Structural Analysis - III

Stiffness Method

Module II

Stiffness method

- Development of stiffness matrices by physical approach – stiffness matrices for truss,beam and frame elements – displacement transformation matrix – development of total stiffness matrix - analysis of simple structures – plane truss beam and plane frame- nodal loads and element loads – lack of fit and temperature effects.

FUNDAMENTALS OF STIFFNESS METHOD

Introduction

- Displacement components are the primary unknowns
- Number of unknowns is equal to the kinematic indeterminacy
- Redundants are the joint displacements, which are automatically specified
- Choice of redundants is unique
- Conducive to computer programming

- Stiffness method (displacements of the joints are the primary unknowns): kinematic indeterminacy

  - kinematic indeterminacy

  - joints: a) where members meet, b) supports, c) free ends
  - joints undergo translations or rotations

  - in some cases joint displacements will be known, from the restraint conditions

  - the unknown joint displacements are the kinematically indeterminate quantities

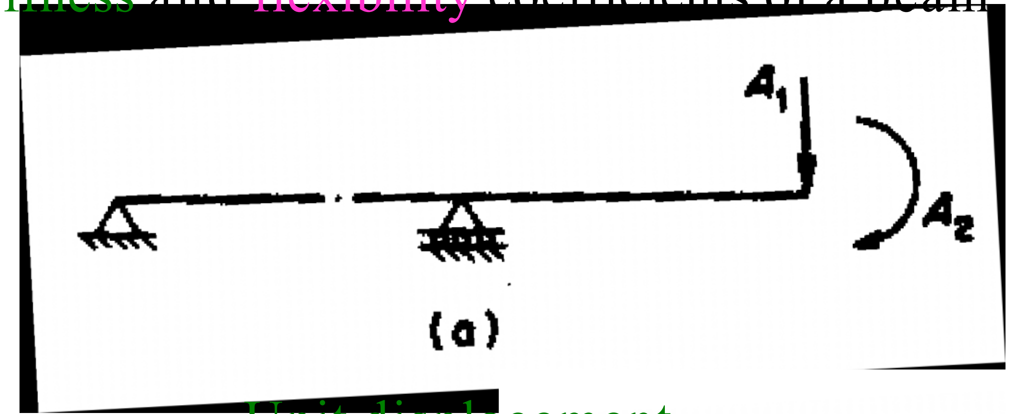
    - degree of kinematic indeterminacy: number of degrees of freedom

- in a truss, the joint rotation is not regarded as a degree of freedom. joint rotations do not have any physical significance as they have no effects in the members of the truss

- in a frame, degrees of freedom due to axial deformations can be neglected

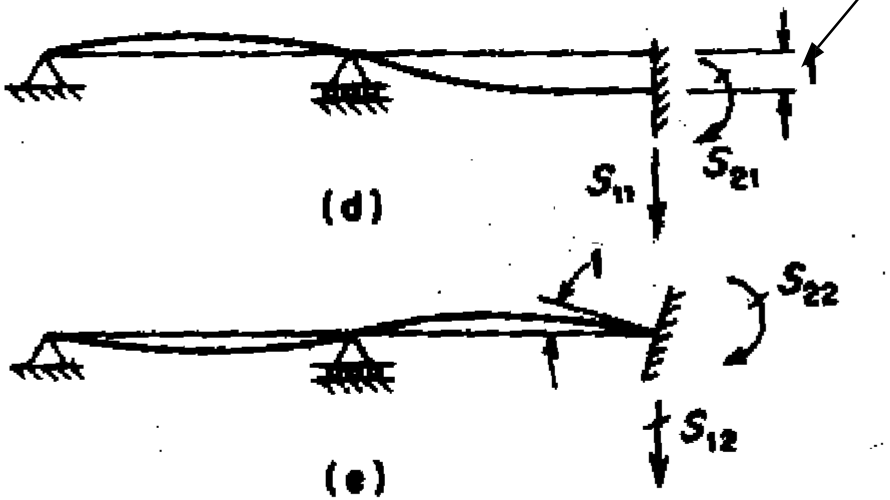
Stiffness coefficients

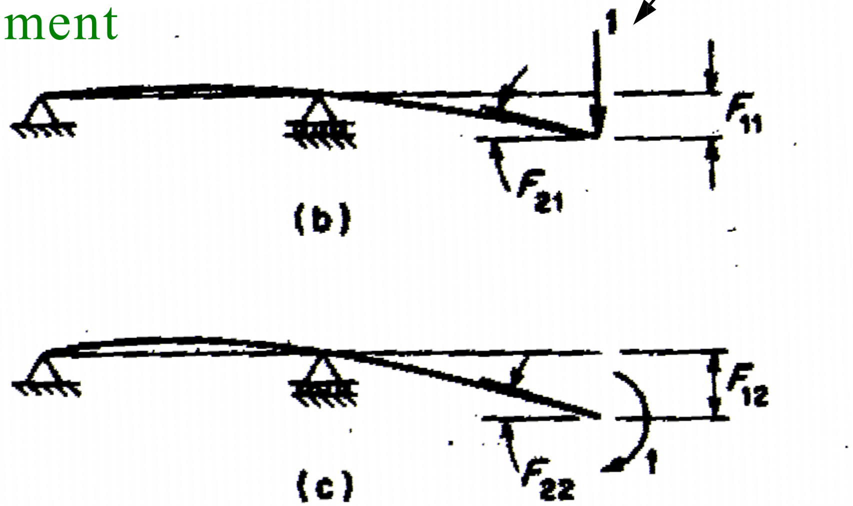
- Example 1: Stiffness and flexibility coefficients of a beam



Unit action

Unit displacement

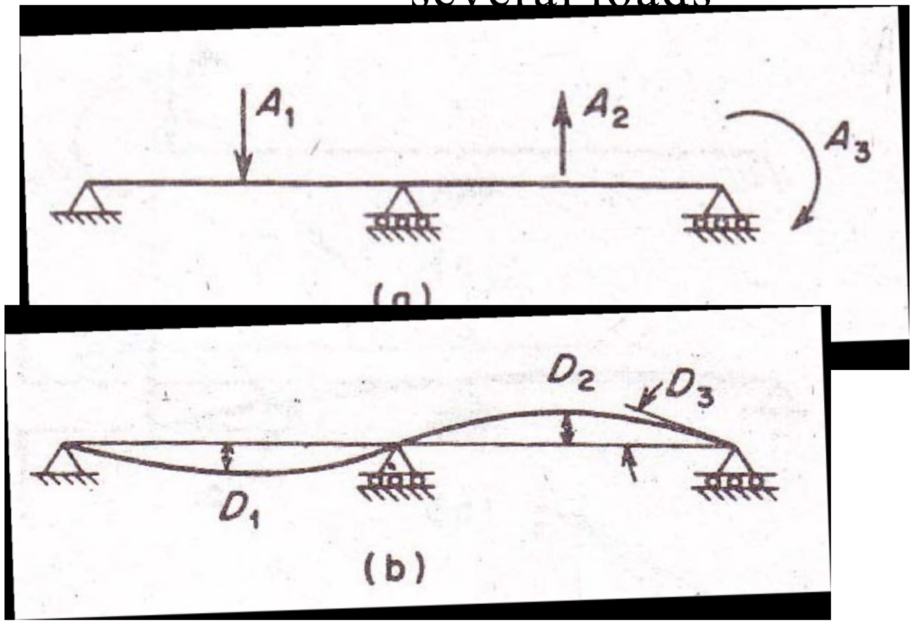


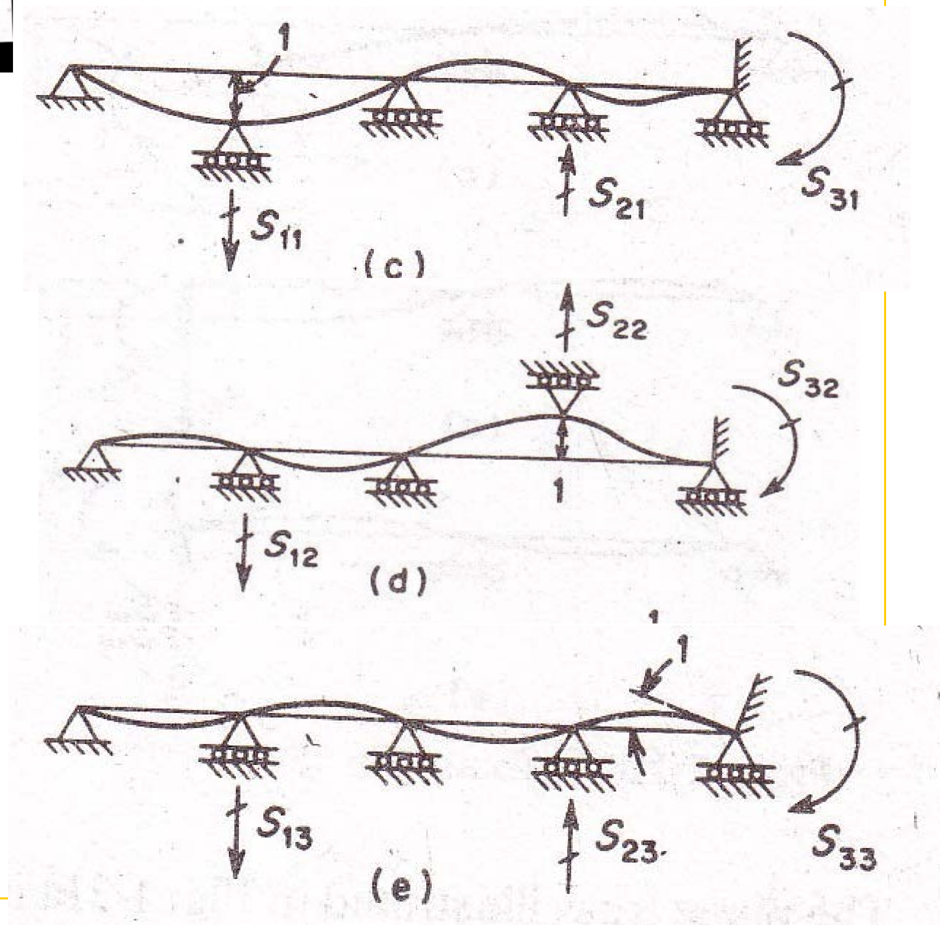


$S_{11}, S_{21}, S_{12}, S_{22}$ Forces due to unit dispts – stiffness coefficients

$F_{11}, F_{21}, F_{12}, F_{22}$ Dispts due to unit forces – flexibility coefficients

•Example 2: Action-displacement equations for a beam subjected to several loads





$$A_1 = A_{11} + A_{12} + A_{13}$$

$$A_1 = S_{11}D_1 + S_{12}D_2 + S_{13}D_3$$

$$A_2 = S_{21}D_1 + S_{22}D_2 + S_{23}D_3$$

$$A_3 = S_{31}D_1 + S_{32}D_2 + S_{33}D_3$$

Stiffness matrix

$$A_1 = S_{11}D_1 + S_{12}D_2 + S_{13}D_3 + \ldots + S_{1n}D_n$$

$$A_2 = S_{21}D_1 + S_{22}D_2 + S_{23}D_3 + \ldots + S_{2n}D_n$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_n = S_{n1}D_1 + S_{n2}D_2 + S_{n3}D_3 + \ldots + S_{nn}D_n$$

○In matrix form,

$$\begin{Bmatrix} A_1 \\ A_2 \\ \cdots \\ A_n \end{Bmatrix}_{n\times 1} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}_{n\times n} \begin{Bmatrix} D_1 \\ D_2 \\ \cdots \\ D_n \end{Bmatrix}_{n\times 1}$$

$$\boldsymbol{A} = \boldsymbol{SD} \qquad \{A\} = [S]\{D\}$$

- Action matrix, Stiffness matrix, Displacement matrix

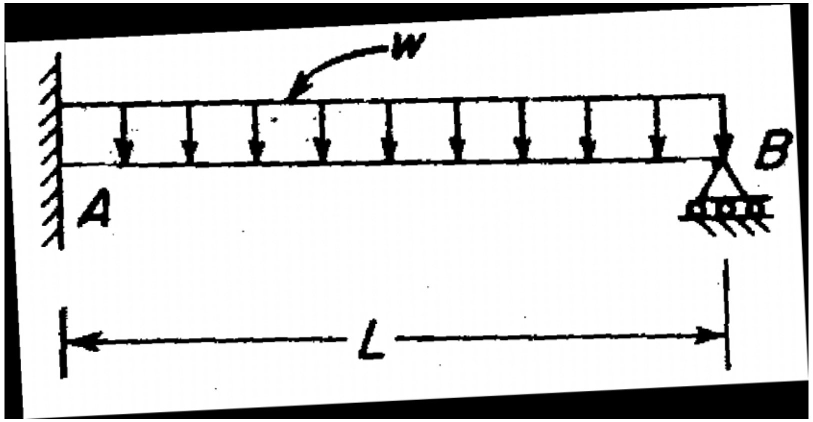
- Stiffness coefficient S_{ij}

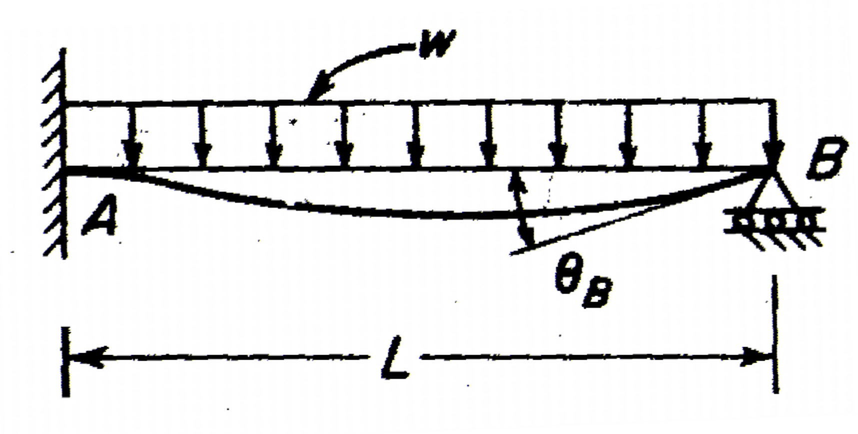
$$\{A\} = [F]^{-1}\{D\} \Rightarrow [S] = [F]^{-1} \qquad [F] = [S]^{-1}$$

Stiffness method

(Direct approach: Explanation using principle of superposition)

Example: **Propped cantilever** (Kinematically indeterminate to first degree)



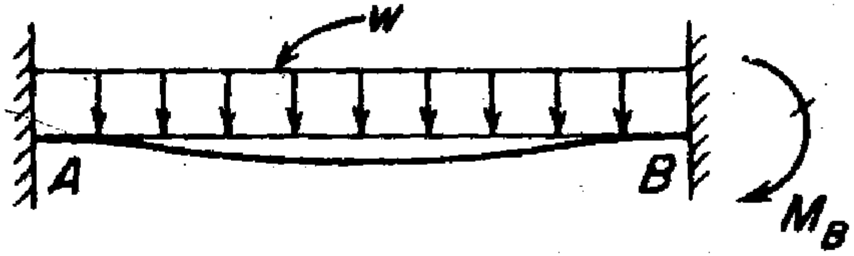


- Required to get θ_B

•degrees of freedom:
one

•Kinematically determinate structure is obtained by restraining all displacements (all displacement components made zero - restrained structure)

Restraint at B causes a reaction of M_B as shown.



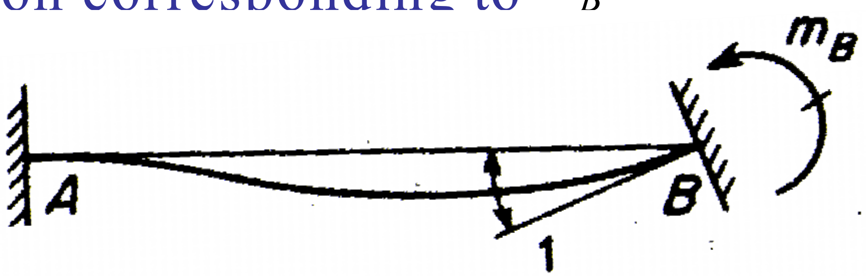
$$M_B = -\frac{wL^2}{12}$$

(Note the sign convention: anticlockwise positive)

The actual rotation at B is θ_B

Hence it is required to induce a rotation of θ_B

•Apply unit rotation corresponding to θ_B



Let the moment required for this unit rotation be m_B

$$m_B = \frac{4EI}{L}$$ anticlockwise

- Moment required to induce a rotation of θ_B is $m_B\theta_B$

$M_B + m_B\theta_B = 0$ (Joint equilibrium equation)

$$-\frac{wL^2}{12} + \frac{4EI}{L}\theta_B = 0 \qquad \therefore \theta_B = -\frac{M_B}{m_B} = \frac{wL^3}{48EI}$$

m_B (Moment required for unit rotation) is the stiffness coefficient here.

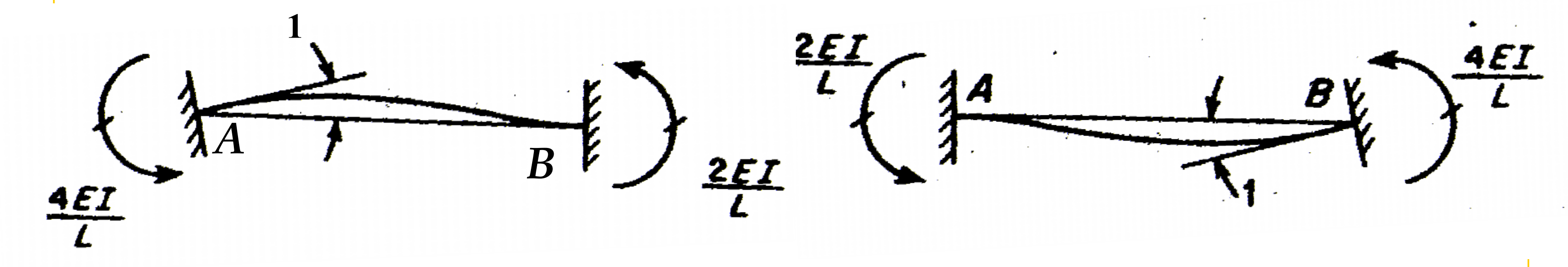
Stiffnesses of prismatic members

Stiffness coefficients of a structure are calculated from the contributions of individual members

Hence it is worthwhile to construct member stiffness matrices

$$\left[S_{Mi}\right] = \left[F_{Mi}\right]^{-1}$$

Memberstiffness matrix for prismatic *beam member* with rotations at the ends as degrees of freedom



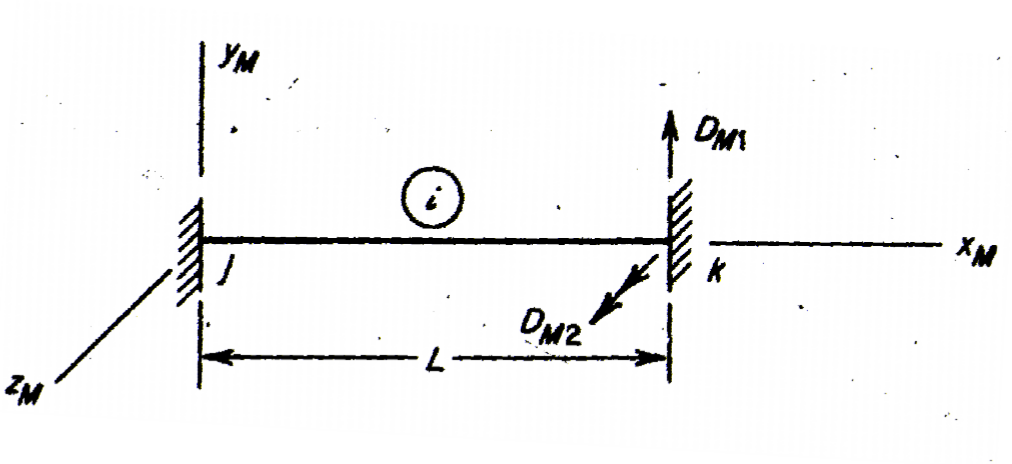
$$[S_{Mi}] = \frac{2EI}{L}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

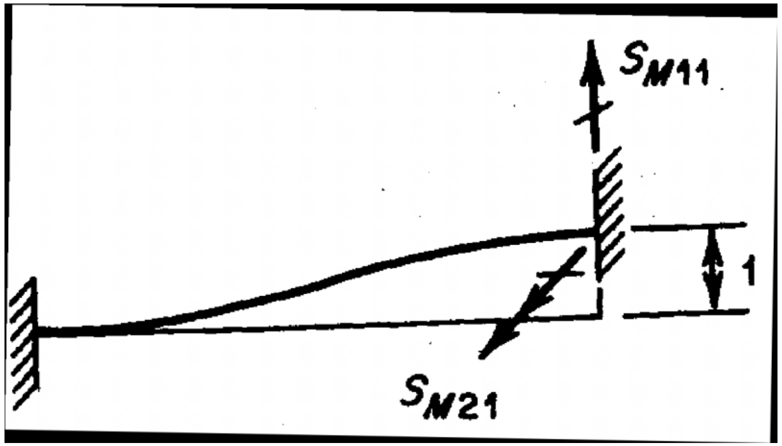
Verification:

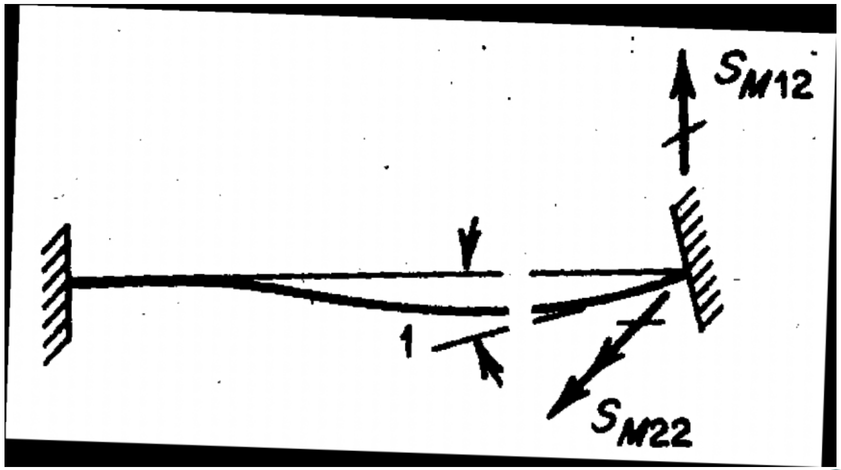
$$[S_{Mi}] = [F_{Mi}]^{-1} = \begin{bmatrix} \frac{L}{3EI} & \frac{-L}{6EI} \\ \frac{-L}{6EI} & \frac{L}{3EI} \end{bmatrix}^{-1} = \left[\frac{L}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right]^{-1} = \frac{6EI}{L(3)}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{2EI}{L}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Member stiffness matrix for prismatic *beam member* with deflection and rotation at one end as degrees of freedom

$$[S_{Mi}] = \begin{bmatrix} S_{M11} & S_{M12} \\ S_{M21} & S_{M22} \end{bmatrix} = \begin{bmatrix} \dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} \\ -\dfrac{6EI}{L^2} & \dfrac{4EI}{L} \end{bmatrix}$$





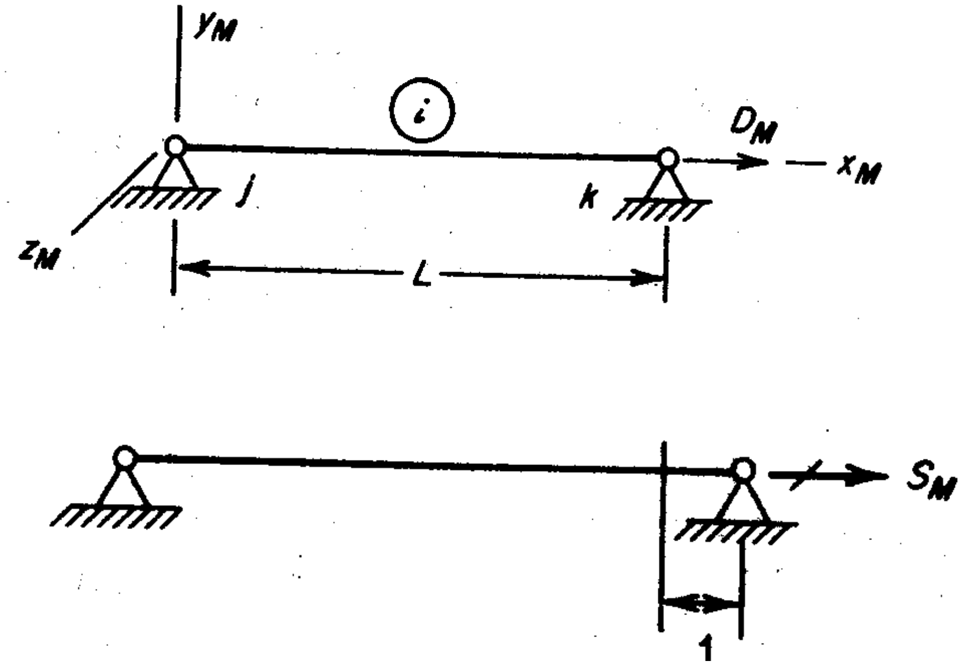


Verification:

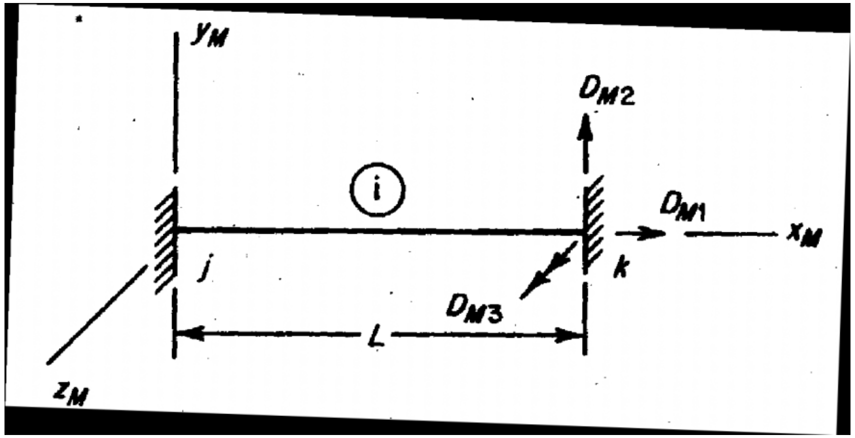
$$[S_{Mi}] = [F_{Mi}]^{-1} = \begin{bmatrix} \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix}^{-1} = \left[\frac{L}{6EI} \begin{bmatrix} 2L^2 & 3L \\ 3L & 6 \end{bmatrix} \right]^{-1} = \frac{6EI}{L(3L^2)} \begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

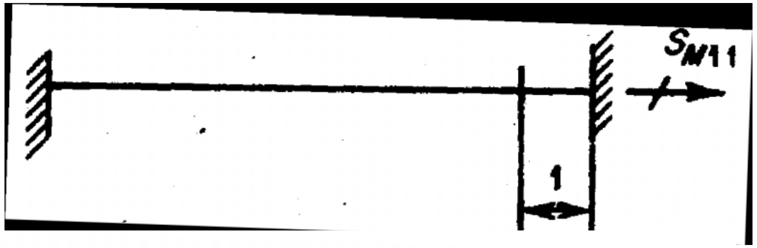
- Truss member

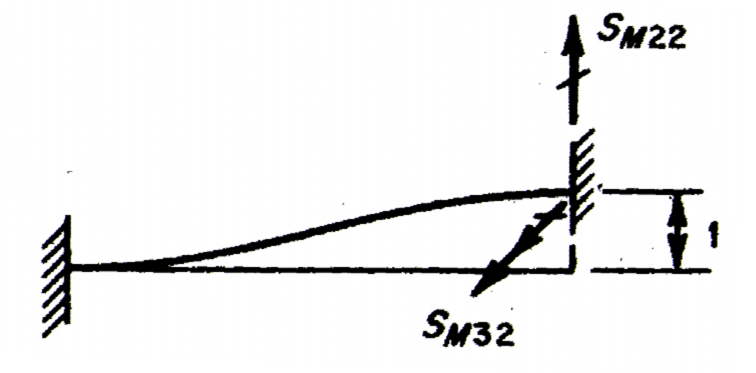
$$[S_{Mi}] = \frac{EA}{L}$$

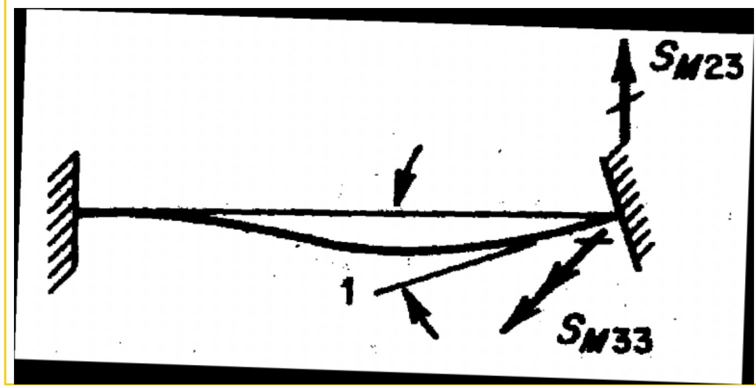


•Plane frame member



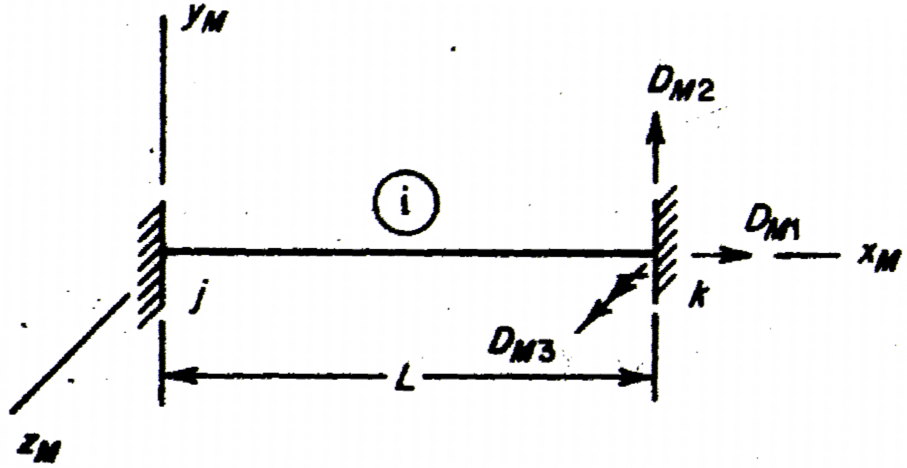


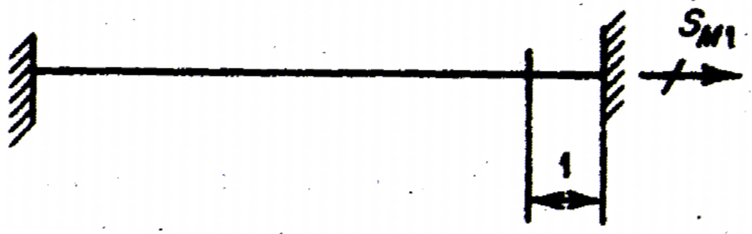


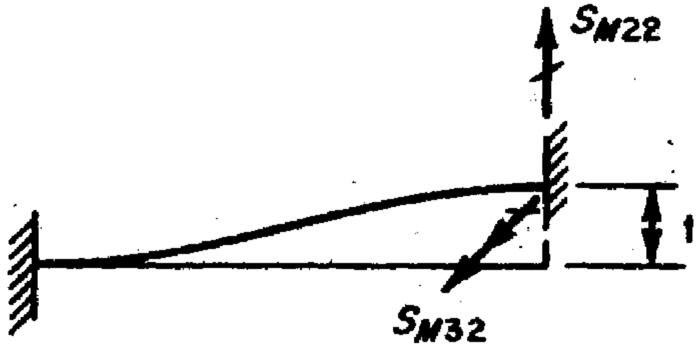


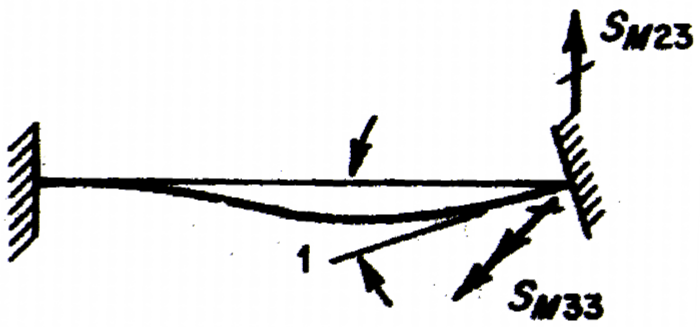
$$[S_{Mi}] = \begin{bmatrix} S_{M11} & S_{M12} & S_{M13} \\ S_{M21} & S_{M22} & S_{M23} \\ S_{M31} & S_{M32} & S_{M33} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

• Grid member



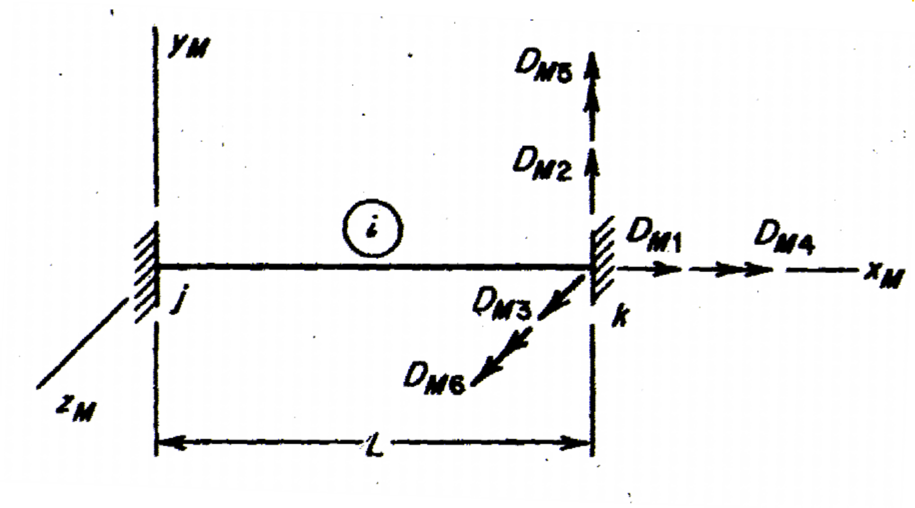






$$[S_{Mi}] = \begin{bmatrix} \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\ 0 & \frac{GJ}{L} & 0 \\ -\frac{6EI}{L^2} & 0 & \frac{4EI}{L} \end{bmatrix}$$

• Space frame member



$$[S_{Mi}] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_Z}{L^3} & 0 & 0 & 0 & -\frac{6EI_Z}{L^2} \\ 0 & 0 & \frac{12EI_Y}{L^3} & 0 & \frac{6EI_Y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_Y}{L^2} & 0 & \frac{4EI_Y}{L} & 0 \\ 0 & -\frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{4EI_Z}{L} \end{bmatrix}$$

Formalization of the Stiffness method

(Explanation using principle of complimentary virtual work)

$$\{A_{Mi}\} = [S_{Mi}]\{D_{Mi}\}$$

Here $\{D_{Mi}\}$ contains *relative* displacements of the *k* end with respect to *j* end of the *i*-th member

If there are *m* members in the structure,

$$\begin{Bmatrix} \{A_{M1}\} \\ \{A_{M2}\} \\ \{A_{M3}\} \\ \vdots \\ \{A_{Mi}\} \\ \vdots \\ \{A_{Mm}\} \end{Bmatrix} = \begin{bmatrix} [S_{M1}] & [0] & [0] & \cdots & [0] & \cdots & [0] \\ [0] & [S_{M2}] & [0] & \cdots & [0] & \cdots & [0] \\ [0] & [0] & [S_{M3}] & \cdots & [0] & \cdots & [0] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ [0] & [0] & [0] & \cdots & [S_{Mi}] & \cdots & [0] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ [0] & [0] & [0] & \cdots & [0] & \cdots & [S_{Mm}] \end{bmatrix} \begin{Bmatrix} \{D_{M1}\} \\ \{D_{M2}\} \\ \{D_{M3}\} \\ \vdots \\ \{D_{Mi}\} \\ \vdots \\ \{D_{Mm}\} \end{Bmatrix}$$

$$\boldsymbol{A}_M = \boldsymbol{S}_M \boldsymbol{D}_M$$

$$\{A_M\} = [S_M]\{D_M\}$$

$[S_M]$ is the *unassembled* stiffness matrix of the entire structure

- Relative end-displacements in $\{D_M\}$ will be related to a vector of joint displacements for the whole structure, $\{D_J\}$

$$\{D_M\} = [C_{MJ}]\{D_J\}$$

$[C_{MJ}]$ displacement transformation matrix (compatibility matrix)

$\{D_J\}$ consists of: free (unknown) joint displacements $\{D_F\}$ and restraint displacements $\{D_R\}$

- If there are no support displacements specified, $\{D_R\}$ will be a null matrix

- Hence, $\{D_M\} = [C_{MJ}]\{D_J\} = \begin{bmatrix} [C_{MF}] & [C_{MR}] \end{bmatrix} \begin{Bmatrix} \{D_F\} \\ \{D_R\} \end{Bmatrix}$

- Elements in displacement transformation matrix (compatibility matrix) $[C_{MJ}]$ are found from compatibility conditions.

$[C_{MF}]$ and $[C_{MR}]$ relate $\{D_M\}$ to $\{D_F\}$ and $\{D_R\}$ respectively

- Each column in the submatrix $[C_{MF}]$ consists of member displacements caused by a unit value of an unknown displacement applied to the *restrained structure*.

- Each column in the submatrix $[C_{MR}]$ consists of member displacements caused by a unit value of a support displacement applied to the *restrained structure*.

- Suppose an arbitrary set of virtual displacements $\{\delta D_M\}$ is applied on the structure.

$$\{\delta D_M\} = [C_{MJ}]\{\delta D_J\} = \begin{bmatrix} [C_{MF}] & [C_{MR}] \end{bmatrix} \begin{Bmatrix} \{\delta D_F\} \\ \{\delta D_R\} \end{Bmatrix}$$

- External virtual work produced by the virtual displacements $\{\delta D_J\}$ and real loads $\{A_J\}$ is

$$\delta W = \{A_J\}^{\mathrm{T}}\{\delta D_J\} = \begin{bmatrix} \{A_F\}^{\mathrm{T}} & \{\delta A_R\}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \delta D_F \\ \delta D_R \end{Bmatrix}$$

- Internal virtual work produced by the virtual (relative) end displacements $\{\delta D_M\}$ and actual member end actions $\{A_M\}$ is

$$\delta U = \{A_M\}^{\mathrm{T}}\{\delta D_M\}$$

- Equating the above two (principle of virtual work),

$$\{A_J\}^{\mathrm{T}}\{\delta D_J\} = \{A_M\}^{\mathrm{T}}\{\delta D_M\}$$

But $\{D_M\} = [C_{MJ}]\{D_J\}$ and $\{A_M\} = [S_M]\{D_M\}$

Also, $\{\delta D_M\} = [C_{MJ}]\{\delta D_J\}$

Hence, $\{A_J\}^{\mathrm{T}}\{\delta D_J\} = \{D_J\}^{\mathrm{T}}[C_{MJ}]^{\mathrm{T}}[S_M]^{\mathrm{T}}[C_{MJ}]\{\delta D_J\}$

$$\{A_J\} = [S_J]\{D_J\}$$

Where, $[S_J] = [C_{MJ}]^T [S_M][C_{MJ}]$, the *assembled stiffness matrix* for the entire structure.

- It is useful to partition $[S_J]$ into submatrices pertaining to free (unknown) joint displacements $\{D_F\}$ and restraint displacements $\{D_R\}$

$$\{A_J\} = [S_J]\{D_J\} \Rightarrow \begin{Bmatrix} \{A_F\} \\ \{A_R\} \end{Bmatrix} = \begin{bmatrix} [S_{FF}] & [S_{FR}] \\ [S_{RF}] & [S_{RR}] \end{bmatrix} \begin{Bmatrix} \{D_F\} \\ \{D_R\} \end{Bmatrix}$$

Where, $[S_{FF}] = [C_{MF}]^T [S_M][C_{MF}]$ $\quad$ $[S_{FR}] = [C_{MF}]^T [S_M][C_{MR}]$

$[S_{RF}] = [C_{MR}]^T [S_M][C_{MF}]$ $\quad$ $[S_{RR}] = [C_{MR}]^T [S_M][C_{MR}]$

$$\{A_F\} = [S_{FF}]\{D_F\} + [S_{FR}]\{D_R\}$$

$$\{A_R\} = [S_{RF}]\{D_F\} + [S_{RR}]\{D_R\}$$

$$\Rightarrow \boxed{\{D_F\} = [S_{FF}]^{-1}\left(\{A_F\} - [S_{FR}]\{D_R\}\right)}$$ Joint displacements

- Support reactions

If actual or equivalent joint loads are applied directly to the supports,

$$\{A_R\} = -\{A_{RC}\} + [S_{RF}]\{D_F\} + [S_{RR}]\{D_R\}$$

$\{A_{RC}\}$ represents combined joint loads (actual and equivalent) applied directly to the supports.

•Member end actions are obtained adding member end actions calculated as above and initial fixed-end actions

$$\text{i.e., } \{A_M\} = \{A_{ML}\} + [S_M][C_{MJ}]\{D_J\}$$

$$\{A_M\} = \{A_{ML}\} + [S_M]\left([C_{MF}]\{D_F\} + [C_{MR}]\{D_R\}\right)$$

where $\{A_{ML}\}$ represents fixed end actions

Important formulae:

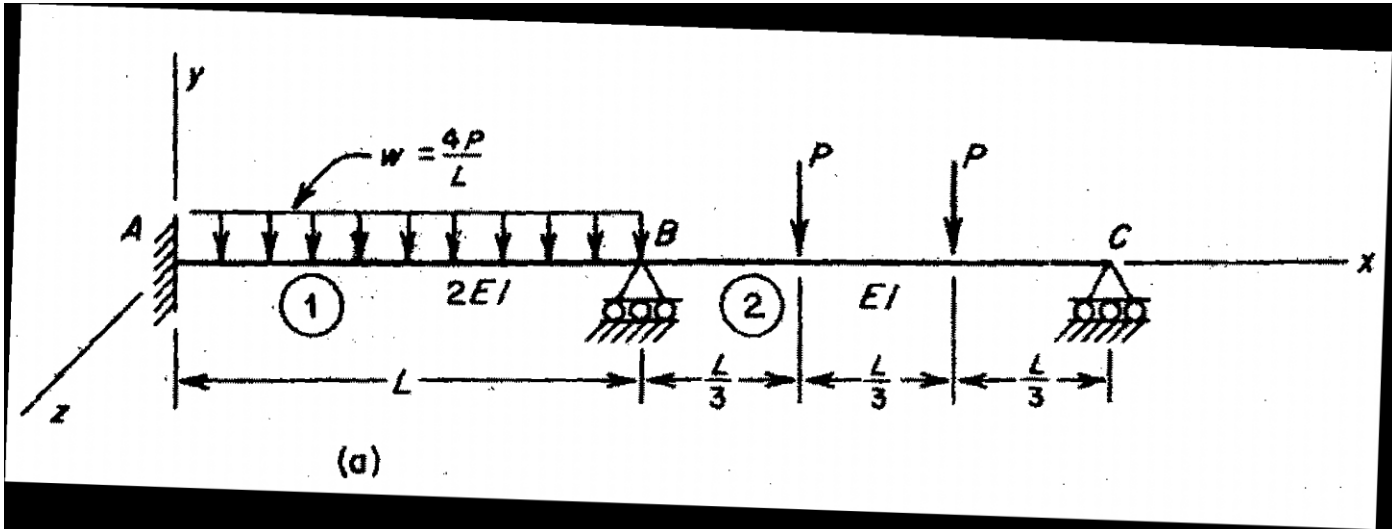
Joint displacements: $$\{D_F\} = [S_{FF}]^{-1}\left(\{A_F\} - [S_{FR}]\{D_R\}\right)$$

Support reactions: $$\{A_R\} = -\{A_{RC}\} + [S_{RF}]\{D_F\} + [S_{RR}]\{D_R\}$$

Member end actions:

$$\{A_M\} = \{A_{ML}\} + [S_M]\left([C_{MF}]\{D_F\} + [C_{MR}]\{D_R\}\right)$$

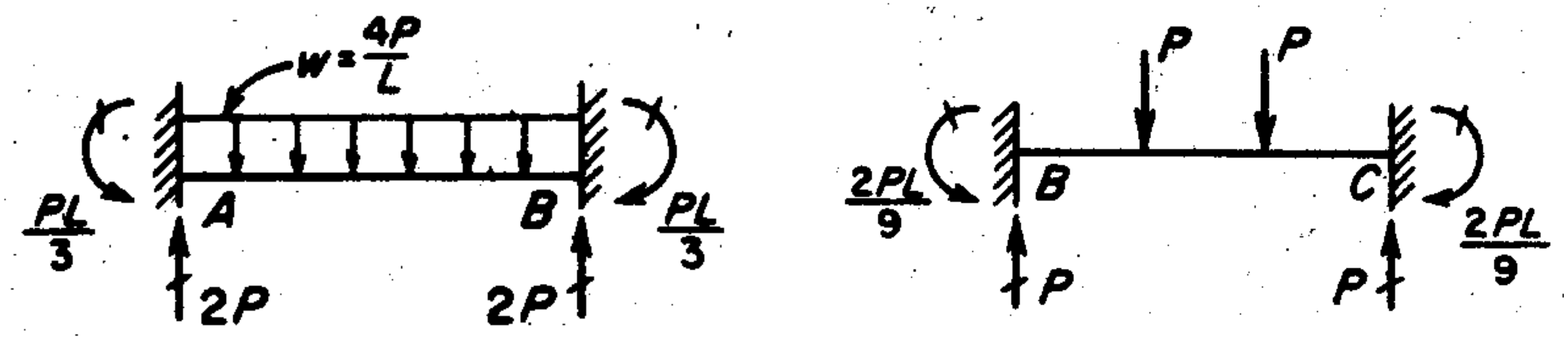
- Problem 1



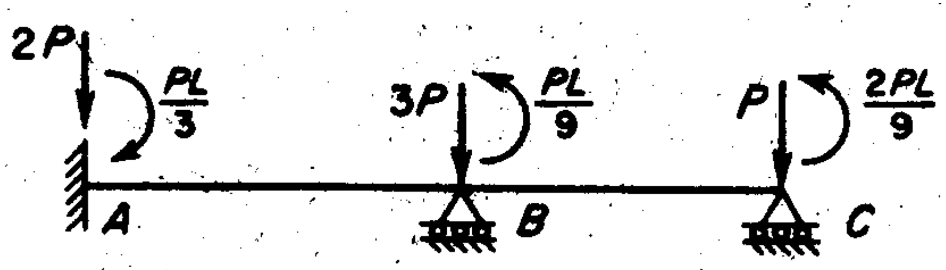
Kinematic indeterminacy = 2

Member stiffness matrix of beam member $[S_{Mi}] = \frac{2EI}{L}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Unassembled stiffness matrix $[S_M] = \frac{2EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

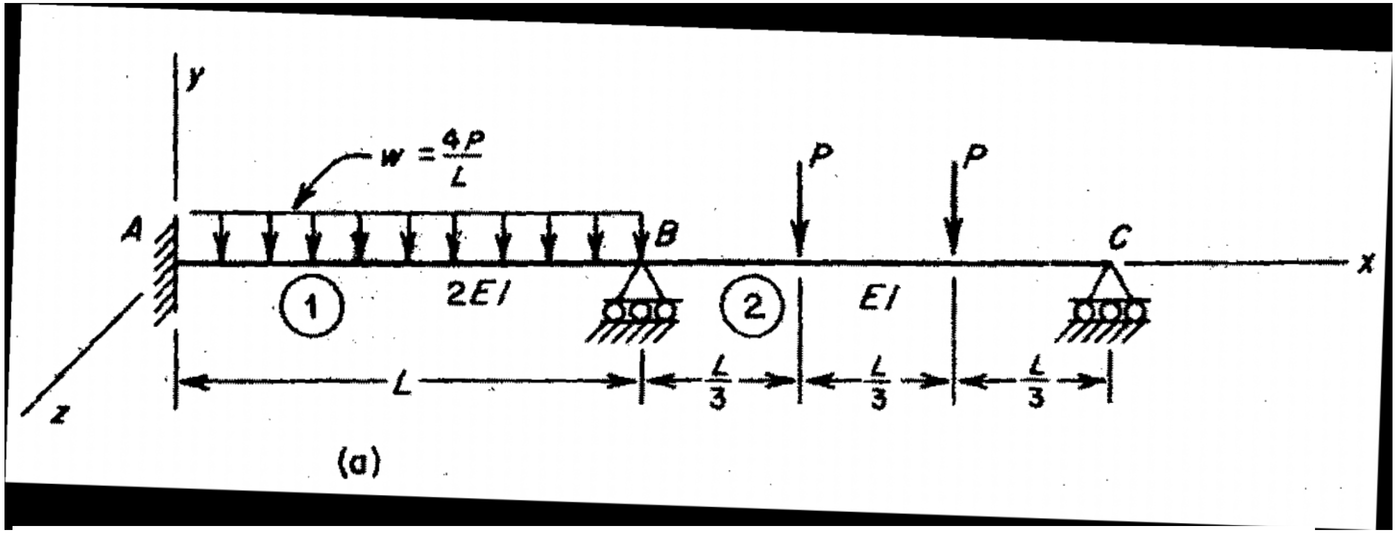


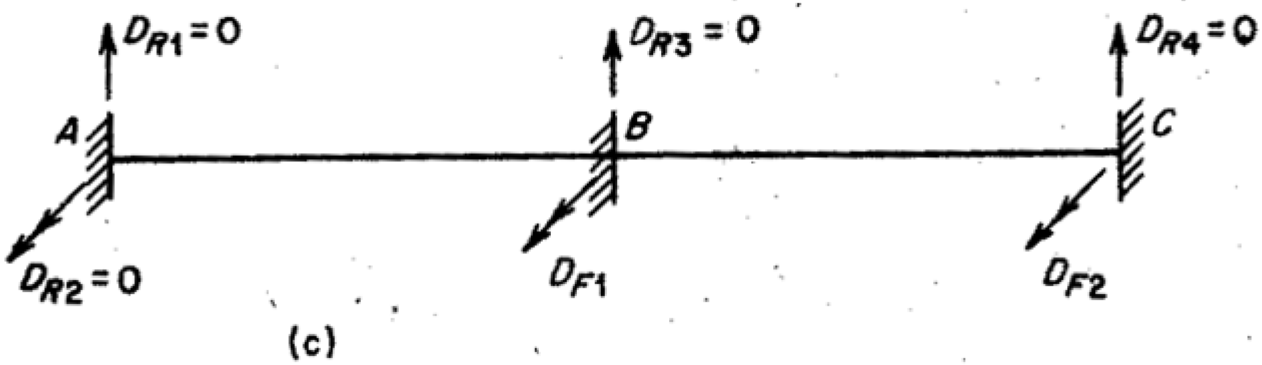
Fixed end actions



(e)

Equivalent joint loads

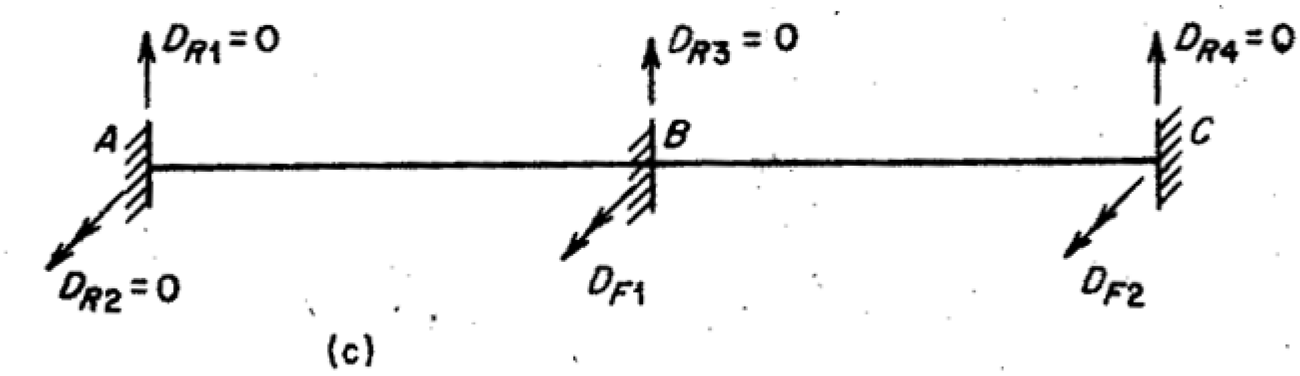




Joint displacements

Free (unknown) joint displacements $\{D_F\}$

Restraint displacements $\{D_R\}$

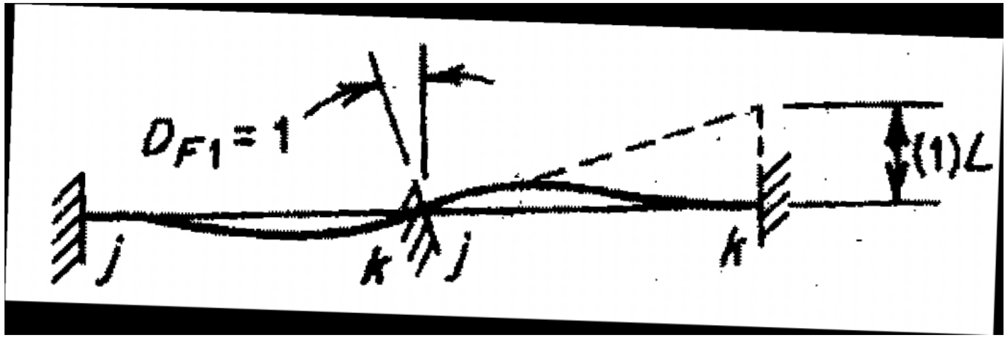


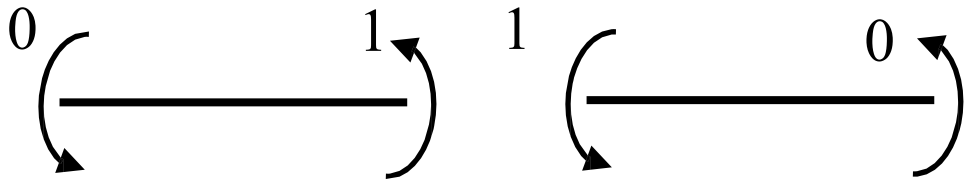
Joint displacements

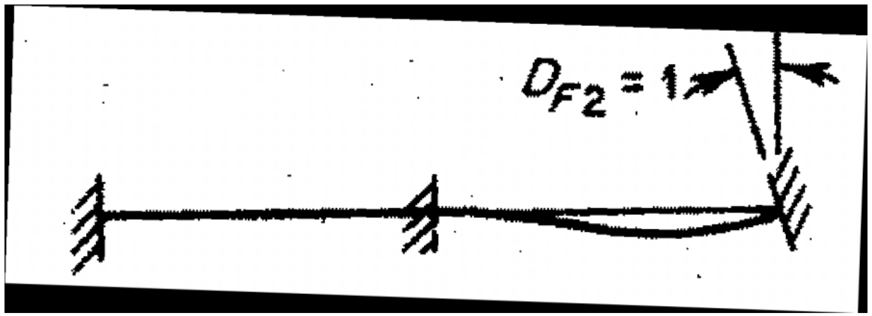
Free (unknown) joint displacements $\{D_F\}$ Restraint displacements $\{D_R\}$

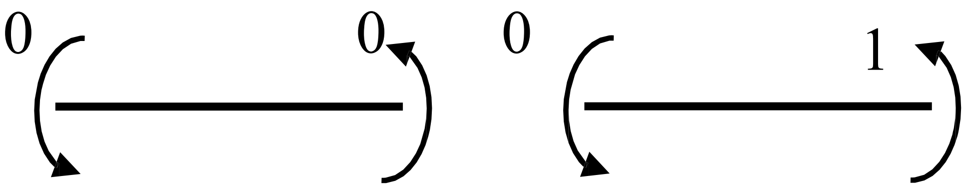
- Each column in the submatrix $[C_{MF}]$ consists of member displacements caused by a unit value of an unknown displacement applied to the *restrained structure*.

- Each column in the submatrix $[C_{MR}]$ consists of member displacements caused by a unit value of a support displacement applied to the *restrained structure*.



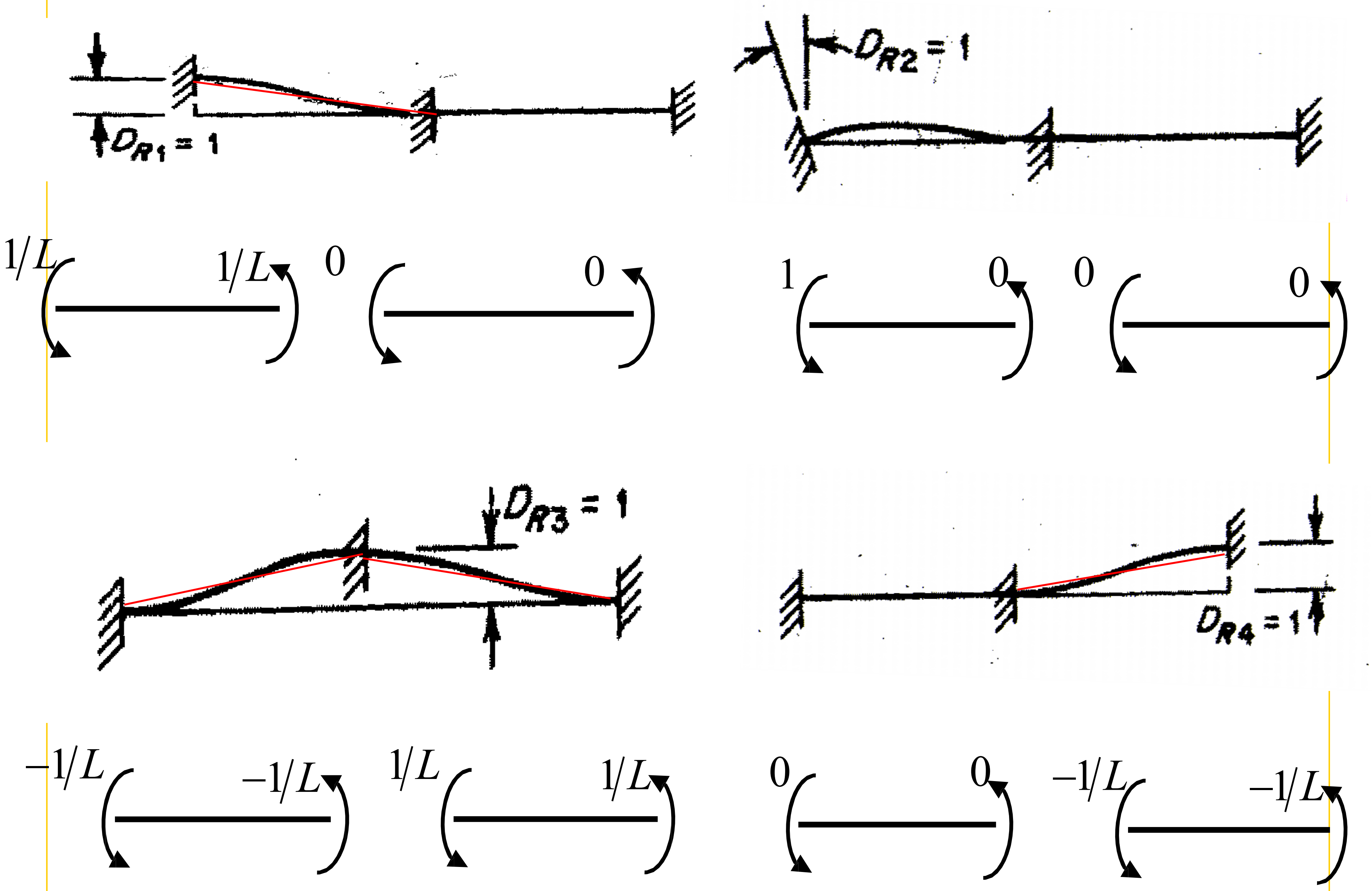






	$D_{F1} = 1$	$D_{F2} = 1$

$$[C_{MF}] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$



$D_{R1} = 1$
1/L
1/L
0
0
$D_{R2} = 1$
1
0
0
0
$D_{R3} = 1$
−1/L
−1/L
1/L
1/L
$D_{R4} = 1$
0
0
−1/L
−1/L

	$D_{R1}=1$	$D_{R2}=1$	$D_{R3}=1$	$D_{R4}=1$

$$[C_{MR}] = \begin{bmatrix} 1/L & 1 & -1/L & 0 \\ 1/L & 0 & -1/L & 0 \\ 0 & 0 & 1/L & -1/L \\ 0 & 0 & 1/L & -1/L \end{bmatrix}$$

	$D_{F1}=1$	$D_{F2}=1$	$D_{R1}=1$	$D_{R2}=1$	$D_{R3}=1$	$D_{R4}=1$

$$[C_{MJ}] = \begin{bmatrix} [C_{MF}] & [C_{MR}] \end{bmatrix} = \frac{1}{L}\left[\begin{array}{cc|cccc} 0 & 0 & 1 & L & -1 & 0 \\ L & 0 & 1 & 0 & -1 & 0 \\ L & 0 & 0 & 0 & 1 & -1 \\ 0 & L & 0 & 0 & 1 & -1 \end{array}\right]$$

Joint displacements

$$\{D_F\} = [S_{FF}]^{-1}\left(\{A_F\} - [S_{FR}]\{D_R\}\right)$$

$\{D_R\}$ is a null matrix, since there are no support displacements

$$\therefore \{D_F\} = [S_{FF}]^{-1}\{A_F\}$$

$$[S_{FF}] = [C_{MF}]^{\mathrm{T}}[S_M][C_{MF}] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{2EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{2EI}{L}\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[S_{FR}] = [C_{MF}]^{T}[S_M][C_{MR}]$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{2EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \frac{1}{L} \begin{bmatrix} 1 & L & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{2EI}{L^3} \begin{bmatrix} 6L & 2L^2 & -3L & -3L \\ 0 & 0 & 3L & -3L \end{bmatrix}$$

Free (unknown) joint displacements $\{D_F\} = \left[\frac{2EI}{L} \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \right]^{-1} \frac{PL}{9} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

$$= \frac{PL^2}{18EI} \cdot \frac{1}{11} \begin{Bmatrix} 0 \\ 11 \end{Bmatrix} = \frac{PL^2}{18EI} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

<u>Support reactions</u> $\{A_R\} = -\{A_{RC}\} + [S_{RF}]\{D_F\} + [S_{RR}]\{D_R\}$

$\{D_R\}$ is a null matrix.

$$\therefore \{A_R\} = -\{A_{RC}\} + [S_{RF}]\{D_F\}$$

$$[S_{RF}] = [C_{MR}]^{T}[S_M][C_{MF}]$$

$$= \frac{1}{L}\begin{bmatrix} 1 & L & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}^{T} \frac{2EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{2EI}{L^2}\begin{bmatrix} 6 & 0 \\ 2L & 0 \\ -3 & 3 \\ -3 & -3 \end{bmatrix}$$

$$[S_{RR}] = [C_{MR}]^{\mathrm{T}}[S_M][C_{MR}]$$

$$= \frac{1}{L}\begin{bmatrix} 1 & L & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}^{T} \frac{2EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \frac{1}{L}\begin{bmatrix} 1 & L & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{2EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 0 \\ 6L & 4L & -6L & 0 \\ -12 & -6L & 18 & -6 \\ 0 & 0 & -6 & 6 \end{bmatrix}$$

$$\therefore \{A_R\} = -\{A_{RC}\} + [S_{RF}]\{D_F\}$$

$$= -\begin{Bmatrix} -2P \\ -\dfrac{PL}{3} \\ -3P \\ -P \end{Bmatrix} + \frac{2EI}{L^2}\begin{bmatrix} 6 & 0 \\ 2L & 0 \\ -3 & 3 \\ 3 & -3 \end{bmatrix} \frac{PL^2}{18EI}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$= -\begin{Bmatrix} -2P \\ -\dfrac{PL}{3} \\ -3P \\ -P \end{Bmatrix} + \frac{PL^2}{18EI}\,\frac{2EI}{L^2}\begin{Bmatrix} 0 \\ 0 \\ 3 \\ -3 \end{Bmatrix} = \begin{Bmatrix} 2P \\ \dfrac{PL}{3} \\ 3P \\ P \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \dfrac{P}{3} \\ -\dfrac{P}{3} \end{Bmatrix} = \begin{Bmatrix} 2P \\ \dfrac{PL}{3} \\ \dfrac{10P}{3} \\ \dfrac{2P}{3} \end{Bmatrix}$$

Member end actions

$$\{A_M\} = \{A_{ML}\} + [S_M]\left([C_{MF}]\{D_F\} + [C_{MR}]\{D_R\}\right)$$

$\{D_R\}$ is a null matrix $\qquad \therefore \{A_M\} = \{A_{ML}\} + [S_M][C_{MF}]\{D_F\}$

$$\{A_M\} = \frac{PL}{9}\begin{Bmatrix}3\\-3\\2\\-2\end{Bmatrix} + \frac{2EI}{L}\begin{bmatrix}4&2&0&0\\2&4&0&0\\0&0&2&1\\0&0&1&2\end{bmatrix}\begin{bmatrix}0&0\\1&0\\1&0\\0&1\end{bmatrix}\frac{PL^2}{18EI}\begin{Bmatrix}0\\1\end{Bmatrix}$$

$$= \frac{PL}{9}\begin{Bmatrix}3\\-3\\2\\-2\end{Bmatrix} + \frac{2EI}{L}\,\frac{PL^2}{18EI}\begin{bmatrix}4&2&0&0\\2&4&0&0\\0&0&2&1\\0&0&1&2\end{bmatrix}\begin{Bmatrix}0\\0\\0\\1\end{Bmatrix} = \frac{PL}{9}\begin{Bmatrix}3\\-3\\2\\-2\end{Bmatrix} + \frac{PL}{9}\begin{Bmatrix}0\\0\\1\\2\end{Bmatrix} = \frac{PL}{3}\begin{Bmatrix}1\\-1\\1\\0\end{Bmatrix}$$

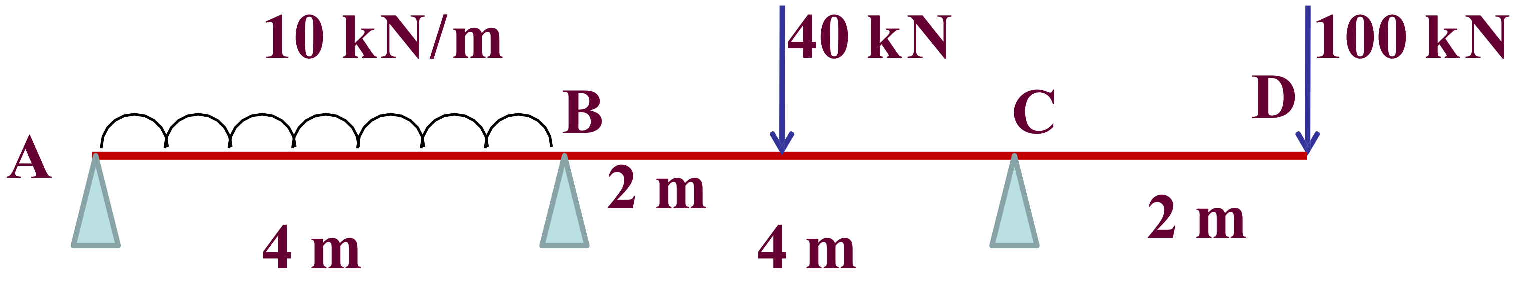
Alternatively, if the entire $[\boldsymbol{S_J}]$ matrix is assembled at a time,

$$[S_J] = [C_{MJ}]^{\mathrm{T}}[S_M][C_{MJ}] = \frac{1}{L}\begin{bmatrix} 0 & L & L & 0 \\ 0 & 0 & 0 & L \\ 1 & 1 & 0 & 0 \\ L & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \frac{2EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \frac{1}{L}\left[\begin{array}{cc|cccc} 0 & 0 & 1 & L & -1 & 0 \\ L & 0 & 1 & 0 & -1 & 0 \\ L & 0 & 0 & 0 & 1 & -1 \\ 0 & L & 0 & 0 & 1 & -1 \end{array}\right]$$

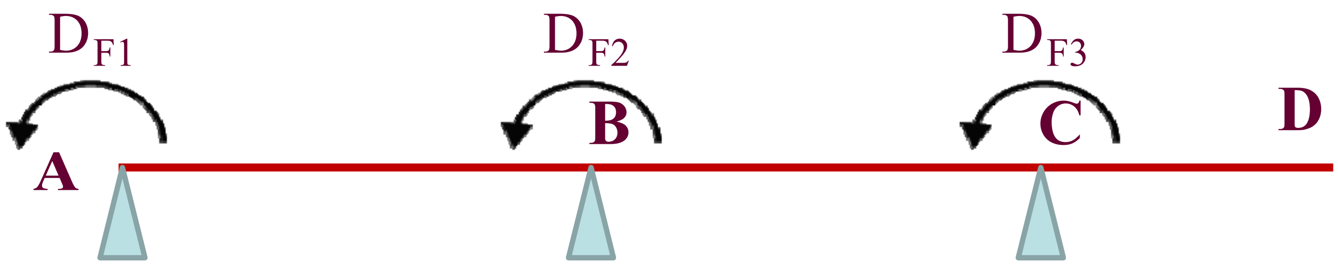
$$= \begin{bmatrix} 0 & L & L & 0 \\ 0 & 0 & 0 & L \\ 1 & 1 & 0 & 0 \\ L & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \frac{2EI}{L^3}\begin{bmatrix} 2L & 0 & 6 & 4L & -6 & 0 \\ 4L & 0 & 6 & 2L & -6 & 0 \\ 2L & L & 0 & 0 & 3 & -3 \\ L & 2L & 0 & 0 & 3 & -3 \end{bmatrix}$$

$$= \frac{2EI}{L^3}\left[\begin{array}{cc|cccc} 6L^2 & L^2 & 6L & 2L^2 & -3L & -3L \\ L^2 & 2L^2 & 0 & 0 & 3L & -3L \\ \hline 6L & 0 & 12 & 6L & -12 & 0 \\ 2L^2 & 0 & 6L & 4L & -6L & 0 \\ -3L & 3L & -12 & -6L & 18 & -6 \\ -3L & -3L & 0 & 0 & -6 & 6 \end{array}\right] = \begin{bmatrix} [S_{FF}] & [S_{FR}] \\ [S_{RF}] & [S_{RR}] \end{bmatrix}$$

• **Problem 2:**



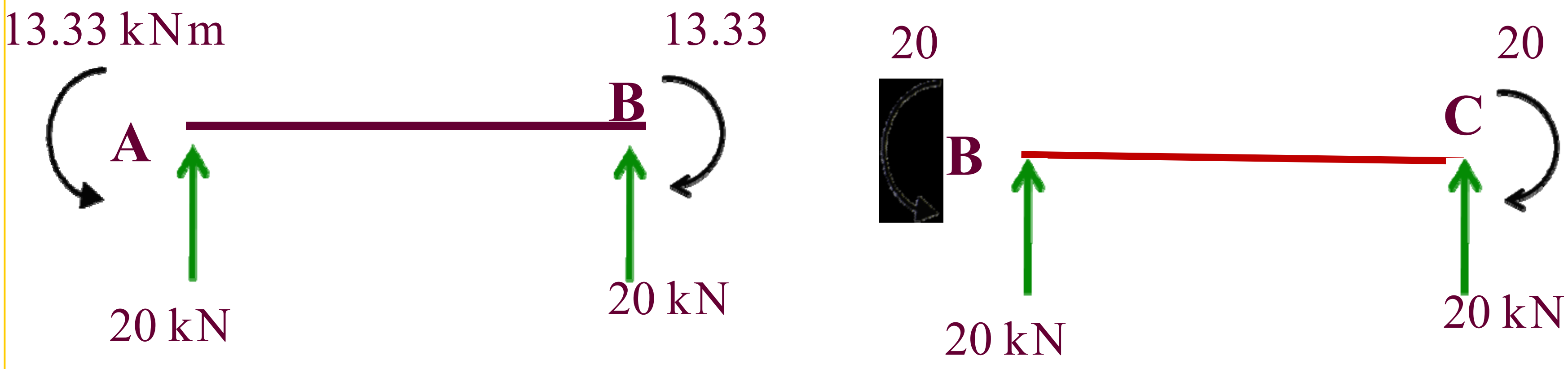
Kinematic indeterminacy = 3 (Not considering joint D in the overhanging portion)



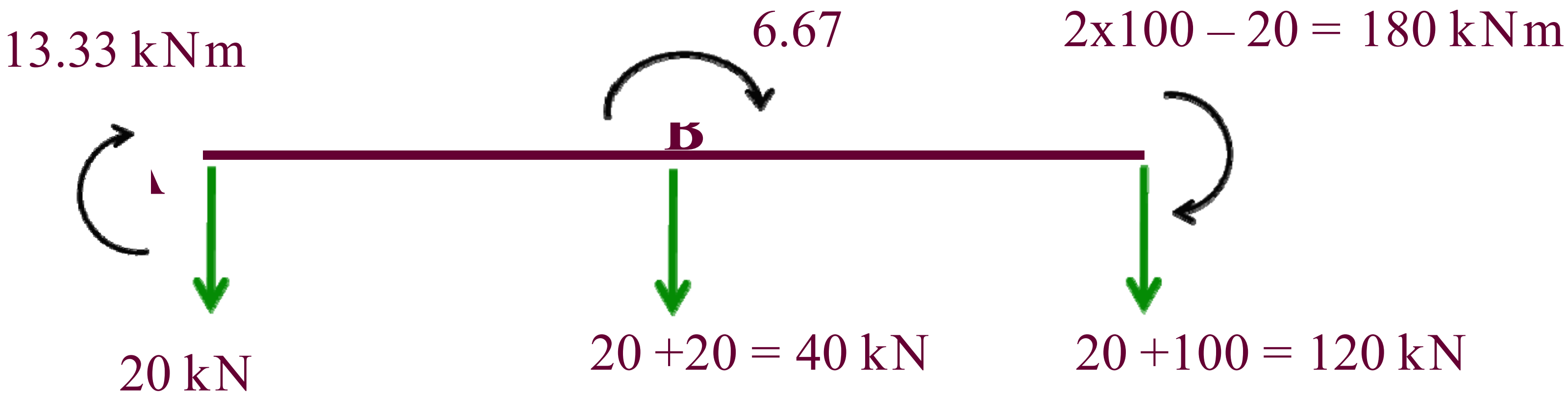
Degrees of freedom

Member stiffness matrix of beam member $$[S_{Mi}] = \frac{2EI}{4}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

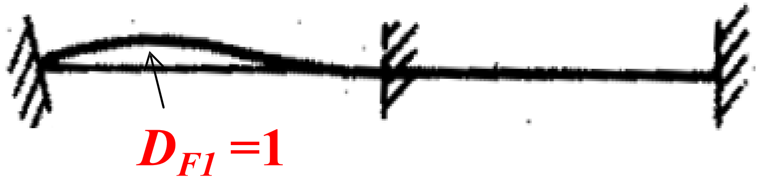
Unassembled stiffness matrix $$[S_M] = \frac{2EI}{4}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

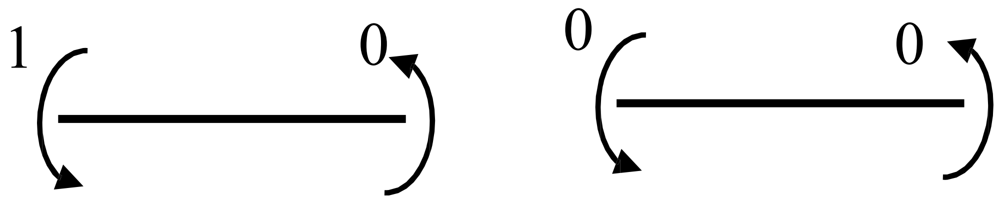


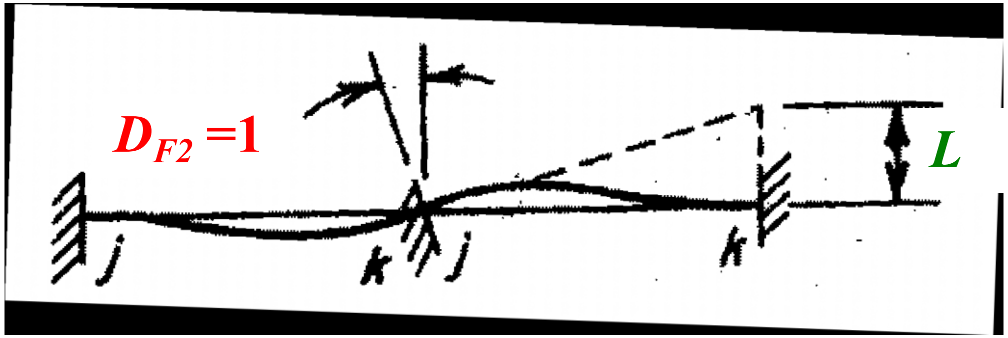
Fixed end actions

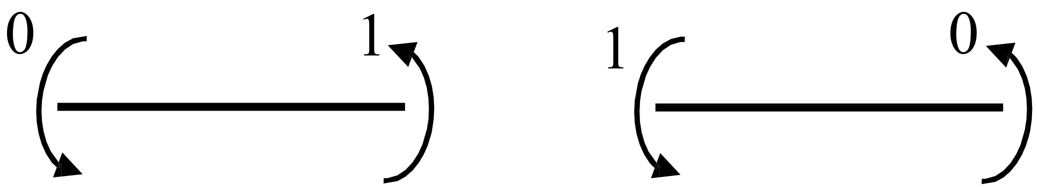


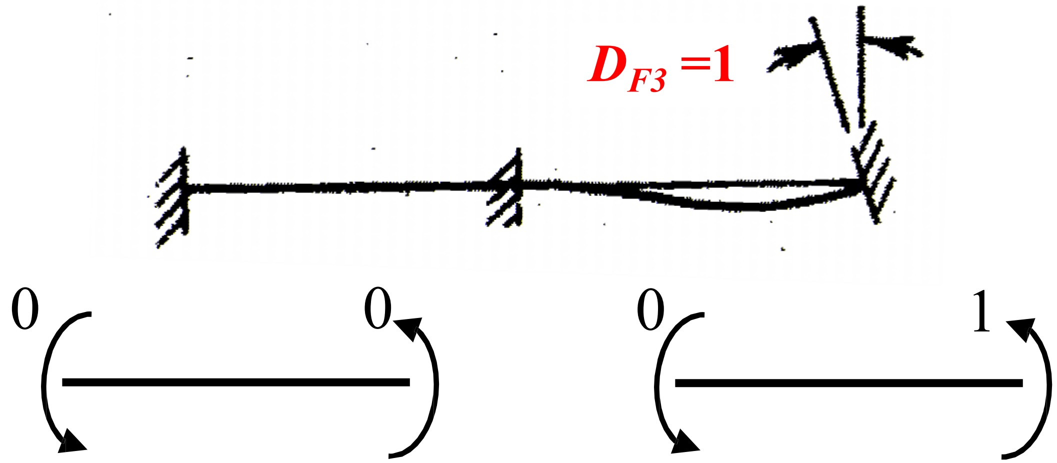
Equivalent joint loads + actual joint loads


D_{F1} =1


1
0
0
0


D_{F2} =1
L
j
k
j
k


0
1
1
0



$$[C_{MF}] = \begin{array}{c} \\ \end{array}\begin{matrix} D_{F1}=1 & D_{F2}=1 & D_{F3}=1 \\ \end{matrix}$$

$$[C_{MF}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Joint displacements

$$\therefore \{D_F\} = [S_{FF}]^{-1}\{A_F\}$$

$\therefore \{D_R\}$ is a null matrix.

$$[S_{FF}] = [C_{MF}]^{T}[S_M][C_{MF}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{T} \frac{2EI}{4} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= EI \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 2 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix}$$

Joint displacements $\{D_F\} = \dfrac{1}{EI}\begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 2 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} -13.33 \\ -6.67 \\ -180 \end{Bmatrix}$

$$= \frac{1}{EI}\begin{Bmatrix} -43.435 \\ 60.33 \\ -210.6 \end{Bmatrix}$$

Member end actions

$$\{A_M\} = \{A_{ML}\} + [S_M]\left([C_{MF}]\{D_F\} + [C_{MR}]\{D_R\}\right)$$

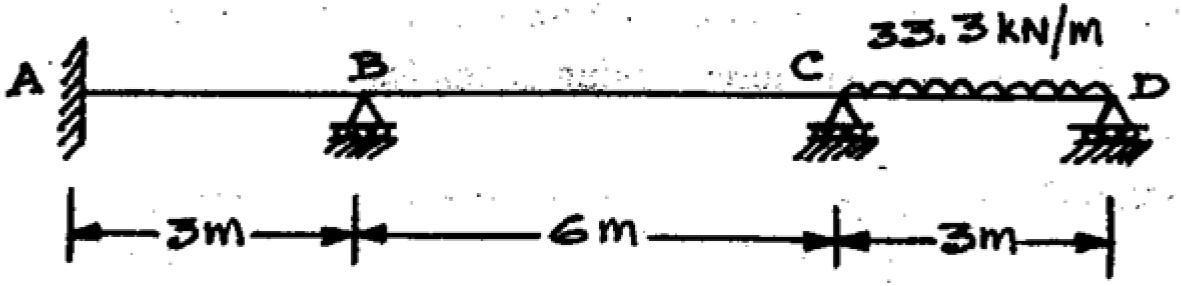
$\{D_R\}$ is a null matrix $\qquad \therefore \{A_M\} = \{A_{ML}\} + [S_M][C_{MF}]\{D_F\}$

$$\{A_M\} = \begin{Bmatrix} 13.33 \\ -13.33 \\ 20 \\ -20 \end{Bmatrix} + \frac{2EI}{4}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\frac{1}{EI}\begin{Bmatrix} -43.435 \\ 60.33 \\ -210.6 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 25 \\ -25 \\ -200 \end{Bmatrix} kNm$$

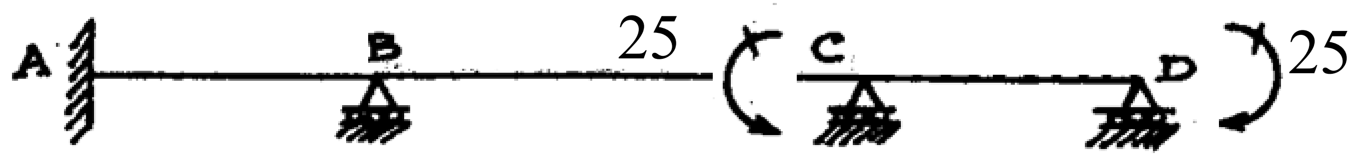
- Problem 3

Analyse the beam. Support B has a downward settlement of 30mm. $EI=5.6\times10^3\,kNm^2$

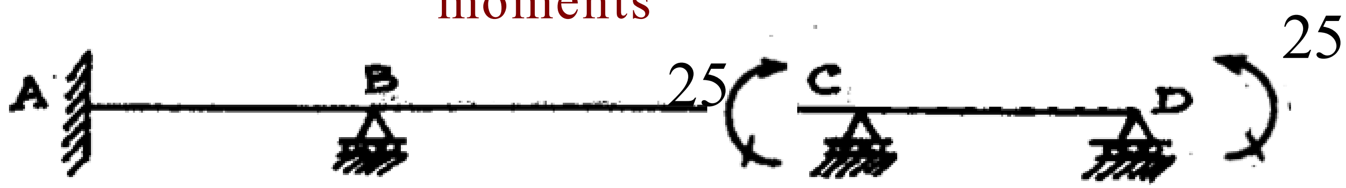


Member stiffness matrix of beam member $\left[S_{Mi}\right]=\dfrac{2EI}{L}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$

Unassembled stiffness matrix $\left[S_M\right]=\dfrac{2EI}{6}\begin{bmatrix}4 & 2 & 0 & 0 & 0 & 0\\ 2 & 4 & 0 & 0 & 0 & 0\\ 0 & 0 & 2 & 1 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 0\\ 0 & 0 & 0 & 0 & 4 & 2\\ 0 & 0 & 0 & 0 & 2 & 4\end{bmatrix}$

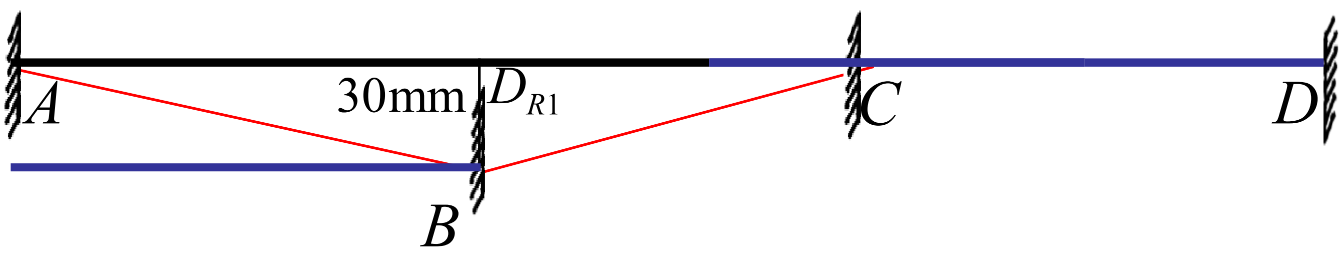


Fixed end moments



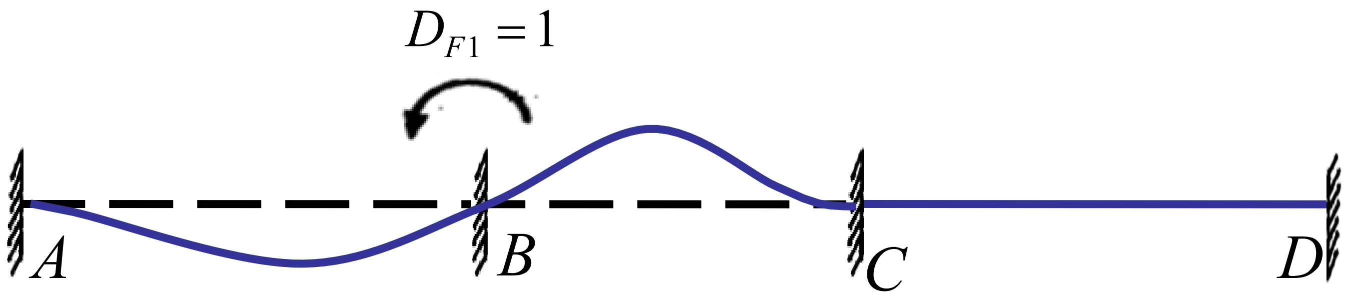
Equivalent joint loads

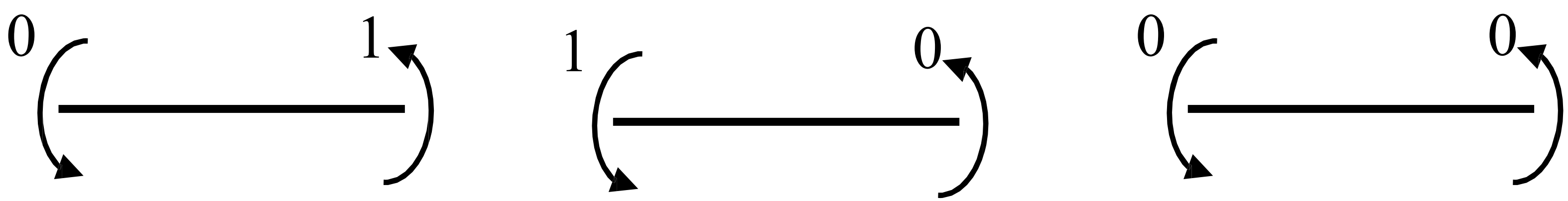
Free (unknown) joint displacements $\{D_F\}$

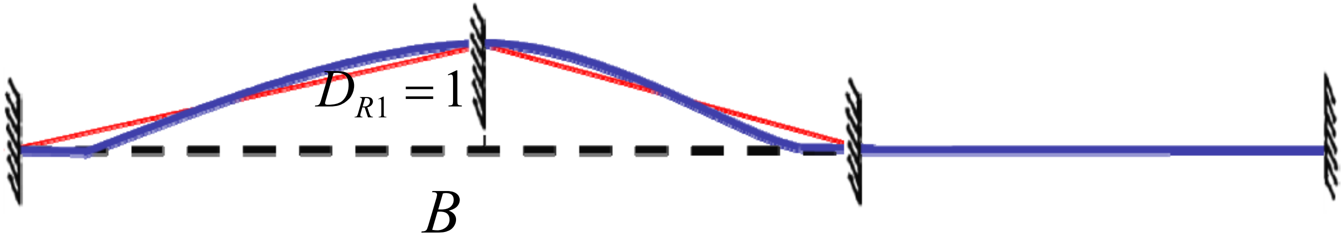


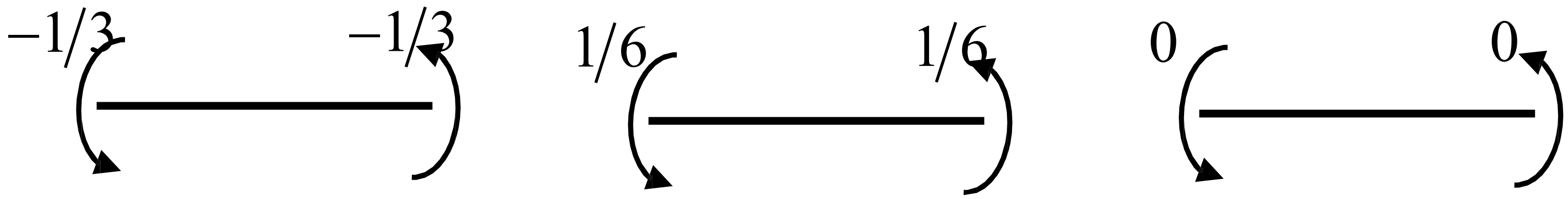
Support settlements (Restraint displacements) $\{D_R\}$

$[C_{MF}]$ and $[C_{MR}]$ consist of member displacements due to unit displacements on the *restrained structure*.









$$[C_{MJ}] = \begin{bmatrix} [C_{MF}] & [C_{MR}] \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} D_{F1}=1 & D_{F2}=1 & D_{F3}=1 & D_{R1}=1 \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & 0 & -1/3 \\ 1 & 0 & 0 & -1/3 \\ 1 & 0 & 0 & 1/6 \\ 0 & 1 & 0 & 1/6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Joint displacements

$$\{D_F\} = [S_{FF}]^{-1}\left(\{A_F\} - [S_{FR}]\{D_R\}\right)$$

$$[S_{FF}] = [C_{MF}]^{\mathrm{T}}[S_M][C_{MF}]$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{T} \frac{EI}{3}\begin{pmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{EI}{3}\begin{pmatrix} 6 & 1 & 0 \\ 1 & 6 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

$$[S_{FR}] = [C_{MF}]^{T} [S_M][C_{MR}]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{T} \frac{EI}{3} \begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} -1/3 \\ -1/3 \\ -1/6 \\ -1/6 \\ 0 \\ 0 \end{bmatrix} = \frac{EI}{3} \begin{bmatrix} -3/2 \\ -1/2 \\ 0 \end{bmatrix}$$

$$\{D_F\} = [S_{FF}]^{-1}\left(\{A_F\} - [S_{FR}]\{D_R\}\right)$$

$$= \left(\frac{EI}{3}\begin{bmatrix} 6 & 1 & 0 \\ 1 & 6 & 2 \\ 0 & 2 & 4 \end{bmatrix}\right)^{-1}\left(\begin{Bmatrix} 0 \\ -25 \\ 25 \end{Bmatrix} - \frac{EI}{3}\begin{Bmatrix} 3/2 \\ 1/2 \\ 0 \end{Bmatrix}\{-0.03\}\right)$$

$$= \frac{3}{116EI}\begin{bmatrix} 20 & -4 & 2 \\ -4 & 24 & -12 \\ 2 & -12 & 35 \end{bmatrix}\left(\begin{Bmatrix} 0 \\ -25 \\ 25 \end{Bmatrix} - \frac{5600}{3}\begin{Bmatrix} -0.045 \\ -0.015 \\ 0 \end{Bmatrix}\right)$$

$$= \frac{3}{116EI}\begin{bmatrix} 20 & -4 & 2 \\ -4 & 24 & -12 \\ 2 & -12 & 35 \end{bmatrix}\left(\begin{Bmatrix} 0 \\ -25 \\ 25 \end{Bmatrix} - \begin{Bmatrix} -84 \\ -28 \\ 0 \end{Bmatrix}\right)$$

$$= \frac{3}{116 \times 5.6 \times 10^3}\begin{Bmatrix} 1642 \\ 108 \\ 671 \end{Bmatrix} = \begin{Bmatrix} 0.007583 \\ 0.0005 \\ 0.0031 \end{Bmatrix}$$

Support reactions

$$\{A_R\} = -\{A_{RC}\} + [S_{RF}]\{D_F\} + [S_{RR}]\{D_R\}$$

$$[S_{RF}] = [C_{MR}]^{\mathrm{T}}[S_M][C_{MF}]$$

$$= [-1/3 \quad -1/3 \quad 1/6 \quad 1/6 \quad 0 \quad 0]\frac{EI}{3}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{EI}{3}[-3/2 \quad 1/2 \quad 0]$$

$$[S_{RR}] = [C_{MR}]^{\mathrm{T}} [S_M][C_{MR}]$$

$$= \begin{bmatrix} -1/3 & -1/3 & 1/6 & 1/6 & 0 & 0 \end{bmatrix} \frac{EI}{3} \begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} \begin{Bmatrix} -1/3 \\ -1/3 \\ 1/6 \\ 1/6 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \left[\frac{EI}{2}\right]$$

$$\{A_R\} = -\{0\} + \frac{EI}{3} \begin{bmatrix} -3/2 & 1/2 & 0 \end{bmatrix} \begin{Bmatrix} -0.007583 \\ 0.0005 \\ 0.0031 \end{Bmatrix} + \left[\frac{EI}{2}\right] \{-0.03\}$$

$$= \frac{EI}{3}(0.0116 - 0.045) = -0.0334 \frac{EI}{3} = -62.35$$

(Support reaction corresponding to D_R. i.e., reaction at B)

Member end actions

$$\{A_M\} = \{A_{ML}\} + [S_M]\left([C_{MF}]\{D_F\} + [C_{MR}]\{D_R\}\right)$$

$$\{A_M\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 25 \\ -25 \end{Bmatrix} + \frac{2EI}{6}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} -0.007583 \\ 0.0005 \\ 0.0031 \end{Bmatrix} + \begin{bmatrix} 1/3 \\ -1/3 \\ 1/6 \\ 1/6 \\ 0 \\ 0 \end{bmatrix}\{-0.03\}\right)$$

$$= \begin{Bmatrix} 83.7 \\ 55.4 \\ -55.4 \\ -40.3 \\ 40.3 \\ 0 \end{Bmatrix}$$

Alternatively, if the entire $[S_J]$ matrix is assembled at a time,

$$[S_J]=[C_{MJ}]^{T}[S_M][C_{MJ}]$$

$$=\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1/3 & -1/3 & 1/6 & 1/6 & 0 & 0 \end{bmatrix} \frac{EI}{3}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} \left[\begin{array}{ccc|c} 0 & 0 & 0 & -1/3 \\ 1 & 0 & 0 & -1/3 \\ 1 & 0 & 0 & 1/6 \\ 0 & 1 & 0 & 1/6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$=\frac{EI}{3}\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1/3 & -1/3 & 1/6 & 1/6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & -2 \\ 4 & 0 & 0 & -2 \\ 2 & 1 & 0 & 1/2 \\ 1 & 2 & 0 & 1/2 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{bmatrix}$$

$$=\frac{EI}{3}\left[\begin{array}{ccc|c} 6 & 1 & 0 & -3/2 \\ 1 & 6 & 2 & 1/2 \\ 0 & 2 & 4 & 0 \\ \hline -3/2 & 1/2 & 0 & 3/2 \end{array}\right] = \begin{bmatrix} [S_{FF}] & [S_{FR}] \\ [S_{RF}] & [S_{RR}] \end{bmatrix}$$

Alternatively, if ALL possible support settlements are accounted for,

$$[C_{MJ}] = \begin{bmatrix} [C_{MF}] & [C_{MR}] \end{bmatrix} = \left[\begin{array}{ccc|ccccc} 0 & 0 & 0 & 1/3 & 1 & -1/3 & 0 & 0 \\ 1 & 0 & 0 & 1/3 & 0 & -1/3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1/6 & -1/6 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/6 & -1/6 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/3 & -1/3 \end{array}\right]$$

$$[S_J] = [C_{MJ}]^{\mathrm{T}} [S_M][C_{MJ}]$$

$$= \left[\begin{array}{ccc|ccccc} 0 & 0 & 0 & 1/3 & 1 & -1/3 & 0 & 0 \\ 1 & 0 & 0 & 1/3 & 0 & -1/3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1/6 & -1/6 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/6 & -1/6 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/3 & -1/3 \end{array}\right]^T \frac{EI}{3} \begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} \left[\begin{array}{ccc|ccccc} 0 & 0 & 0 & 1/3 & 1 & -1/3 & 0 & 0 \\ 1 & 0 & 0 & 1/3 & 0 & -1/3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1/6 & -1/6 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/6 & -1/6 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/3 & -1/3 \end{array}\right]$$

$$
= \frac{EI}{3}\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1/3 & -1/3 & 1/6 & 1/6 & 0 & 0 \\ 0 & 0 & -1/6 & -1/6 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & -1/3 & -1/3 \end{bmatrix}
\begin{bmatrix} 2 & 0 & 0 & 2 & 4 & -2 & 0 & 0 \\ 4 & 0 & 0 & 2 & 2 & -2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1/2 & -1/2 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 4 & 2 & 0 & 0 & 0 & 2 & -2 \\ 0 & 2 & 4 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}
$$

$$
= \frac{EI}{3}\left[\begin{array}{ccc|ccccc} 6 & 1 & 0 & 2 & 2 & -3/2 & -1/2 & 0 \\ 1 & 6 & 2 & 0 & 0 & 1/2 & 3/2 & -2 \\ 0 & 2 & 4 & 0 & 0 & 0 & 2 & -2 \\ \hline 2 & 0 & 0 & 4/3 & 2 & -4/3 & 0 & 0 \\ 2 & 0 & 0 & 2 & 4 & -2 & 0 & 0 \\ -3/2 & 1/2 & 0 & -4/3 & -2 & 3/2 & -1/6 & 0 \\ -1/2 & 3/2 & 2 & 0 & 0 & -1/6 & 3/2 & -4/3 \\ 0 & -2 & -2 & 0 & 0 & 0 & -4/3 & 4/3 \end{array}\right]
= \begin{bmatrix} [S_{FF}] & [S_{FR}] \\ [S_{RF}] & [S_{RR}] \end{bmatrix}
$$

Joint displacements

$$\{D_F\} = [S_{FF}]^{-1}\left(\{A_F\} - [S_{FR}]\{D_R\}\right)$$

$$= \left(\frac{EI}{3}\begin{bmatrix} 6 & 1 & 0 \\ 1 & 6 & 2 \\ 0 & 2 & 4 \end{bmatrix}\right)^{-1}\left(\begin{Bmatrix} 0 \\ -25 \\ -25 \end{Bmatrix} - \frac{EI}{3}\begin{bmatrix} 2 & 2 & -3/2 & -1/2 & 0 \\ 0 & 0 & 1/2 & 3/2 & -2 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.03 \\ 0 \\ 0 \\ 0 \end{Bmatrix}\right)$$

$$= \frac{3}{116EI}\begin{bmatrix} 20 & -4 & 2 \\ -4 & 24 & -12 \\ 2 & -12 & 35 \end{bmatrix}\left(\begin{Bmatrix} 0 \\ -25 \\ -25 \end{Bmatrix} - \frac{5600}{3}\begin{Bmatrix} 0.045 \\ -0.015 \\ 0 \end{Bmatrix}\right)$$

$$= \frac{3}{116EI}\begin{bmatrix} 20 & -4 & 2 \\ -4 & 24 & -12 \\ 2 & -12 & 35 \end{bmatrix}\begin{Bmatrix} -84 \\ 3 \\ -25 \end{Bmatrix} = \frac{3}{116EI}\begin{Bmatrix} -1642 \\ 108 \\ 671 \end{Bmatrix} = \begin{Bmatrix} -0.007583 \\ 0.0005 \\ 0.0031 \end{Bmatrix}$$

Support reactions

$$\{A_R\} = -\{A_{RC}\} + [S_{RF}]\{D_F\} + [S_{RR}]\{D_R\}$$

$$\{A_R\} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -50 \\ -50 \end{Bmatrix} + \frac{EI}{3}\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ -3/2 & 1/2 & 0 \\ -1/2 & 3/2 & 2 \\ 0 & -2 & -2 \end{bmatrix}\begin{Bmatrix} -0.007583 \\ 0.0005 \\ 0.0031 \end{Bmatrix} + \frac{EI}{3}\begin{bmatrix} 4/3 & 2 & -4/3 & 0 & 0 \\ 2 & 4 & -2 & 0 & 0 \\ -4/3 & -2 & 3/2 & -1/6 & 0 \\ 0 & 0 & -1/6 & 3/2 & -4/3 \\ 0 & 0 & 0 & -4/3 & 4/3 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.03 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{A_R\} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -50 \\ -50 \end{Bmatrix} + \begin{Bmatrix} -28.373 \\ -28.373 \\ 21.653 \\ 19.973 \\ -13.44 \end{Bmatrix} + \begin{Bmatrix} 74.667 \\ 112 \\ -84 \\ 9.333 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 46.294 \\ 83.627 \\ -62.347 \\ 79.31 \\ 36.56 \end{Bmatrix}$$

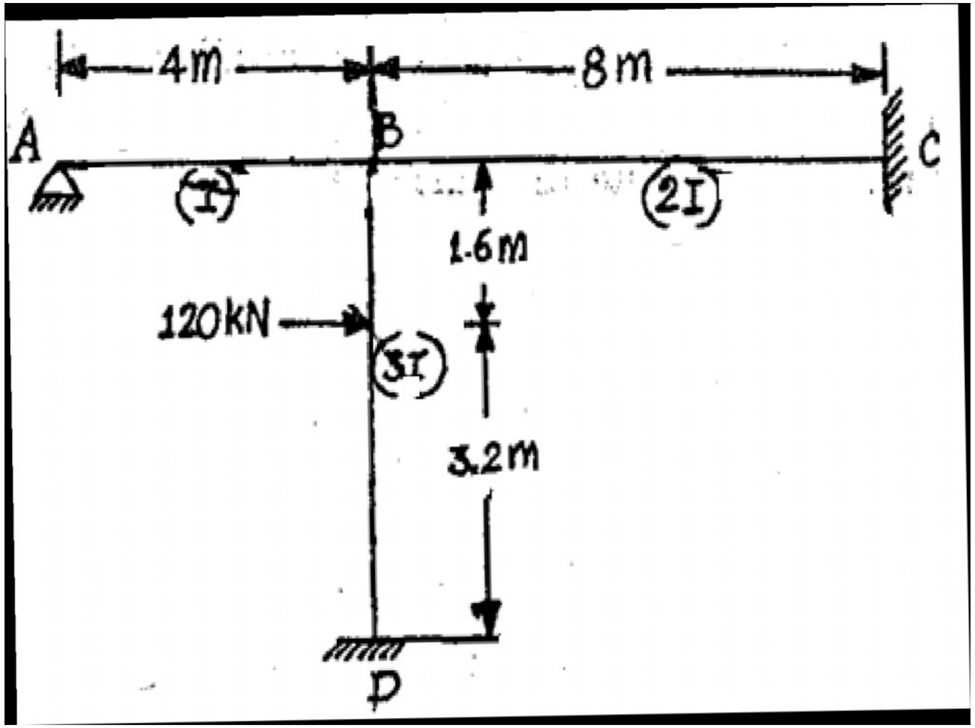
Member end actions

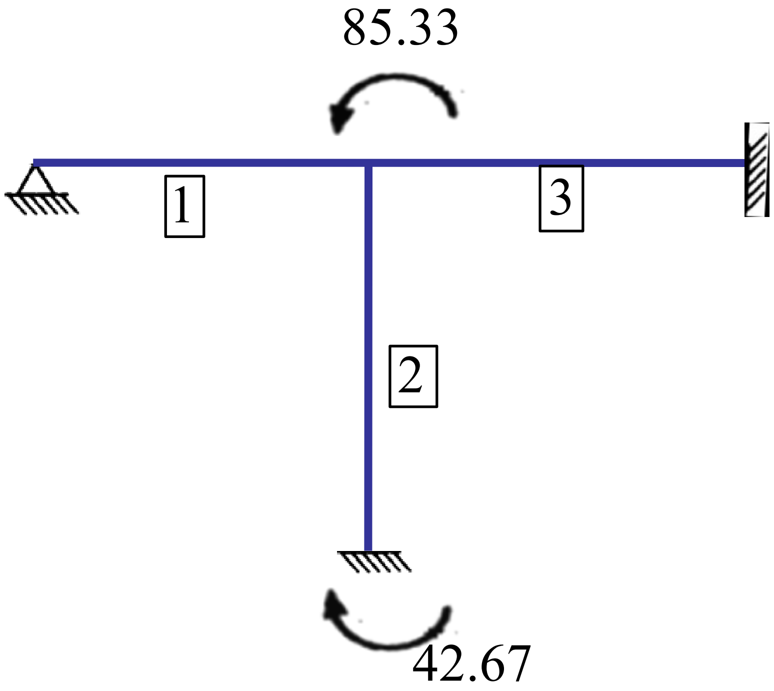
$$\{A_M\} = \{A_{ML}\} + [S_M]\left([C_{MF}]\{D_F\} + [C_{MR}]\{D_R\}\right)$$

$$\{A_M\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 25 \\ -25 \end{Bmatrix} + \frac{2EI}{6}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} -0.007583 \\ 0.0005 \\ 0.0031 \end{Bmatrix} + \begin{bmatrix} 1/3 & 1 & -1/3 & 0 & 0 \\ 1/3 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1/6 & -1/6 & 0 \\ 0 & 0 & 1/6 & -1/6 & 0 \\ 0 & 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 0 & 1/3 & -1/3 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.03 \\ 0 \\ 0 \end{Bmatrix}\right)$$

$$\{A_M\} = \begin{Bmatrix} 83.7 \\ 55.4 \\ -55.4 \\ -40.3 \\ 40.3 \\ 0 \end{Bmatrix}$$

• Problem 4



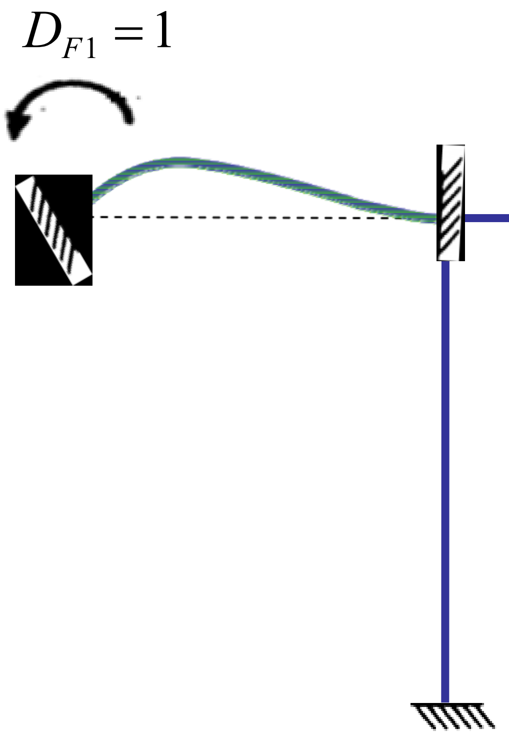


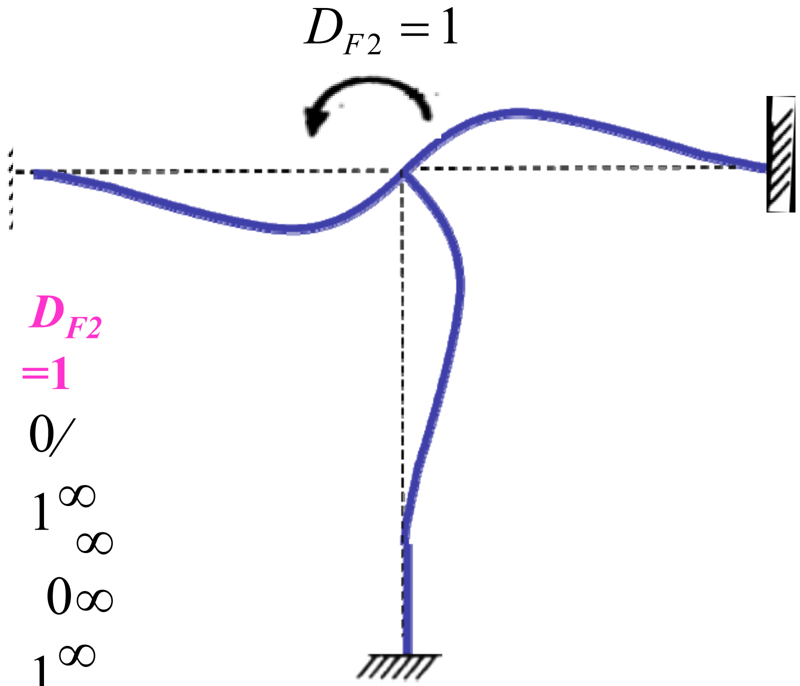
Equivalent joint loads

Unassembled stiffness matrix

$$[S_M] = 2EI \begin{bmatrix} 2/4 & 1/4 & 0 & 0 & 0 & 0 \\ 1/4 & 2/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6/4.8 & 3/4.8 & 0 & 0 \\ 0 & 0 & 3/4.8 & 6/4.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4/8 & 2/8 \\ 0 & 0 & 0 & 0 & 2/8 & 4/8 \end{bmatrix}$$

• $[C_{MF}]$ consists of member displacements due to unit displacements on the *restrained structure*.





$$[C_{MF}] = \begin{bmatrix} D_{F1}=1 & D_{F2}=1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Joint displacements

$$\{D_F\} = [S_{FF}]^{-1}\left(\{A_F\} - [S_{FR}]\{D_R\}\right)$$

$\{D_F\} = [S_{FF}]^{-1}\{A_F\}$ since there are no support displacements.

$$[S_{FF}] = [C_{MF}]^{\mathrm{T}}[S_M][C_{MF}]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} 2EI \begin{bmatrix} 2/4 & 1/4 & 0 & 0 & 0 & 0 \\ 1/4 & 2/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6/4.8 & 3/4.8 & 0 & 0 \\ 0 & 0 & 3/4.8 & 6/4.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4/8 & 2/8 \\ 0 & 0 & 0 & 0 & 2/8 & 4/8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{EI}{8}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 4 & 8 \\ 0 & 20 \\ 0 & 10 \\ 0 & 8 \\ 0 & 4 \end{bmatrix} = \frac{EI}{2}\begin{bmatrix} 2 & 1 \\ 1 & 9 \end{bmatrix}$$

$$\therefore \{D_F\} = [S_{FF}]^{-1}\{A_F\} = \left(\frac{EI}{2}\begin{bmatrix} 2 & 1 \\ 1 & 9 \end{bmatrix}\right)^{-1}\begin{Bmatrix} 0 \\ 85.333 \end{Bmatrix}$$

$$= \frac{8}{272EI}\begin{bmatrix} 36 & -4 \\ -4 & 8 \end{bmatrix}\begin{Bmatrix} 0 \\ 85.333 \end{Bmatrix} = \frac{1}{34EI}\begin{Bmatrix} -341.332 \\ 682.664 \end{Bmatrix} = \frac{1}{EI}\begin{Bmatrix} -10.039 \\ 20.078 \end{Bmatrix}$$

Member end actions

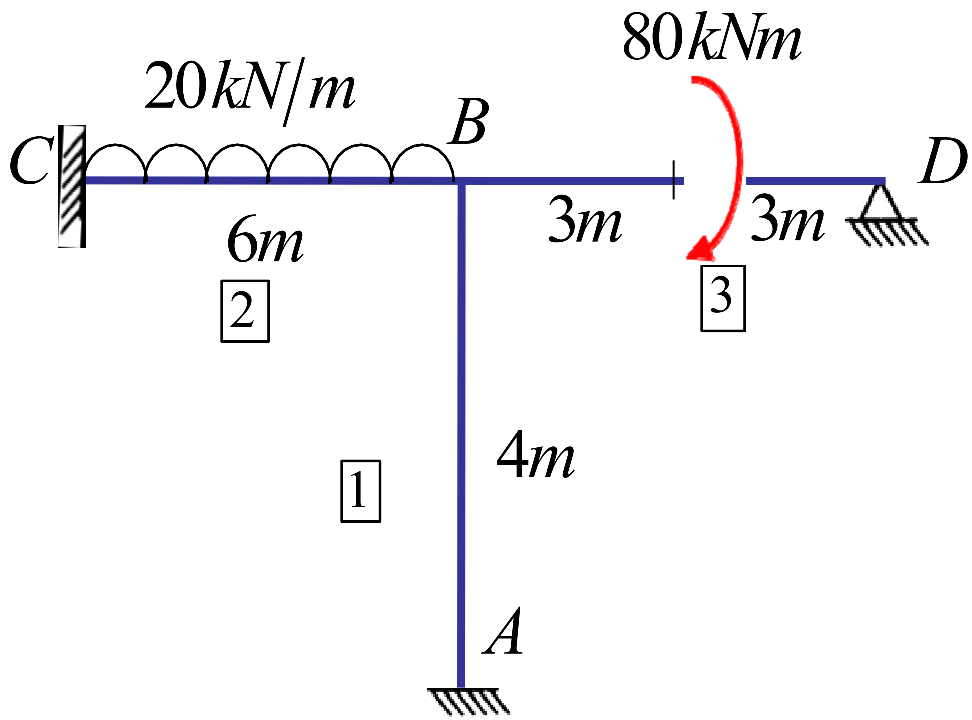
$\{D_R\}$ is a null matrix

$$\{A_M\}=\{A_{ML}\}+[S_M]\left([C_{MF}]\{D_F\}+[C_{MR}]\{D_R\}\right)$$

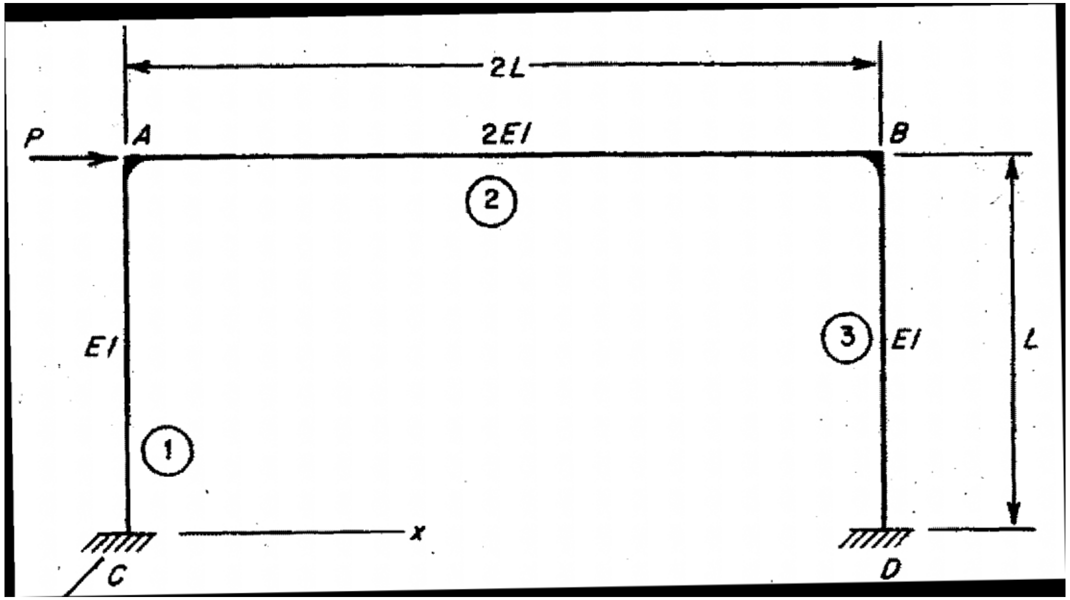
$$=\begin{Bmatrix}0\\0\\42.667\\-85.333\\0\\0\end{Bmatrix}+2EI\begin{bmatrix}2/4 & 1/4 & 0 & 0 & 0 & 0\\1/4 & 2/4 & 0 & 0 & 0 & 0\\0 & 0 & 6/4.8 & 3/4.8 & 0 & 0\\0 & 0 & 3/4.8 & 6/4.8 & 0 & 0\\0 & 0 & 0 & 0 & 4/8 & 2/8\\0 & 0 & 0 & 0 & 2/8 & 4/8\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\\0 & 1\\0 & 0\\0 & 1\\0 & 0\end{bmatrix}\frac{1}{EI}\begin{Bmatrix}-10.039\\20.078\end{Bmatrix}$$

$$=\begin{Bmatrix}0\\0\\42.667\\-85.333\\0\\0\end{Bmatrix}+\frac{1}{8}\begin{Bmatrix}0\\120.47\\200.78\\401.56\\160.624\\80.312\end{Bmatrix}=\begin{Bmatrix}0\\15.059\\67.765\\-35.138\\20.078\\10.039\end{Bmatrix}$$

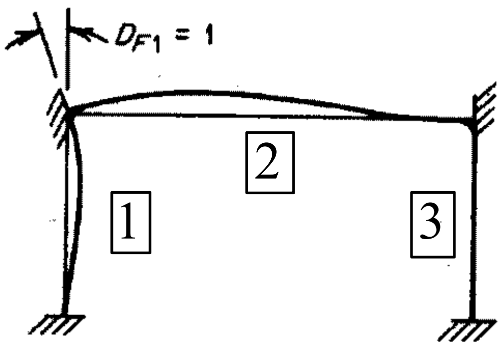
- **Homework 2:**



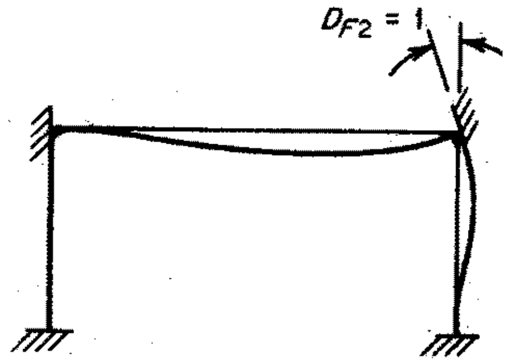
- Problem 5



Unassembled stiffness matrix $[S_M] = \frac{2EI}{L}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

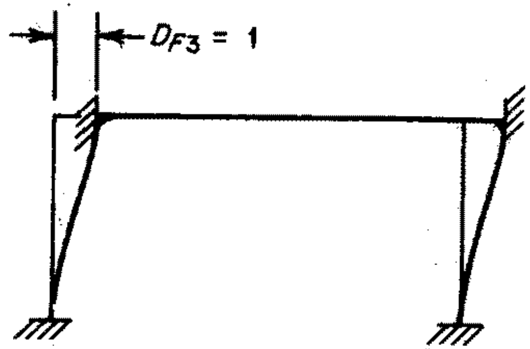


(b)



(c)

$$
[C_{MF}] = \begin{bmatrix} D_{F1}=1 & D_{F2}=1 & D_{F3}=1 \\ 0 & 0 & 1/L \\ 1 & 0 & 1/L \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1/L \\ 0 & 0 & 1/L \end{bmatrix}
$$



(d)

$$[S_{FF}] = [C_{MF}]^{T} [S_M][C_{MF}]$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1/L & 1/L & 0 & 0 & 1/L & 1/L \end{bmatrix} \frac{2EI}{L} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1/L \\ 1 & 0 & 1/L \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1/L \\ 0 & 0 & 1/L \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1/L & 1/L & 0 & 0 & 1/L & 1/L \end{bmatrix} \frac{2EI}{L} \begin{bmatrix} 1 & 0 & 3/L \\ 2 & 0 & 3/L \\ 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 3/L \\ 0 & 1 & 3/L \end{bmatrix} = \frac{2EI}{L} \begin{bmatrix} 4 & 1 & 3/L \\ 1 & 4 & 3/L \\ 3/L & 3/L & 12/L^2 \end{bmatrix}$$

Joint displacements

$$\{D_F\} = [S_{FF}]^{-1}\left(\{A_F\} - [S_{FR}]\{D_R\}\right)$$

$\{D_F\} = [S_{FF}]^{-1}\{A_F\}$, since there are no support displacements.

$$\{D_F\} = \left[\frac{2EI}{L}\begin{bmatrix} 4 & 1 & 3/L \\ 1 & 4 & 3/L \\ 3/L & 3/L & 12/L^2 \end{bmatrix}\right]^{-1}\begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix} = \frac{L}{84EI}\begin{bmatrix} 13 & -3 & -3L \\ -3 & 13 & -3L \\ -3L & -3L & 5L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix} = \frac{PL^2}{84EI}\begin{Bmatrix} -3 \\ -3 \\ 5L \end{Bmatrix}$$

Member end actions

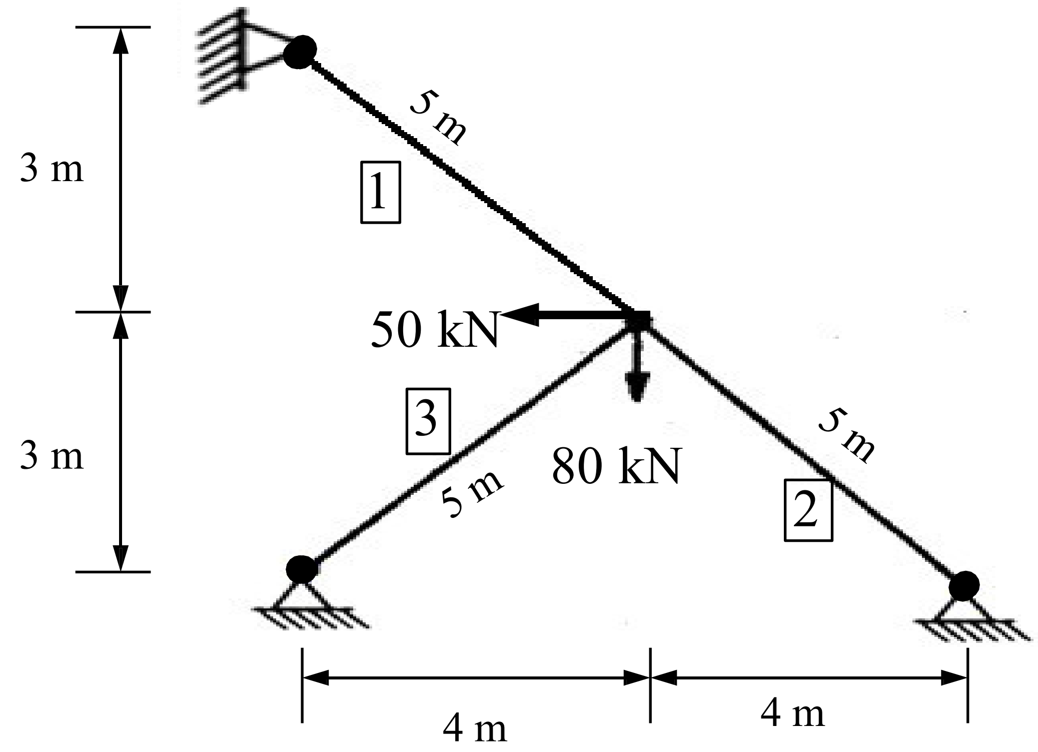
$$\{A_M\} = \{A_{ML}\} + [S_M]\left([C_{MF}]\{D_F\} + [C_{MR}]\{D_R\}\right)$$

$$\{A_M\} = [S_M][C_{MF}]\{D_F\}$$

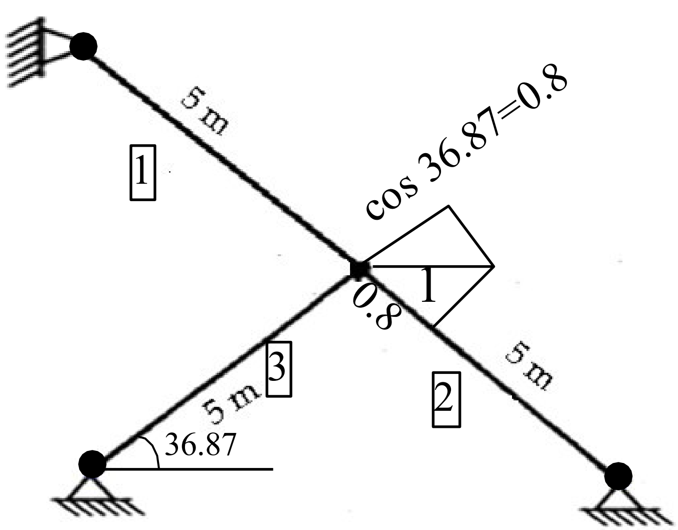
$$\{A_M\} = \frac{2EI}{L}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1/L \\ 1 & 0 & 1/L \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1/L \\ 0 & 0 & 1/L \end{bmatrix}\frac{PL^2}{84EI}\begin{Bmatrix} -3 \\ -3 \\ 5L \end{Bmatrix}$$

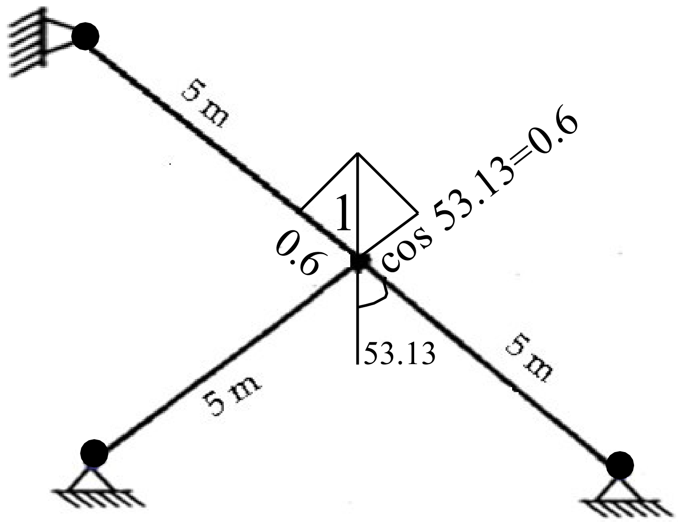
$$= \frac{2EI}{L}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}\frac{PL^2}{84EI}\begin{Bmatrix} 5 \\ 2 \\ -3 \\ -3 \\ 2 \\ 5 \end{Bmatrix} = \frac{2EI}{L}\frac{PL^2}{84EI}\begin{Bmatrix} 12 \\ 9 \\ -9 \\ -9 \\ 9 \\ 12 \end{Bmatrix} = \frac{PL}{14}\begin{Bmatrix} 4 \\ 3 \\ -3 \\ -3 \\ 3 \\ 4 \end{Bmatrix}$$

•Problem 6:



Unassembled stiffness matrix $[S_M] = \begin{bmatrix} 0.2 & 0.0 & 0.0 \\ 0.0 & 0.2 & 0.0 \\ 0.0 & 0.0 & 0.2 \end{bmatrix}$





$$[C_{MF}] = \begin{bmatrix} D_{F1}=1 & D_{F2}=1 \\ 0.8 & -0.6 \\ -0.8 & 0.6 \\ 0.8 & 0.6 \end{bmatrix}$$

Joint displacements

$$\{D_F\} = [S_{FF}]^{-1}\left(\{A_F\} - [S_{FR}]\{D_R\}\right)$$

$\{D_F\} = [S_{FF}]^{-1}\{A_F\}$, since there are no support displacements.

$$[S_{FF}] = [C_{MF}]^{T}[S_M][C_{MF}] = \begin{bmatrix} 0.384 & -0.096 \\ -0.096 & 0.216 \end{bmatrix}$$

$$\{D_F\} = \begin{bmatrix} 2.930 & 1.302 \\ 1.302 & 5.208 \end{bmatrix} \begin{Bmatrix} 50 \\ 80 \end{Bmatrix} = \begin{Bmatrix} 250.651 \\ 481.771 \end{Bmatrix}$$

Member Forces:

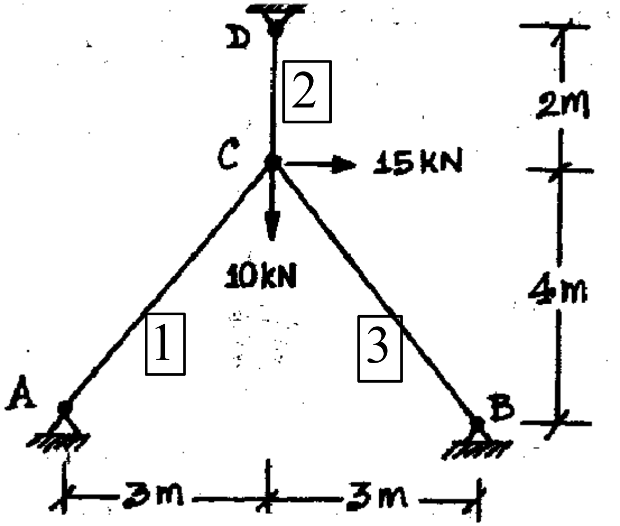
$$\{A_M\} = \{A_{ML}\} + [S_M]\left([C_{MF}]\{D_F\} + [C_{MR}]\{D_R\}\right)$$

$$= [S_M][C_{MF}]\{D_F\}$$

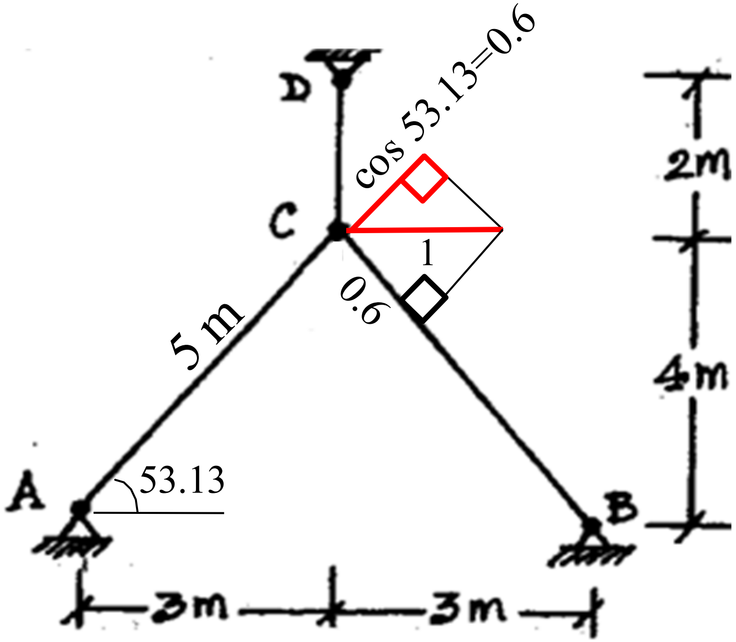
$$= \begin{bmatrix} 0.2 & 0.0 & 0.0 \\ 0.0 & 0.2 & 0.0 \\ 0.0 & 0.0 & 0.2 \end{bmatrix} \begin{bmatrix} 0.8 & -0.6 \\ -0.8 & 0.6 \\ 0.8 & 0.6 \end{bmatrix} \begin{Bmatrix} -250.651 \\ -481.771 \end{Bmatrix}$$

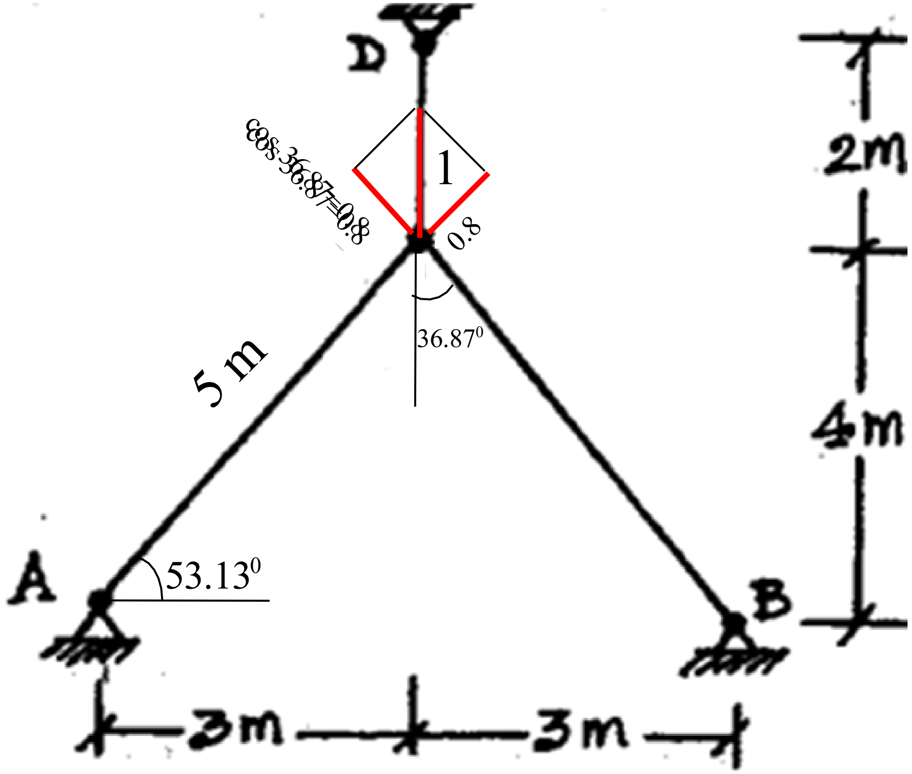
$$= \begin{Bmatrix} 17.708 \\ -17.708 \\ -97.917 \end{Bmatrix}$$

•Problem 7:



Unassembled stiffness matrix $[S_M] = \begin{bmatrix} 0.2 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.2 \end{bmatrix}$





$$[C_{MF}] = \begin{array}{cc} D_{F1}=1 & D_{F2}=1 \\ 0.600 & 0.800 \\ 0.000 & -1.000 \\ -0.600 & 0.800 \end{array}$$

Joint displacements

$$\{D_F\} = [S_{FF}]^{-1}\left(\{A_F\} - [S_{FR}]\{D_R\}\right)$$

$\{D_F\} = [S_{FF}]^{-1}\{A_F\}$, since there are no support displacements.

$$[S_{FF}] = [C_{MF}]^{T}[S_M][C_{MF}] = \begin{bmatrix} 0.144 & 0.000 \\ 0.000 & 0.756 \end{bmatrix}$$

$$\{D_F\} = \begin{bmatrix} 6.944 & 0.000 \\ 0.000 & 1.323 \end{bmatrix} \begin{Bmatrix} 15.000 \\ 10.000 \end{Bmatrix} = \begin{Bmatrix} 104.167 \\ 13.228 \end{Bmatrix}$$

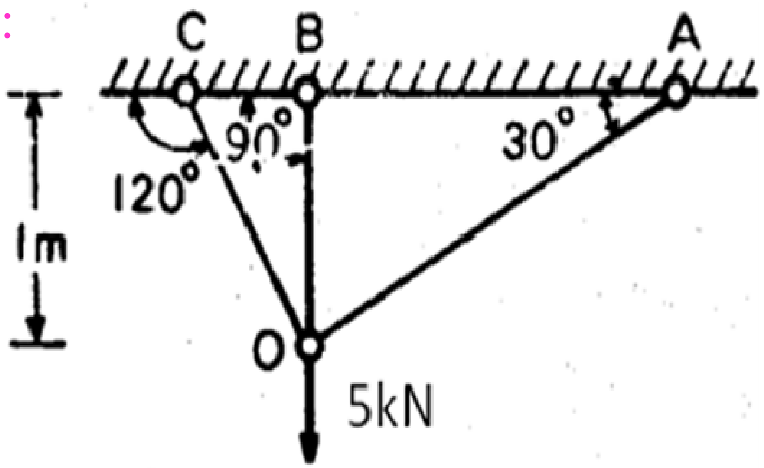
Member Forces:

$$\{A_M\} = \{A_{ML}\} + [S_M]\left([C_{MF}]\{D_F\} + [C_{MR}]\{D_R\}\right)$$

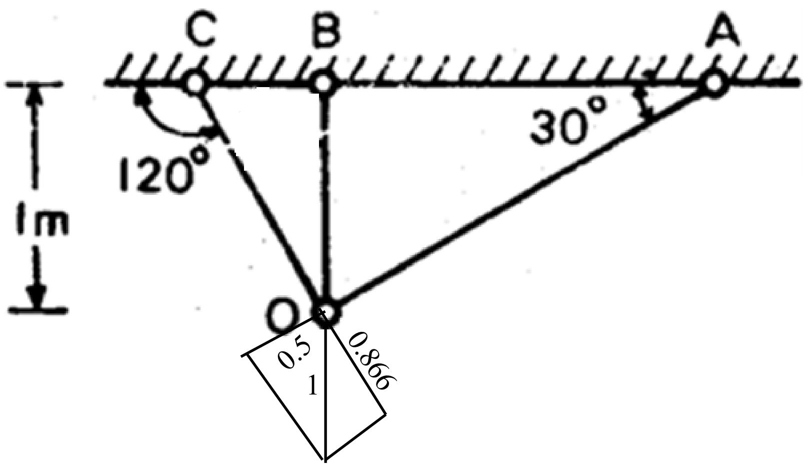
$$\{A_M\} = [S_M][C_{MF}]\{D_F\}$$

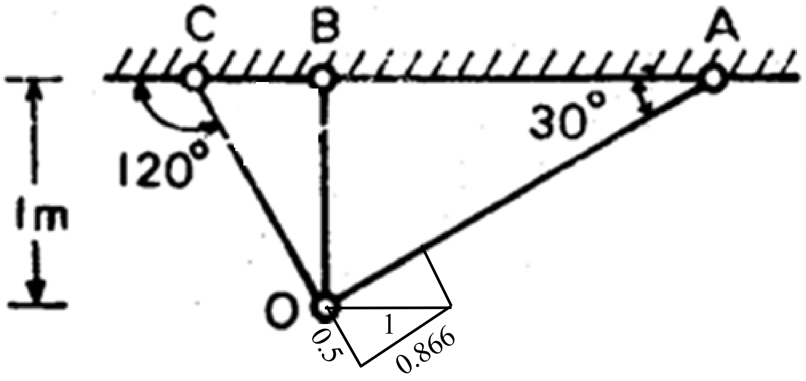
$$\{A_M\} = \begin{bmatrix} 0.2 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.2 \end{bmatrix} \begin{bmatrix} 0.600 & 0.800 \\ 0.000 & -1.000 \\ 0.600 & 0.800 \end{bmatrix} \begin{Bmatrix} 104.167 \\ 13.228 \end{Bmatrix} = \begin{Bmatrix} 10.4 \\ 6.6 \\ 14.6 \end{Bmatrix}$$

• Problem 8 (Homework 3):



Unassembled stiffness matrix $[S_M] = \begin{bmatrix} 0.866 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 0.500 \end{bmatrix}$





$D_{F1}=1$ **$D_{F2}=1$**

$$[C_{MF}] = \begin{bmatrix} 0.866 & 0.500 \\ 1.000 & 0.000 \\ 0.500 & -0.866 \end{bmatrix}$$

Joint displacements

$$\{D_F\} = [S_{FF}]^{-1}\left(\{A_F\} - [S_{FR}]\{D_R\}\right)$$

$\{D_F\} = [S_{FF}]^{-1}\{A_F\}$, since there are no support displacements.

$$[S_{FF}] = [C_{MF}]^{T}[S_M][C_{MF}] = \begin{bmatrix} 1.774 & 0.158 \\ 0.158 & 0.591 \end{bmatrix}$$

$$\{D_F\} = \begin{bmatrix} 0.577 & -0.155 \\ -0.155 & 1.732 \end{bmatrix} \begin{Bmatrix} 5 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 2.887 \\ -0.773 \end{Bmatrix}$$

Member Forces:

$$\{A_M\} = \{A_{ML}\} + [S_M]\left([C_{MF}]\{D_F\} + [C_{MR}]\{D_R\}\right)$$

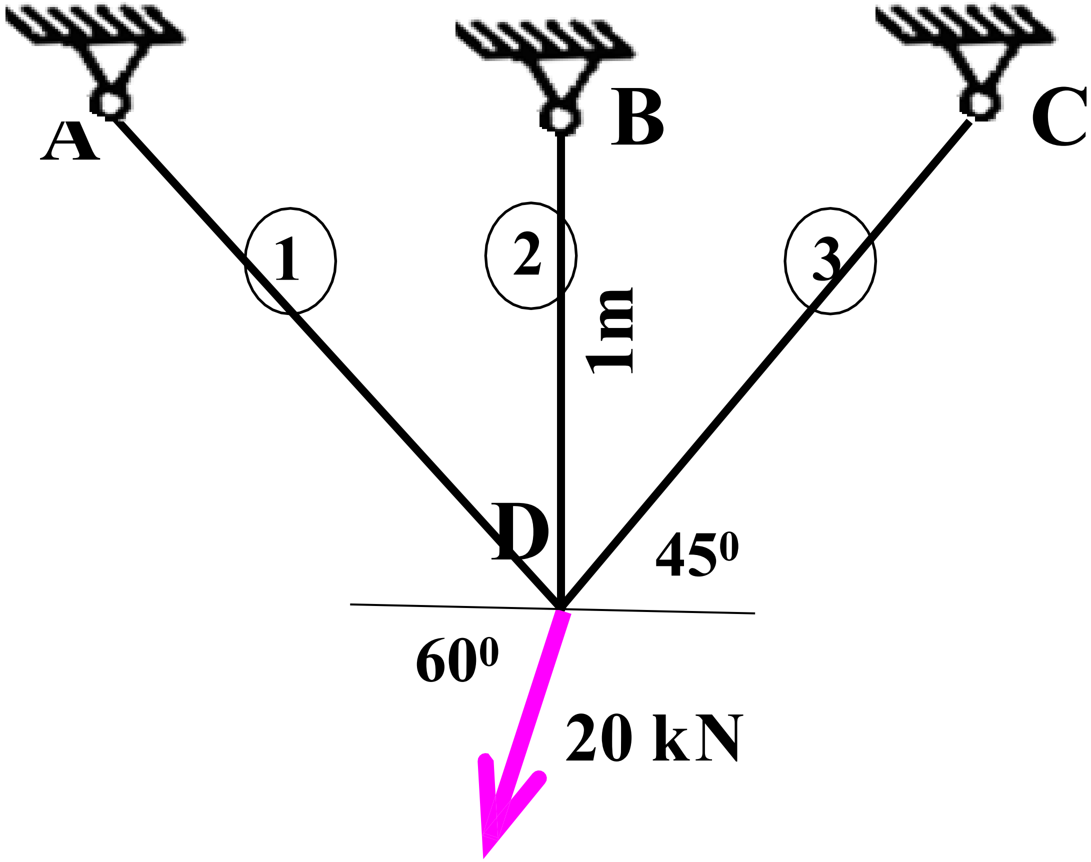
$$\{A_M\} = [S_M][C_{MF}]\{D_F\}$$

$$\{A_M\} = \begin{bmatrix} 0.866 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 0.500 \end{bmatrix} \begin{bmatrix} 0.866 & 0.500 \\ 1.000 & 0.000 \\ 0.500 & -0.866 \end{bmatrix} \begin{Bmatrix} 2.887 \\ 0.773 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1.830 \\ 2.887 \\ 3.057 \end{Bmatrix}$$

- **Homework 4:**

AE **is constant.**



Summary

Stiffness method

- Development of stiffness matrices by physical approach – stiffness matrices for truss, beam and frame elements – displacement transformation matrix – development of total stiffness matrix - analysis of simple structures – plane truss beam and plane frame- nodal loads and element loads – lack of fit and temperature effects.