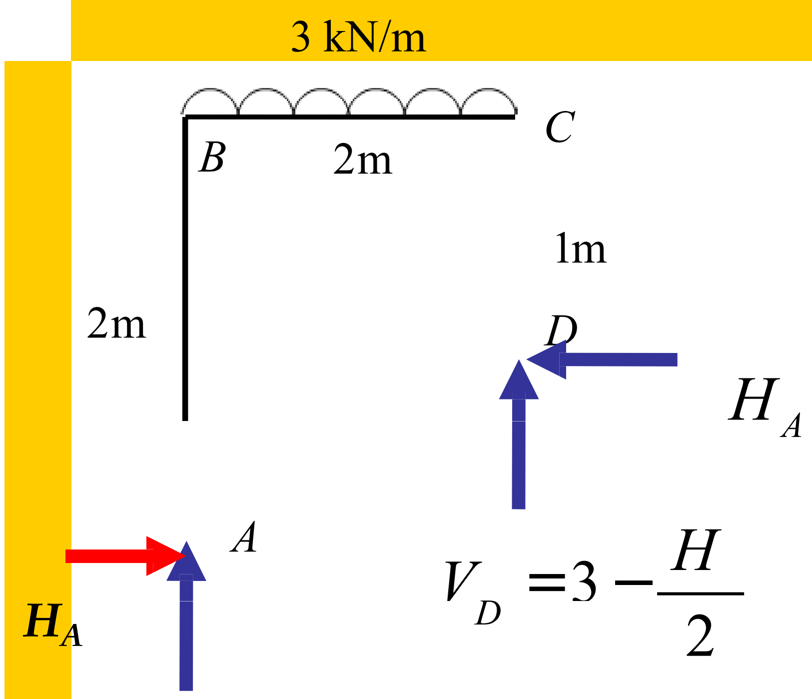
$$\Rightarrow \quad R_B = \frac{5wL}{4}$$

Example 2: Using Castigliano's Theorem, analyse the frame shown in figure.





Let H_A be the redundant.



$$V_D = 3 - \frac{H}{2}$$

$$V_A = 3 + \frac{H_A}{2}$$

From **A** to **B**,

$$M_x = -H_A x$$

$$\frac{\partial M}{\partial H_A} = -x$$

From **B** to **C**,

$$M_x = \left(3 + \frac{H_A}{2}\right)x - 2H_A - \frac{3x^2}{2}$$

$$\frac{\partial M}{\partial H_A} = \frac{x}{2} - 2$$

From **D** to **C**,

$$M_x = -H_A x$$

$$\frac{\partial M}{\partial H_A} = -x$$

$$\therefore \int \left(\frac{M}{EI}\right)\frac{\partial M}{\partial H_A}\,dx = 0$$

$$\Rightarrow \frac{1}{EI}\left\{\int_0^2 (-H_A x)(-x)\,dx + \int_0^2 \left(3x + \frac{H_A x}{2} - 2H_A - \frac{3x^2}{2}\right)\left(\frac{x}{2} - 2\right)dx + \int_0^1 (-H_A x)(-x)\,dx\right\} = 0$$

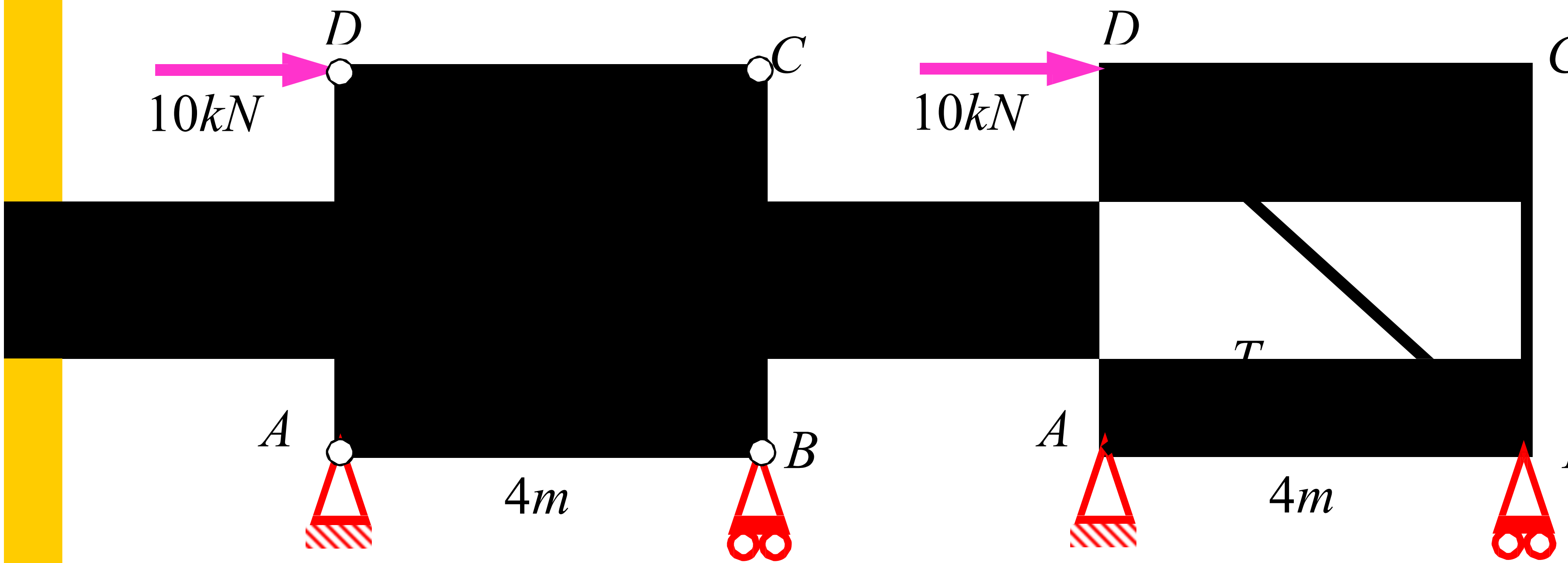
$$\left[\frac{H_A x^3}{3}\right]_0^2 + \left[\frac{x^3}{2} + \frac{H_A x^3}{12} - \frac{H_A x^2}{2} - \frac{3x^4}{16} - 3x^2 - \frac{H_A x^2}{2} + 4H_A x + x^3\right]_0^2 + \left[\frac{H_A x^3}{3}\right]_0^1 = 0$$

$$\frac{8H_A}{3} + \left[\frac{8}{2} + \frac{8H_A}{12} - \frac{4H_A}{2} - 3 - 12 - 2H_A + 8H_A + 8\right] + \frac{H_A}{3} = 0$$

$$H_A = 0.39\ kN \qquad V_A = 3 + \frac{H_A}{2} = 3.19\ kN$$

$$V_D = 3 - \frac{H_A}{2} = 2.8\ kN$$

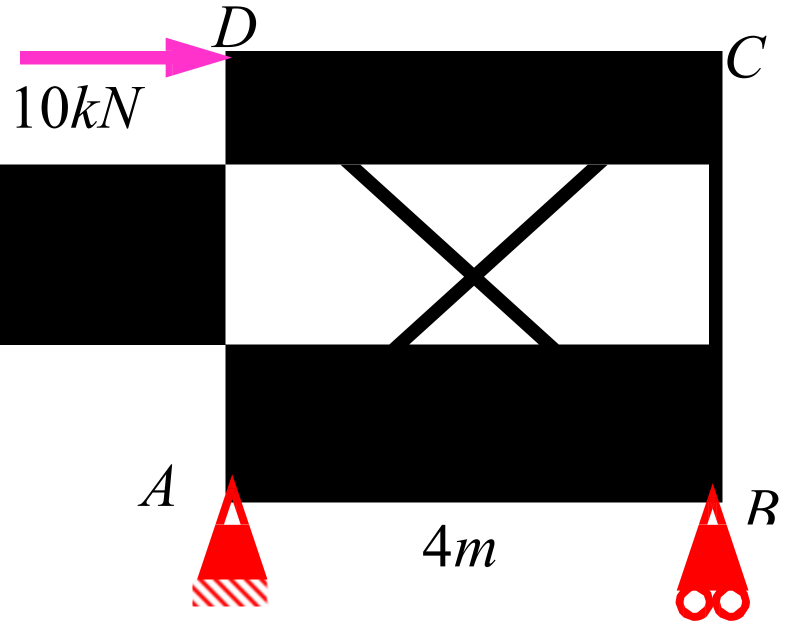
Example 4: Using Castigliano’s Theorem, analyse the truss shown in figure. ***AE*** is constant.



$$\sum P.\frac{\partial P}{\partial T}.\frac{L}{AE} = 0$$

$$T = 6.25kN$$

Example 4: Do the above problem using the method of **consistent deformation.**



$$T = -\frac{\sum \frac{P'kL}{AE}}{\sum \frac{k^2L}{AE}}$$

$$P = P' + kT$$

Note 1: If there are two internal redundants in a truss, in method of **consistent deformation,**

$$\sum \frac{P'k_1L}{AE} + T_1\sum \frac{k_1^2L}{AE} + T_2\sum \frac{k_1k_2L}{AE} = 0$$

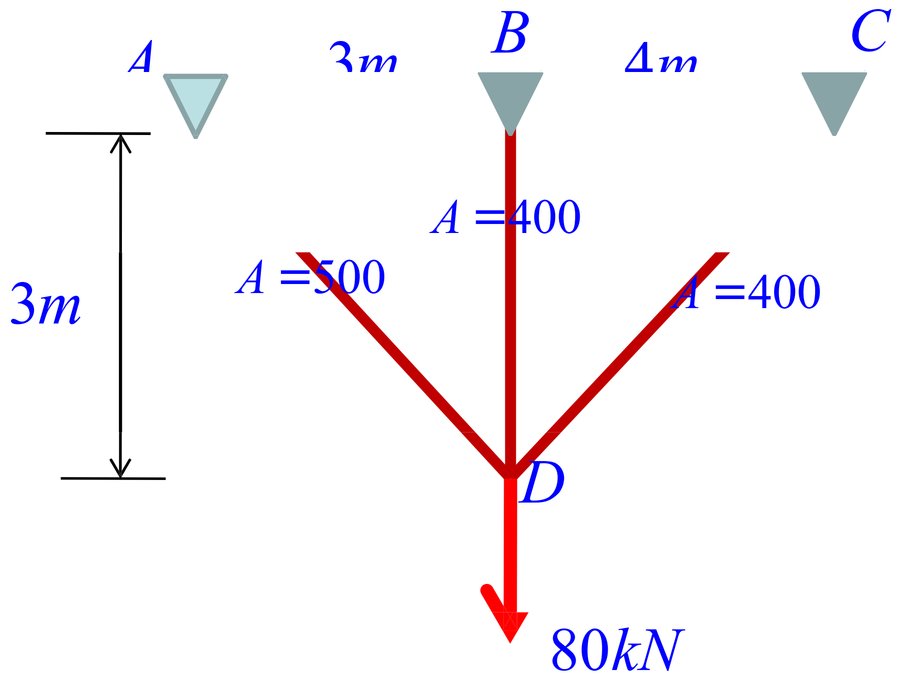
$$\sum \frac{P'k_2L}{AE} + T_1\sum \frac{k_1k_2L}{AE} + T_2\sum \frac{k_2^2L}{AE} = 0$$

Note 2: If there are both internal redundants external redundants in a truss, in method of **consistent deformation,**

$$\sum \frac{P'k_1L}{AE} + T_1\sum \frac{k_1^2L}{AE} + V_B\sum \frac{k_1k_BL}{AE} = 0$$

$$\sum \frac{P'k_BL}{AE} + T_1\sum \frac{k_1k_BL}{AE} + V_B\sum \frac{k_B^2L}{AE} = 0$$

Example 3: Using Castigliano's Theorem, analyse the pin-jointed truss shown in figure.



Internally indeterminate to degree 1.

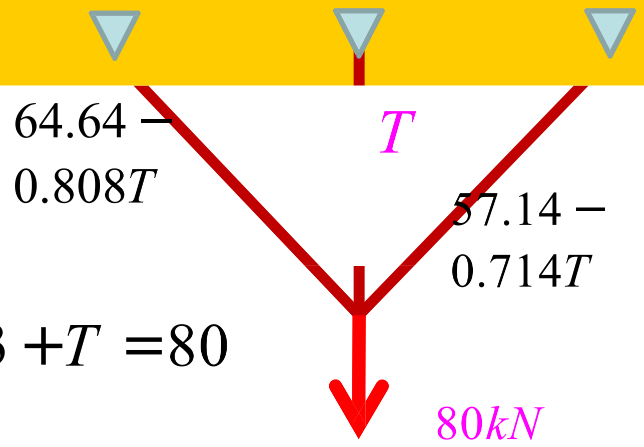
Take force in ***BD*** as redundant.

Assume force in ***BD*** is ***T***

$$\sum P.\frac{\partial P}{\partial T}.\frac{L}{AE}=0$$

$$F_{AD}\cos 45 = F_{CD}\sin 53.13$$

$$F_{AD}\cos 45 + F_{CD}\cos 53.13 + T = 80$$



$$F_{CD}\sin 53.13 + F_{CD}\cos 53.13 + T = 80$$

$$F_{CD} = 57.14 - 0.714T$$

$$F_{AD} = (57.14 - 0.714T)\frac{\sin 53.13}{\cos 45} = 64.64 - 0.808T$$

$$\sum P.\frac{\partial P}{\partial T}.\frac{L}{AE} = 0$$

$$\left(\frac{4.243(57.14 - 0.714T)}{500E}\right)\times(-0.714) + \left(\frac{3T}{400E}\right)\times 3 + \left(\frac{5(64.64 - 0.808T)}{400E}\right)\times(-0.808) = 0$$

$$\therefore T = 49.09kN$$

Summary

Statically and kinematically indeterminate structures

- Degree of static indeterminacy, Degree of kinematic indeterminacy, Force and displacement method of analysis

Force method of analysis

- Method of **consistent deformation**-Analysis of fixed and continuous beams

- Clapeyron's **theorem of three moments**-Analysis of fixed and continuous beams

- **Principle of minimum strain energy**-Castigliano's second theorem-Analysis of beams, plane trusses and plane frames.