Structural Analysis - III

Introduction to Matrix Methods

Module I

Matrix analysis of structures

- Definition of flexibility and stiffness influence coefficients – development of flexibility matrices by physical approach & energy principle.

Flexibility method

- Flexibility matrices for truss, beam and frame elements – load transformation matrix-development of total flexibility matrix of the structure –analysis of simple structures – plane truss, continuous beam and plane frame- nodal loads and element loads – lack of fit and temperature effects.

Force method and Displacement method

- These methods are applicable to discretized structures of all types

- **Force method** (Flexibility method)
    - Actions are the primary unknowns
    - Static indeterminacy: excess of unknown actions than the available number of equations of static equilibrium

- **Displacement method** (Stiffness method)

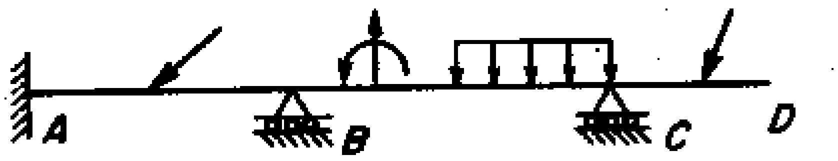
  - Displacements of the joints are the primary unknowns

  - Kinematic indeterminacy: number of independent translations and rotations (the unknown joint displacements)

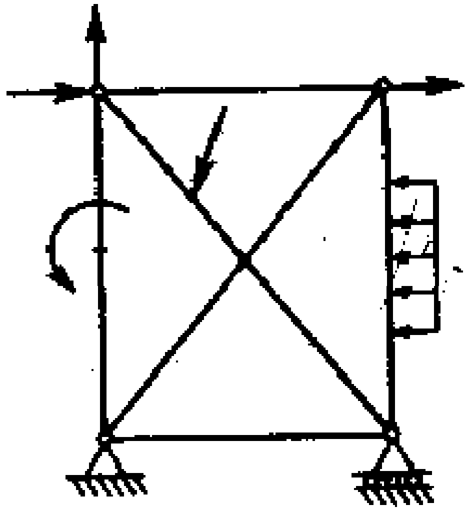
  - More suitable for computer programming

Types of Framed Structures

- a. Beams: may support bending moment, shear force and axial force



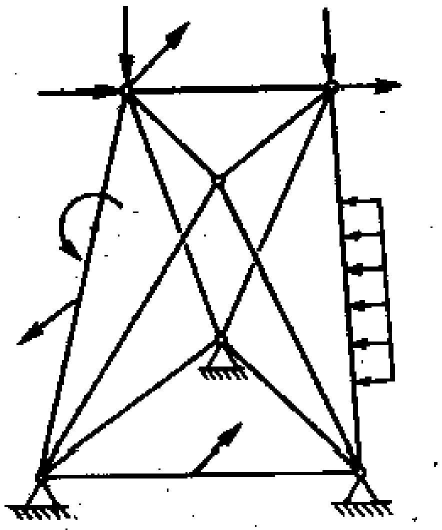
(a)



(b)

- b. Plane trusses: hinge joints; In addition to axial forces, a member CAN have bending moments and shear forces if it has loads directly acting on them, in addition to joint loads

•c. Space trusses: hinge joints; any couple acting on a member should have moment vector perpendicular to the axis of the member, since a truss member is incapable of supporting a twisting moment



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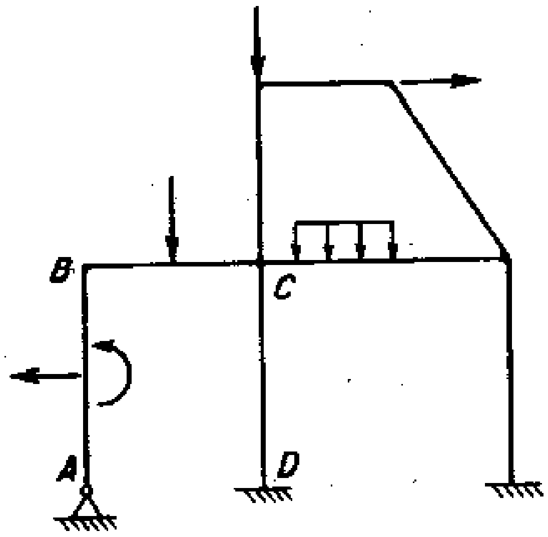
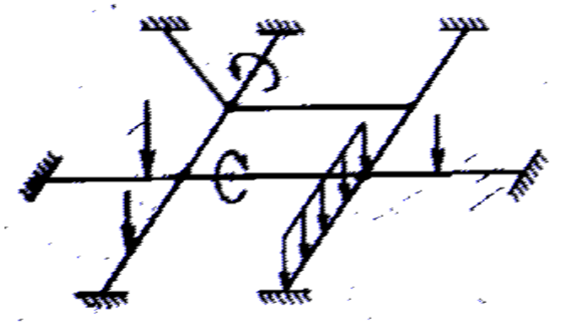


(d)

•d. Plane frames: Joints are rigid; all forces in the plane of the frame, all couples normal to the plane of the frame

•e. Grids: all forces normal to the plane of the grid, all couples in the plane of the grid (includes bending and torsion)



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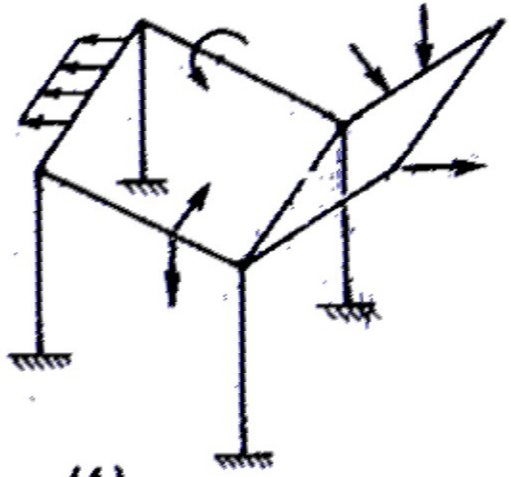


(f)

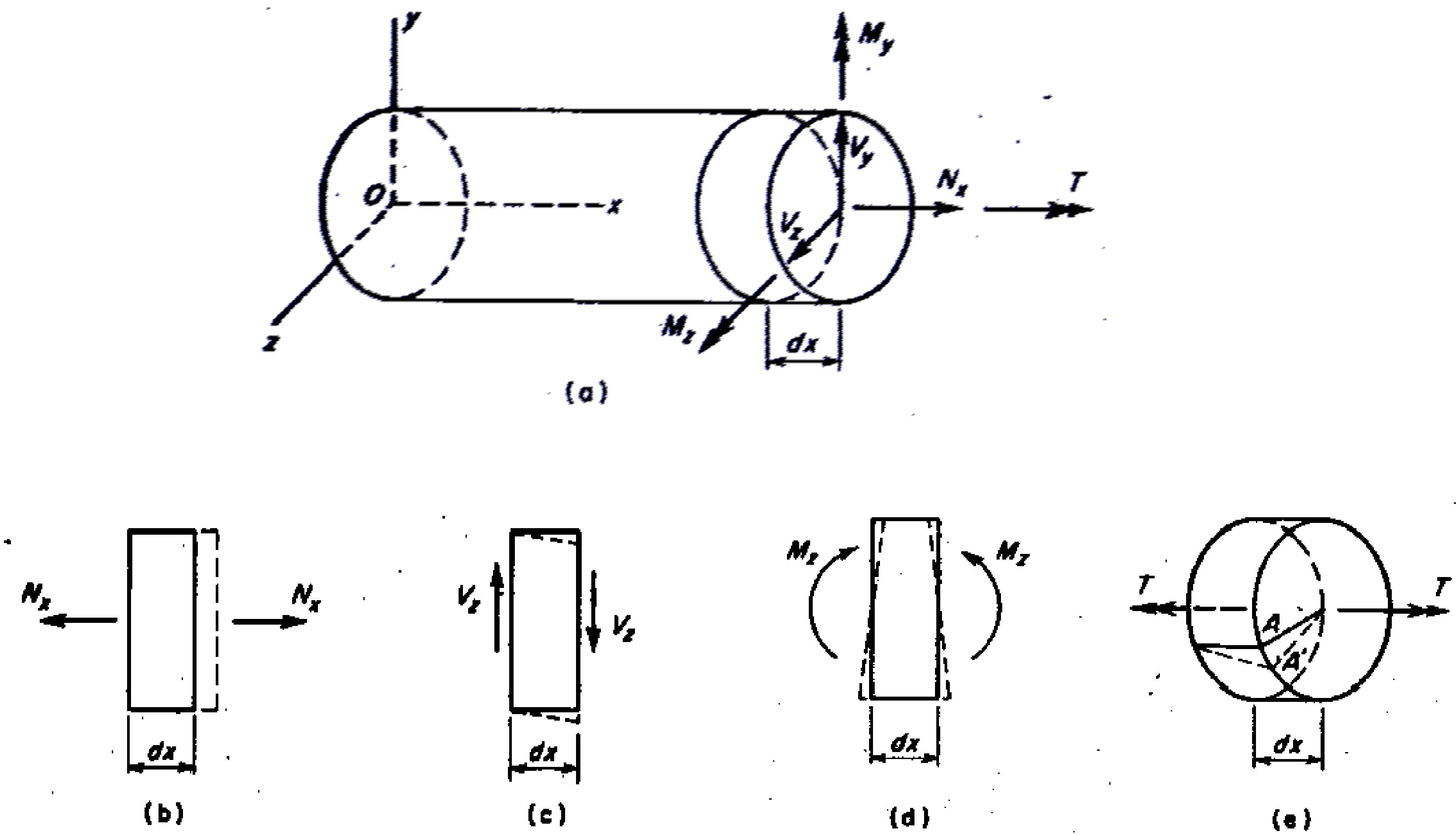
•f. Space frames: most general framed structure; may support bending moment, shear force, axial force and torsion

Deformations in Framed Structures

Three forces: N_x, V_y, V_z Three couples: T_x, M_y, M_z

- Significant deformations in framed structures:

Structure	**Significant deformations**
Beams	flexural
Plane trusses	axial
Space trusses	axial
Plane frames	flexural and axial
Grids	flexural and torsional
Space frames	axial, flexural and torsional



Types of deformations in framed structures
b) axial c) shearing d) flexural e) torsional

Static indeterminacy

- **Beam:**

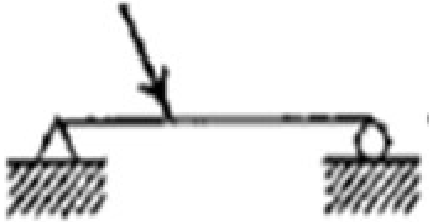
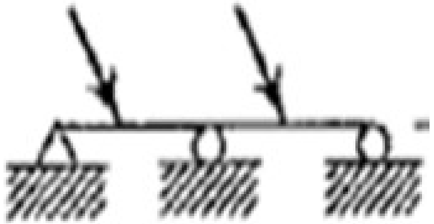
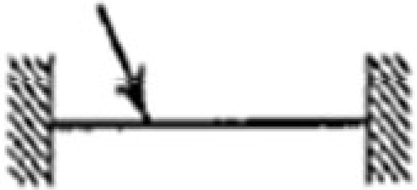
•Static indeterminacy = Reaction components - number of eqns available

$$E = R - 3$$

- Examples:

•Single span beam with both ends hinged with inclined loads

- Continuous beam
- Propped cantilever
- Fixed beam

	Structure	Static Indeterminacy
(a)		0
(b)		1
(c)		3

- **Rigid frame (Plane):**

  - External indeterminacy = Reaction components - number of eqns available $E = R - 3$

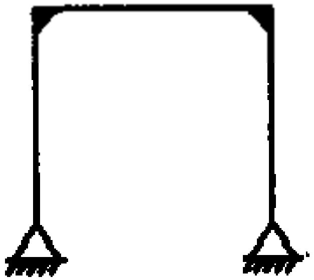
  - Internal indeterminacy = $3 \times$ closed frames $I = 3a$

  - Total indeterminacy
    = External indeterminacy + Internal indeterminacy

    $$T = E + I = (R - 3) + 3a$$

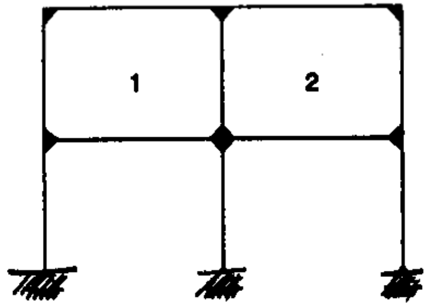
  - Note: An internal hinge will provide an additional eqn

Example 1



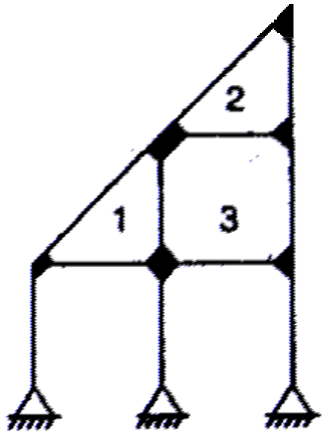
$$T = E + I = (R-3) + 3a$$
$$= (2\times 2 - 3) + 3\times 0 = 1$$

Example 2



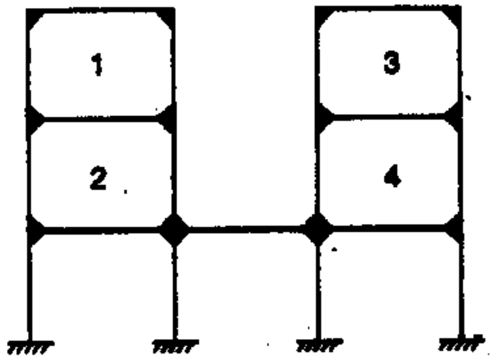
$$T = E + I = (R-3) + 3a$$
$$= (3\times 3 - 3) + 3\times 2 = 12$$

Example 3

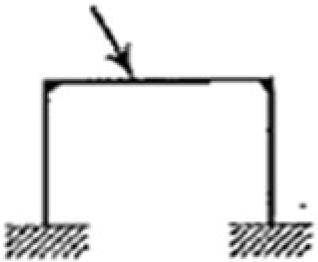
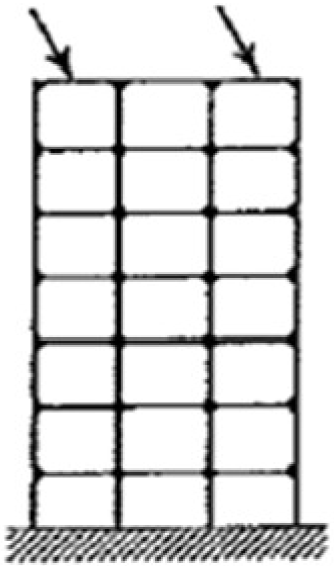


$$T = E + I = (R-3) + 3a$$
$$= (3\times 2 - 3) + 3\times 3 = 12$$

Example 4



$$T = E + I = (R-3) + 3a$$
$$= (4\times 3 - 3) + 3\times 4 = 21$$

Structure	Static Indeterminacy
	3
	63

- **Rigid frame (Space):**

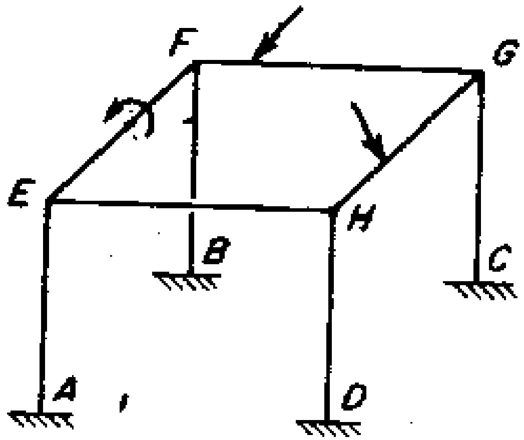
  - External indeterminacy = Reaction components - number of eqns available

$$E = R - 6$$

  - Internal indeterminacy = $6 \times$ closed frames

Example 1

$$T = E + I = (R - 6) + 6a$$
$$= (4 \times 6 - 6) + 6 \times 1 = 24$$



If axial deformations are neglected, static indeterminacy is not affected since the same number of actions still exist in the structure

- **Plane truss (general):**
  - External indeterminacy

    = Reaction components - number of eqns available

$$E = R - 3$$

  - Minimum 3 members and 3 joints.
  - A ny additional joint requires 2 additional members.
  - Hence, number of members for stability,

$$m = 3 + 2\left(j - 3\right) = 2j - 3$$

- Hence, internal indeterminacy, $I = m - (2j - 3)$
- Total (Internal and external) indeterminacy

$$T = E + I = R - 3 + m - (2j - 3)$$
$$= m + R - 2j$$

- m : number of members
- R : number of reaction components
- j : number of joints

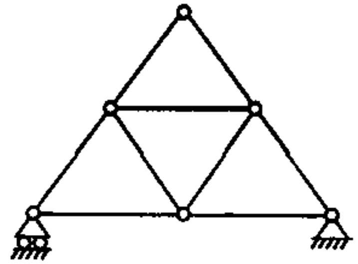
- Note: Internal hinge will provide additional eqn

Example 1

$$T = m + R - 2j = 9 + 3 - 2 \times 6 = 0$$

$$E = R - 3 = 3 - 3 = 0$$

$$I = T - E = 0$$

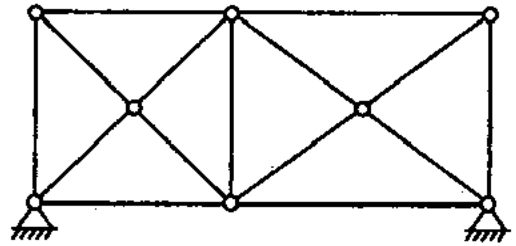


Example 2

$$T = m + R - 2j = 15 + 4 - 2 \times 8 = 3$$

$$E = R - 3 = 4 - 3 = 1$$

$$I = T - E = 2$$



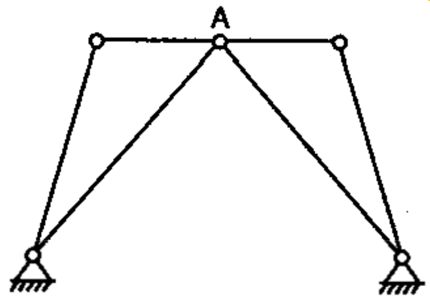
Example 3

$$T = m + R - 2j = 6 + 4 - 2 \times 5 = 0$$

$$E = R - (3 + 1) = 4 - 4 = 0$$

Hinge at A

$$I = T - E = 0$$

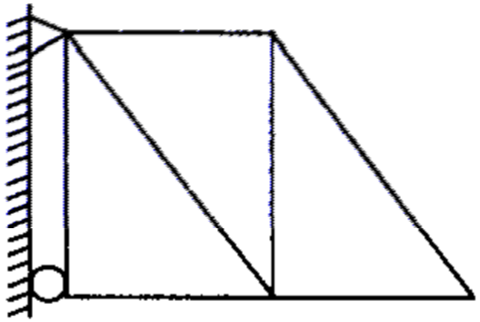


Example 4

$$T = m + R - 2j = 7 + 3 - 2 \times 5 = 0$$

$$E = R - 3 = 3 - 3 = 0$$

$$I = T - E = 0$$

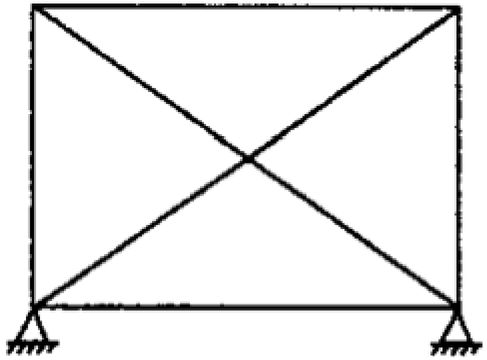


Example 5

$$T = m + R - 2j = 6 + 4 - 2 \times 4 = 2$$

$$E = R - 3 = 4 - 3 = 1$$

$$I = T - E = 1$$

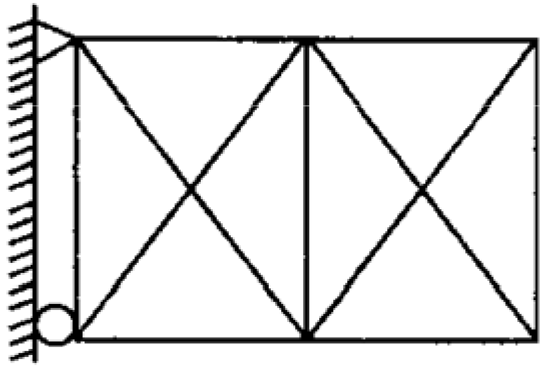


Example 6

$$T = m + R - 2j = 11 + 3 - 2 \times 6 = 2$$

$$E = R - 3 = 3 - 3 = 0$$

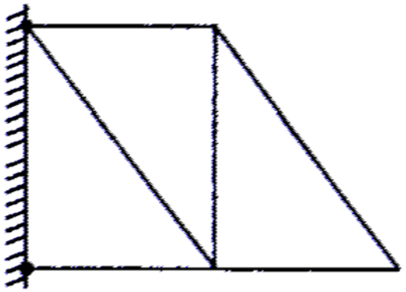
$$I = T - E = 2$$



- Wall or roof attached pin jointed plane truss (Exception to the above general case):
  - Internal indeterminacy $I = m - 2j$

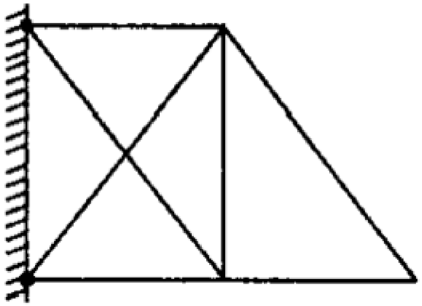
- External indeterminacy = 0 (Since, once the member forces are determined, reactions are determinable)

Example 1



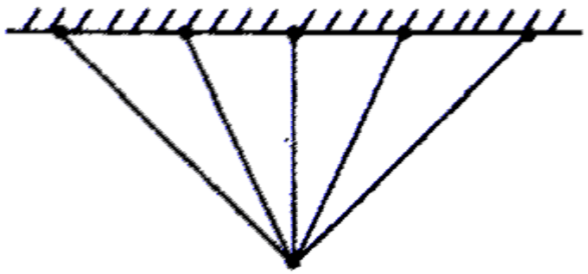
$$T = I = m - 2j$$
$$= 6 - 2 \times 3 = 0$$

Example 2



$$T = I = m - 2j$$
$$= 7 - 2 \times 3 = 1$$

Example 3

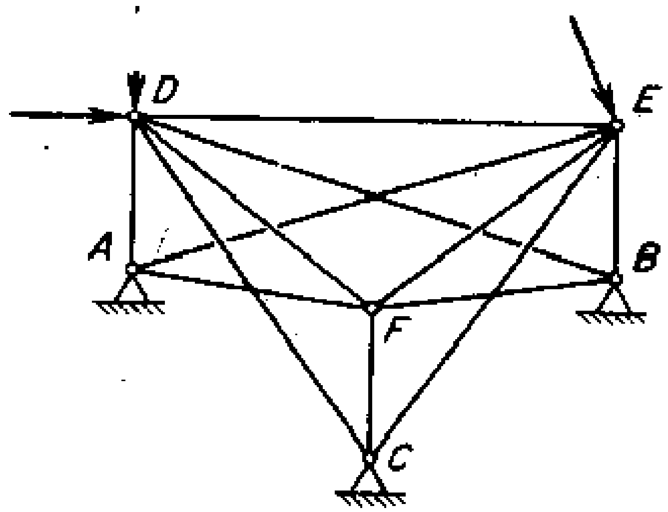


$$T = I = m - 2j$$
$$= 5 - 2 \times 1 = 3$$

- Space Truss:
    - External indeterminacy = Reaction components - number of equations available $E = R - 6$

    - Total (Internal and external) indeterminacy $T = m + R - 3j$

Example 1



- Total (Internal and external) indeterminacy

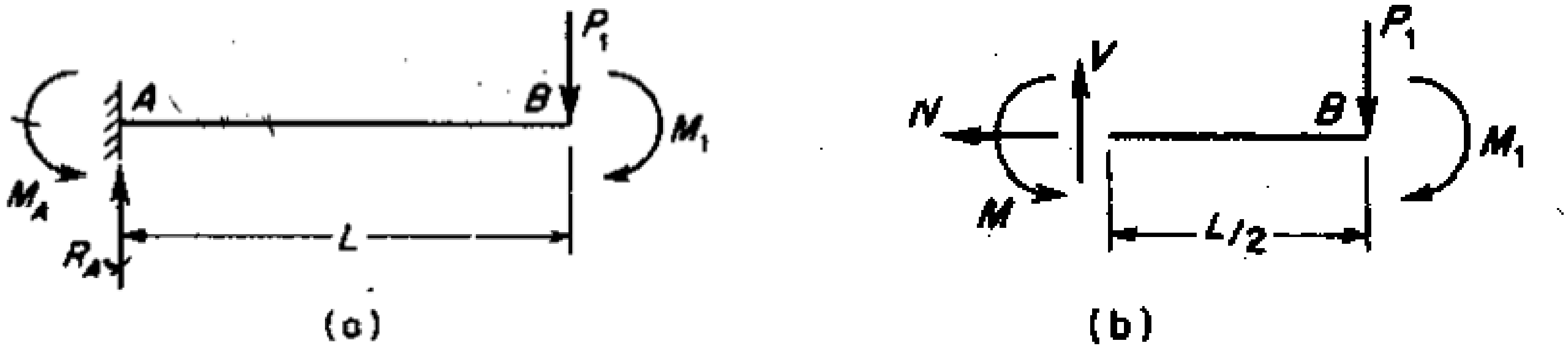
$$T = m + R - 3j$$

$$\therefore T = 12 + 9 - 3 \times 6 = 3$$

$$E = R - 6 = 9 - 6 = 3$$

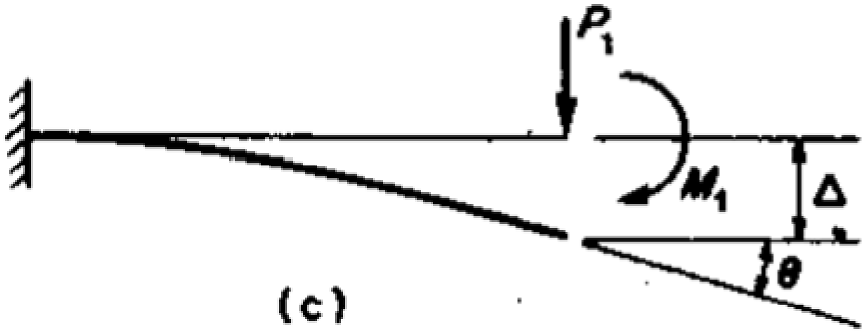
Actions and displacements

- Actions:
  - External actions (Force or couple or combinations) and
  - Internal actions (Internal stress resultants – BM, SF, axial forces, twisting moments)

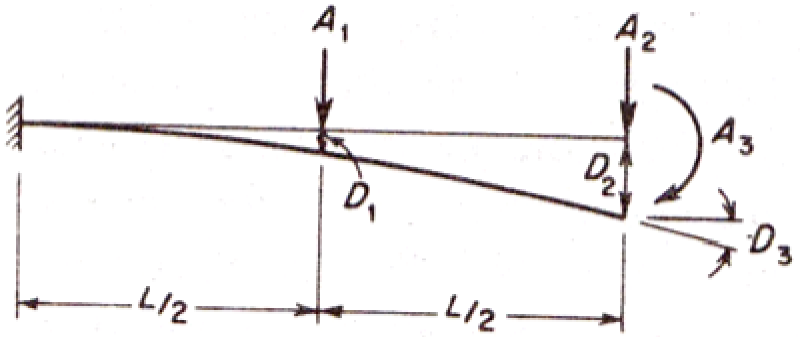


• Displacements: A translation or rotation at some point

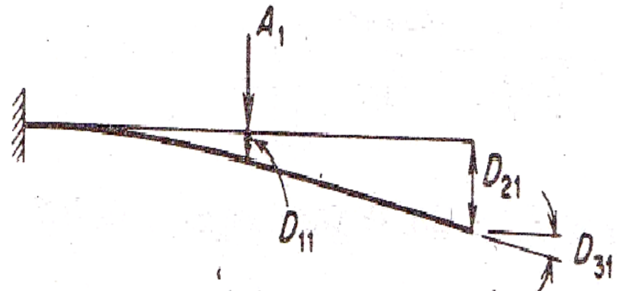
• Displacement *corresponding* to an action: Need not be *caused* by that action



- Notations for actions and displacements:



(a)



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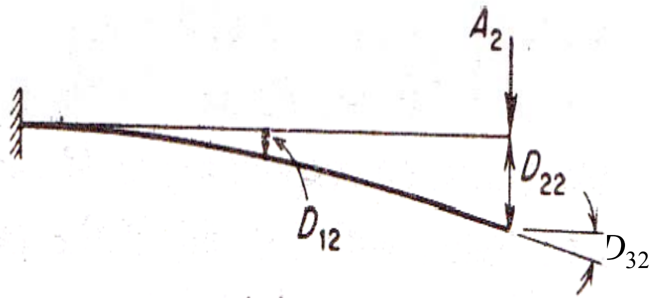

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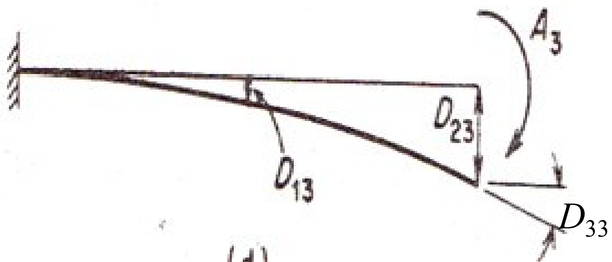


(d)

Equilibrium

- Resultant of all actions (a force, a couple or both) must vanish for static equilibrium

- Resultant *force* vector must be zero; resultant *moment* vector must be zero

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$

$$\sum M_x = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 0$$

- For 2-dimensional problems (forces are in one plane and couples have vectors normal to the plane),

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = 0$$

- In stiffness method, the basic equations to be solved are the equilibrium conditions at the joints

Compatibility

- Compatibility conditions: Conditions of continuity of displacements throughout the structure

- Eg: at a rigid connection between two members, the displacements (translations and rotations) of both members must be the same

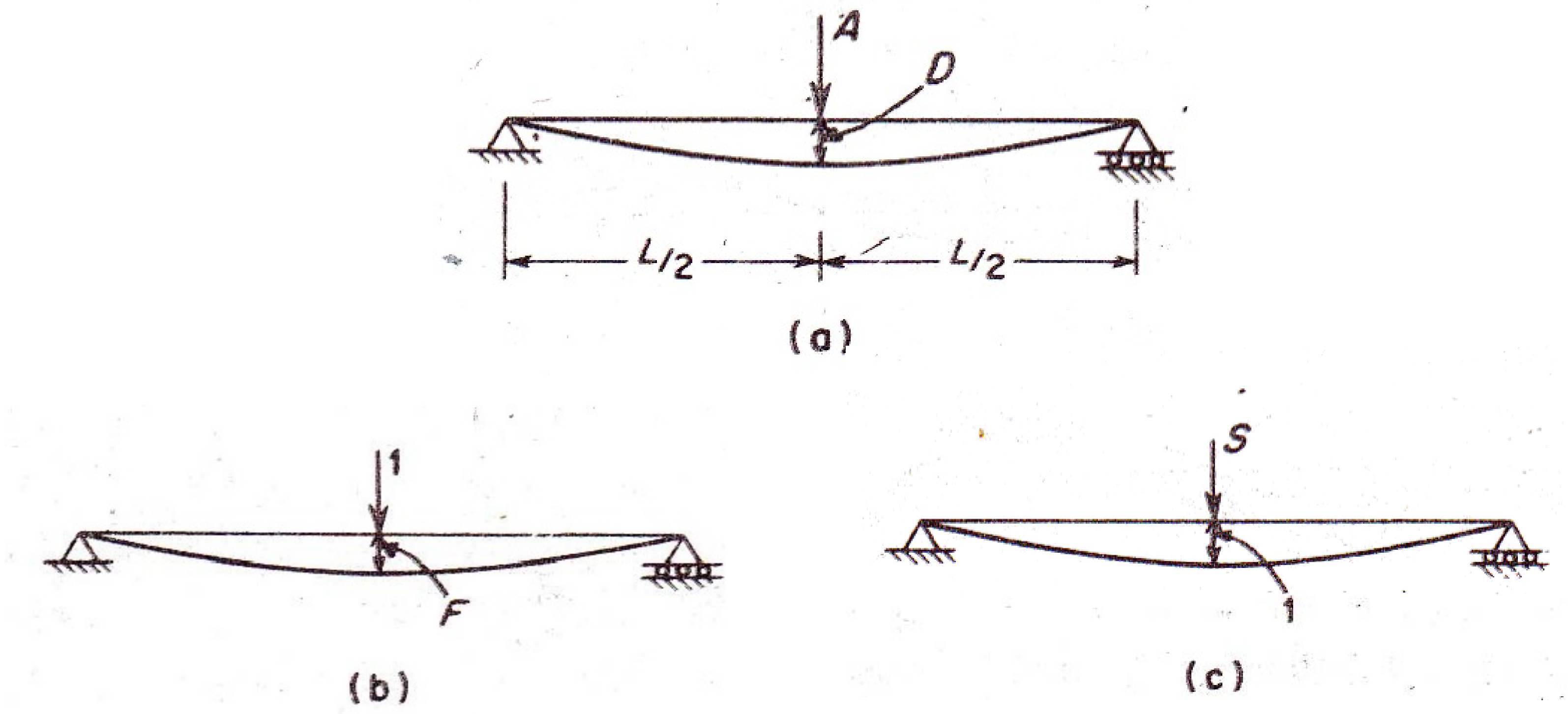
  - In flexibility method, the basic equations to be solved are the compatibility conditions

Action and displacement equations

- Spring: $D = FA$ $A = SD$
- Stiffness $S = F^{-1}$
- Flexibility: $F = S^{-1}$
- The above equations apply to any linearly elastic structure

•Example 1:

Flexibility and stiffness of a beam subjected to a single load

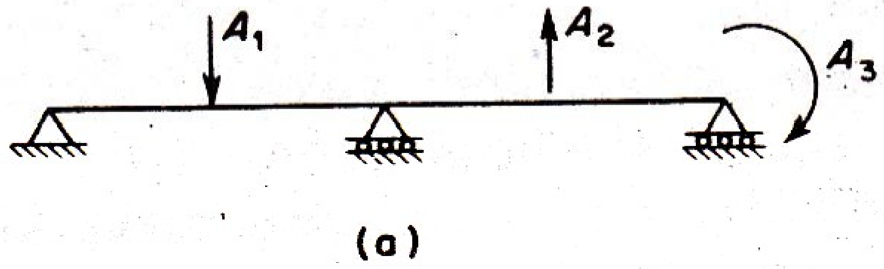


Flexibility $F = \dfrac{L^3}{48EI}$

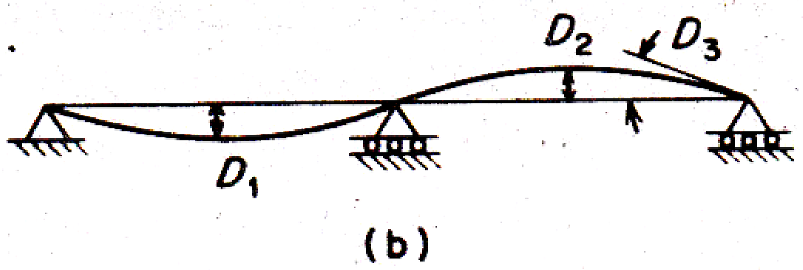
Stiffness $S = \dfrac{48EI}{L^3}$

- Example 2:

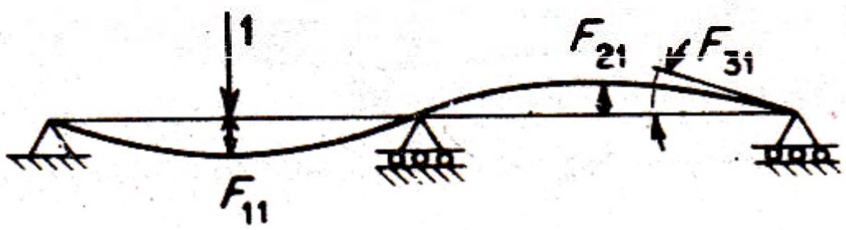
Flexibility coefficients of a beam subjected to several loads



Actions on the beam



Deformations corresponding to actions



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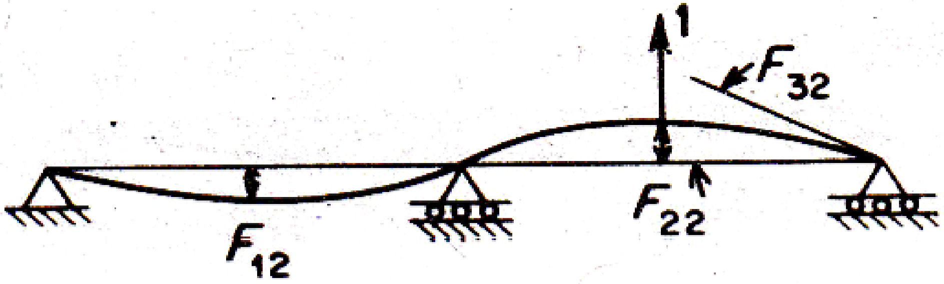
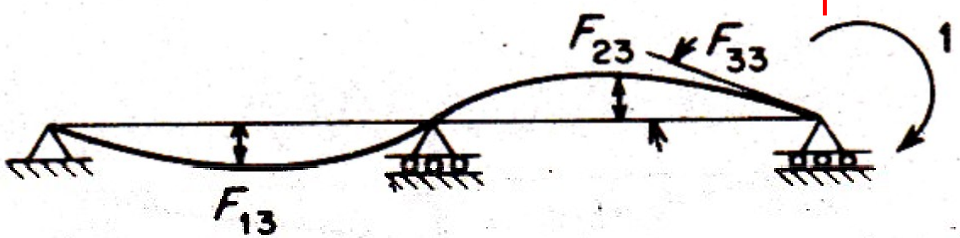


(d)



(e)

Unit load applied corresponding to each action, separately

$$D_1 = D_{11} + D_{12} + D_{13} \qquad D_2 = D_{21} + D_{22} + D_{23} \qquad D_3 = D_{31} + D_{32} + D_{33}$$

$$D_1 = F_{11}A_1 + F_{12}A_2 + F_{13}A_3$$

$$D_2 = F_{21}A_1 + F_{22}A_2 + F_{23}A_3$$

$$D_3 = F_{31}A_1 + F_{32}A_2 + F_{33}A_3$$

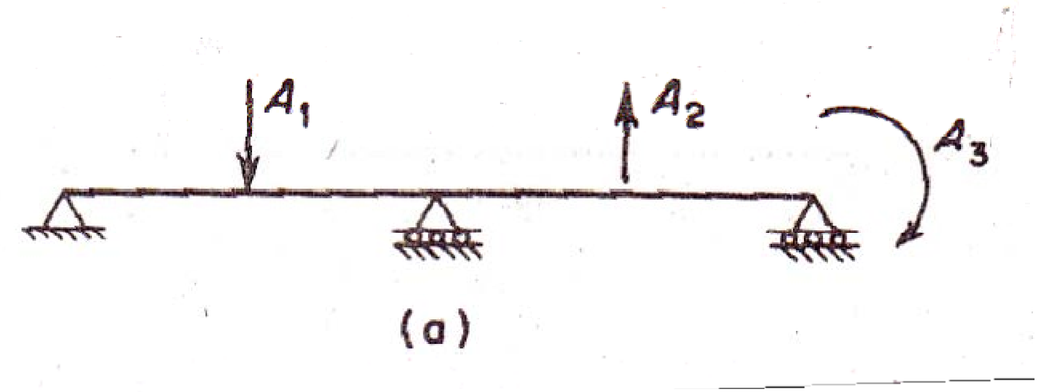
F_{11}, F_{12}, F_{13} etc. → Flexibility coefficients

- Flexibility coefficient F_{12}: Displacement corresponding to A_1 caused by a unit value of A_2.

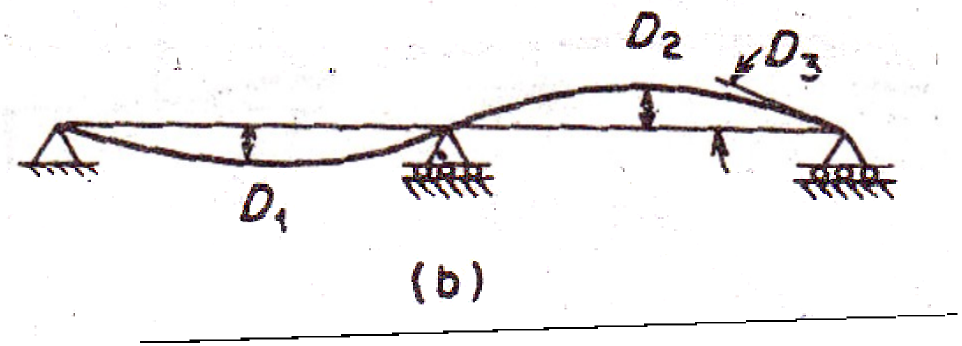
- In general, flexibility coefficient $\boldsymbol{F_{ij}}$ is the displacement corresponding to $\boldsymbol{A_i}$ caused by a unit value of $\boldsymbol{A_j}$.

• Example 3:

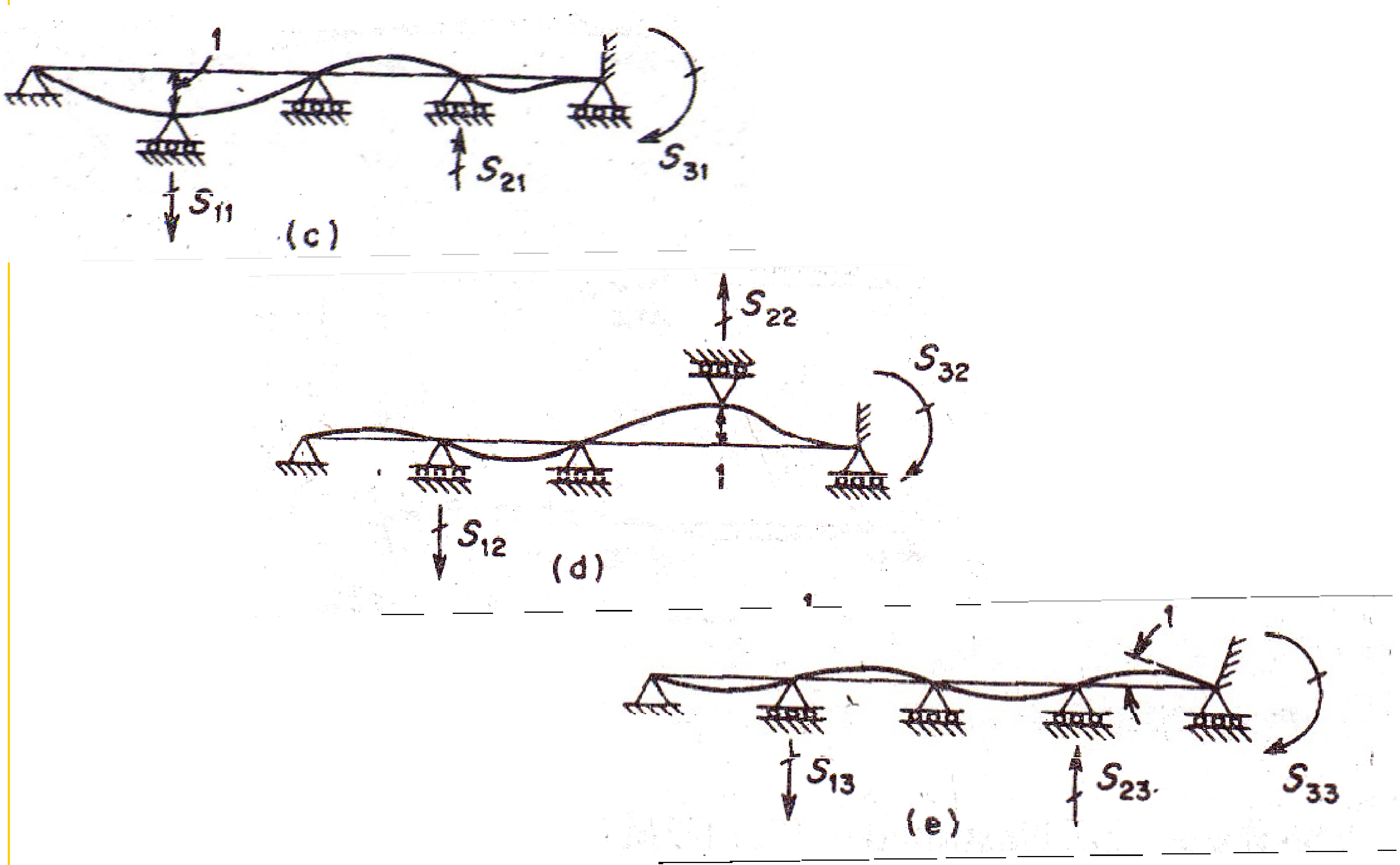
Stiffness coefficients of a beam subjected to several loads



Actions on the beam



Deformations corresponding to actions



Unit displacement applied corresponding to each DOF, separately, keeping all other displacements zero

$$A_1 = A_{11} + A_{12} + A_{13}$$

$$A_1 = S_{11}D_1 + S_{12}D_2 + S_{13}D_3$$

$$A_2 = S_{21}D_1 + S_{22}D_2 + S_{23}D_3$$

$$A_3 = S_{31}D_1 + S_{32}D_2 + S_{33}D_3$$

S_{11}, S_{12}, S_{13} etc. → Stiffness coefficients:

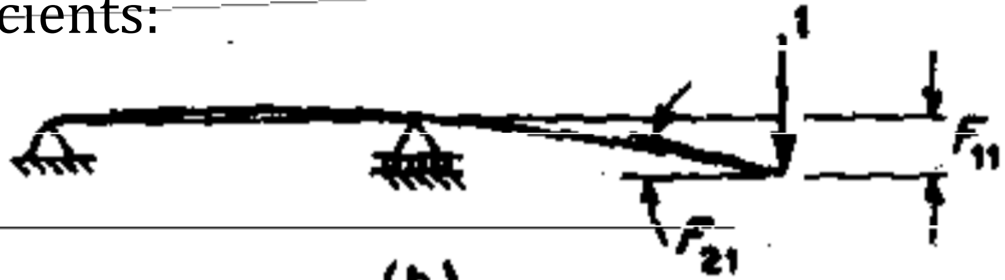
•Stiffness coefficient S_{12}: Action corresponding to D_1 caused by a unit value of D_2.

•In general, stiffness coefficient S_{ij} is the action corresponding to D_i caused by a unit value of D_j.

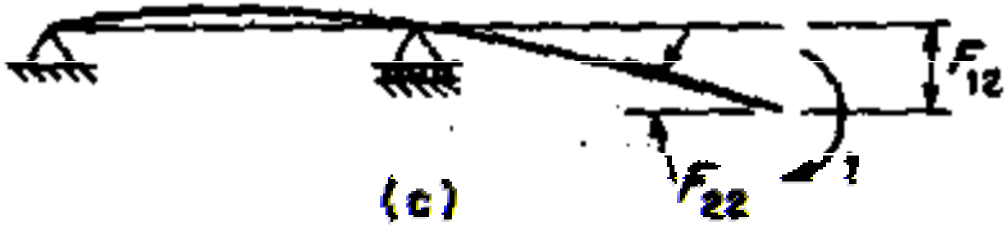
- Example 4: Flexibility and stiffness coefficients of a beam

(a)

Flexibility coefficients:

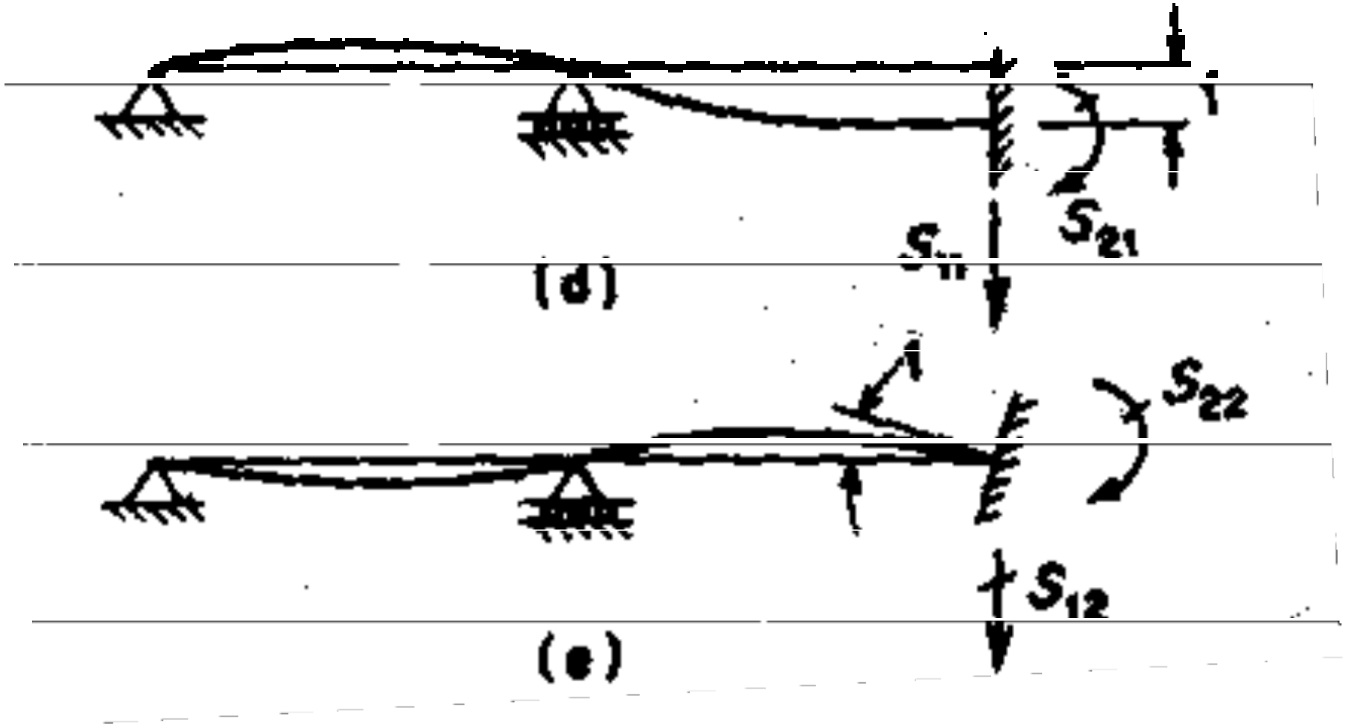


(b)

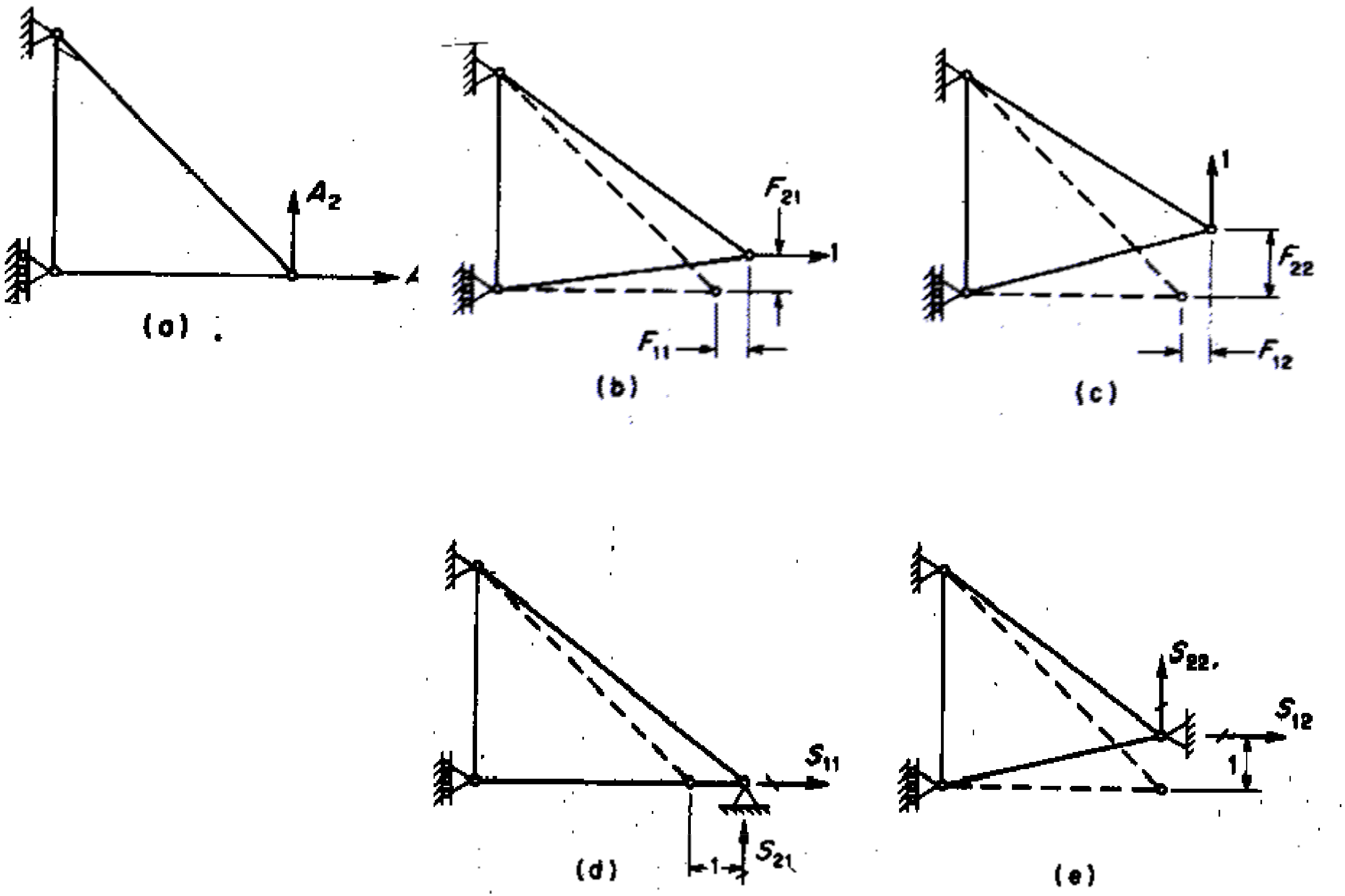


(c)

Stiffness coefficients:



• Example 5: Flexibility and stiffness coefficients of a truss



Flexibility and stiffness matrices

1. Flexibility matrix

- The compatibility equations are:

$$D_1 = F_{11}A_1 + F_{12}A_2 + F_{13}A_3 + \ldots + F_{1n}A_n$$

$$D_2 = F_{21}A_1 + F_{22}A_2 + F_{23}A_3 + \ldots + F_{2n}A_n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$D_n = F_{n1}A_1 + F_{n2}A_2 + F_{n3}A_3 + \ldots + F_{nn}A_n$$

- In matrix form,

$$\begin{Bmatrix} D_1 \\ D_2 \\ \ldots \\ D_n \end{Bmatrix}_{n\times 1} = \begin{bmatrix} F_{11} & F_{12} & \ldots & F_{1n} \\ F_{21} & F_{22} & \ldots & F_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ F_{n1} & F_{n2} & \ldots & F_{nn} \end{bmatrix}_{n\times n} \begin{Bmatrix} A_1 \\ A_2 \\ \ldots \\ A_n \end{Bmatrix}_{n\times 1}$$

$$\boldsymbol{D} = \boldsymbol{F}\boldsymbol{A}$$

$$\{D\} = [F]\{A\}$$

$\{D\}$ Displacement matrix (vector),

$[F]$ Flexibility matrix,

$\{A\}$ Action matrix (vector)

F_{ij} are the flexibility coefficients

2. Stiffness matrix

- The equilibrium equations are:

$$A_1 = S_{11}D_1 + S_{12}D_2 + S_{13}D_3 + \ldots + S_{1n}D_n$$

$$A_2 = S_{21}D_1 + S_{22}D_2 + S_{23}D_3 + \ldots + S_{2n}D_n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_n = S_{n1}D_1 + S_{n2}D_2 + S_{n3}D_3 + \ldots + S_{nn}D_n$$

- In matrix form,

$$\underset{n\times 1}{\begin{Bmatrix} A_1 \\ A_2 \\ \ldots \\ A_n \end{Bmatrix}} = \underset{n\times n}{\begin{bmatrix} S_{11} & S_{12} & \ldots & S_{1n} \\ S_{21} & S_{22} & \ldots & S_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ S_{n1} & S_{n2} & \ldots & S_{nn} \end{bmatrix}} \underset{n\times 1}{\begin{Bmatrix} D_1 \\ D_2 \\ \ldots \\ D_n \end{Bmatrix}}$$

$$A = SD$$

$$\{A\} = [S]\{D\}$$

$\{A\}$ Action matrix (vector) ,

$[S]$ Stiffness matrix,

$\{D\}$ Displacement matrix (vector)

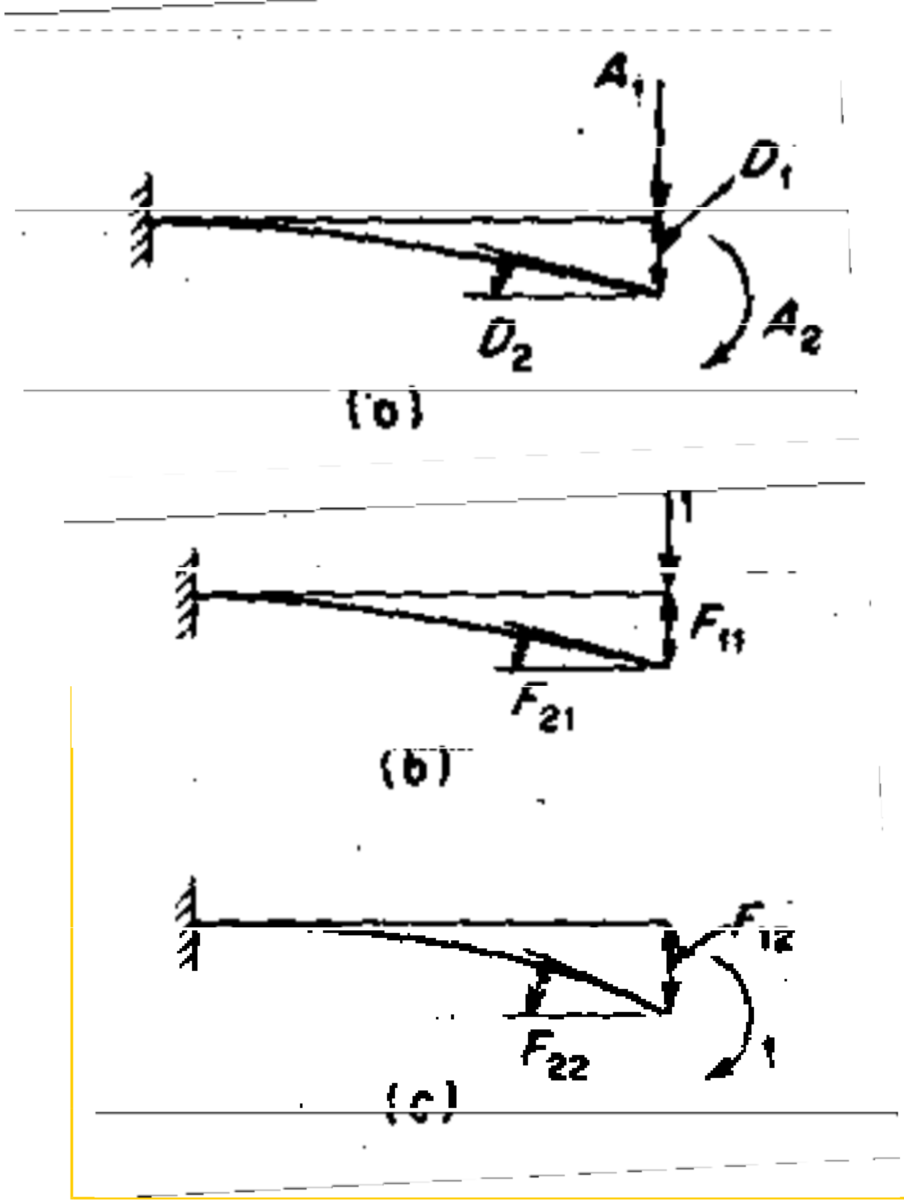
S_{ij} are the stiffness coefficients

- Relationship between flexibility and stiffness matrices

$$A = S \quad D = S \quad F \quad A$$

$$\therefore F = S^{-1}$$

• Example: Cantilever element



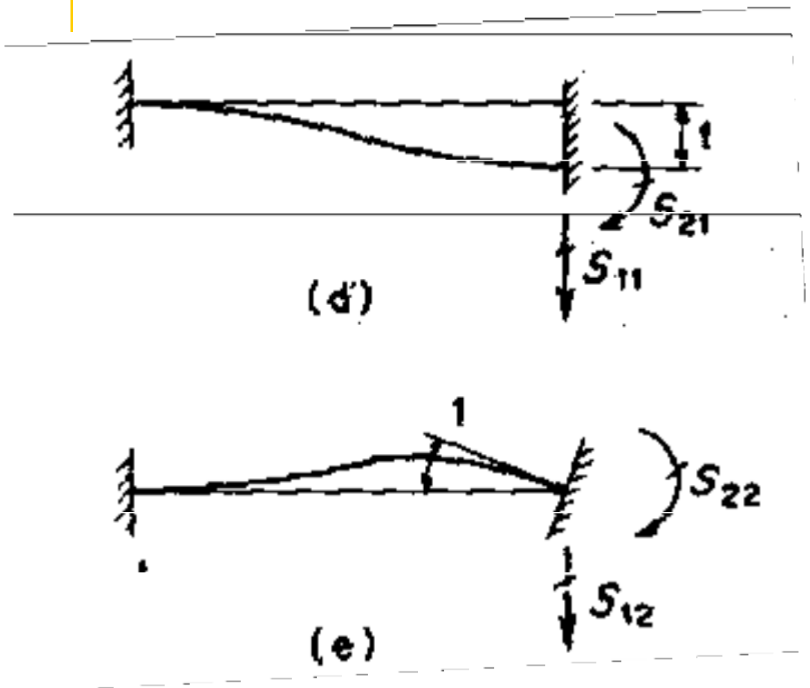
$$F_{11} = \frac{L^3}{3EI}; \quad F_{21} = F_{12} = \frac{L^2}{2EI}; F_{22} = \frac{L}{EI}$$

$$D_1 = \frac{L^3}{3EI} A_1 + \frac{L^2}{2EI} A_2$$

$$D_2 = \frac{L^2}{2EI} A_1 + \frac{L}{EI} A_2$$

$$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{bmatrix} \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$$

$[F]$



$$S_{11} = \frac{12EI}{L^3}; \quad S_{21} = S_{12} = \frac{-6EI}{L^2}; \quad S_{22} = \frac{4EI}{L}$$

$$A_1 = \frac{12EI}{L^3} D_1 - \frac{6EI}{L^2} D_2$$

$$A_2 = \frac{-6EI}{L^2} D_1 + \frac{4EI}{L} D_2$$

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}$$

$[S$

$$[F][S] = [S][F] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I]$$

- Flexibility matrix and stiffness matrix are relating actions and *corresponding* displacements

The flexibility matrix $[F]$ obtained for a structure analysed by flexibility method may not be the inverse of the stiffness matrix $[S]$ obtained for the same structure analysed by stiffness method because different sets of actions and corresponding displacements may be utilized in the two methods.

Equivalent joint loads

- Analysis by flexibility and stiffness methods requires that loads must act only at joints.

- Thus, loads acting on the members (i.e., loads that are not acting at the joints) must be replaced by equivalent loads acting at the joints.

- The loads that are determined from loads on the members are called equivalent joint loads.

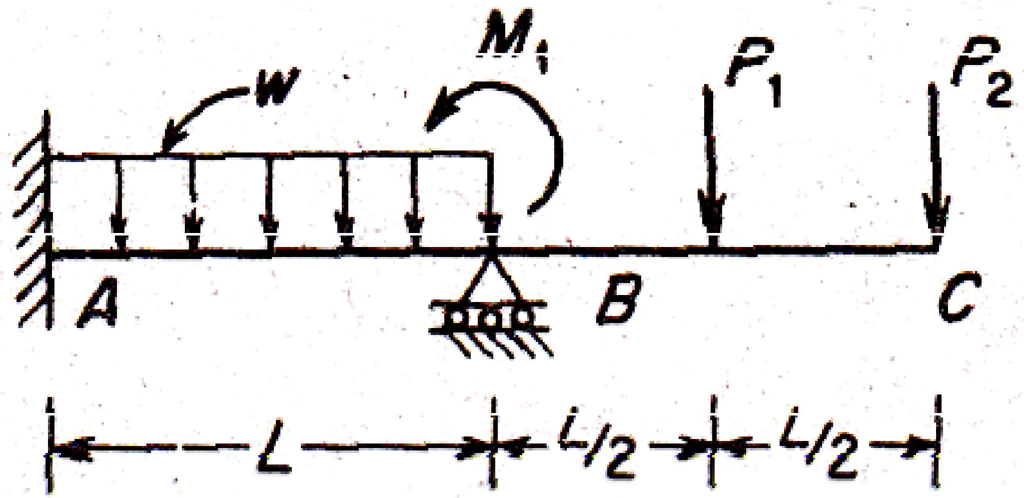
- Equivalent joint loads are added to the actual joint loads to get combined joint loads.

- Analysis carried out for combined joint loads

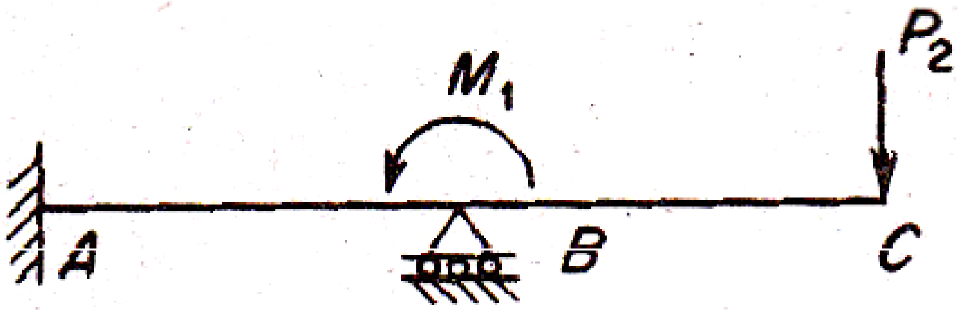
- Combined joint loads can be evaluated in such a manner that the resulting displacements of the structure are same as the displacements produced by the actual loads

- This is achieved thru the use of fixed end actions to get equivalent joint loads

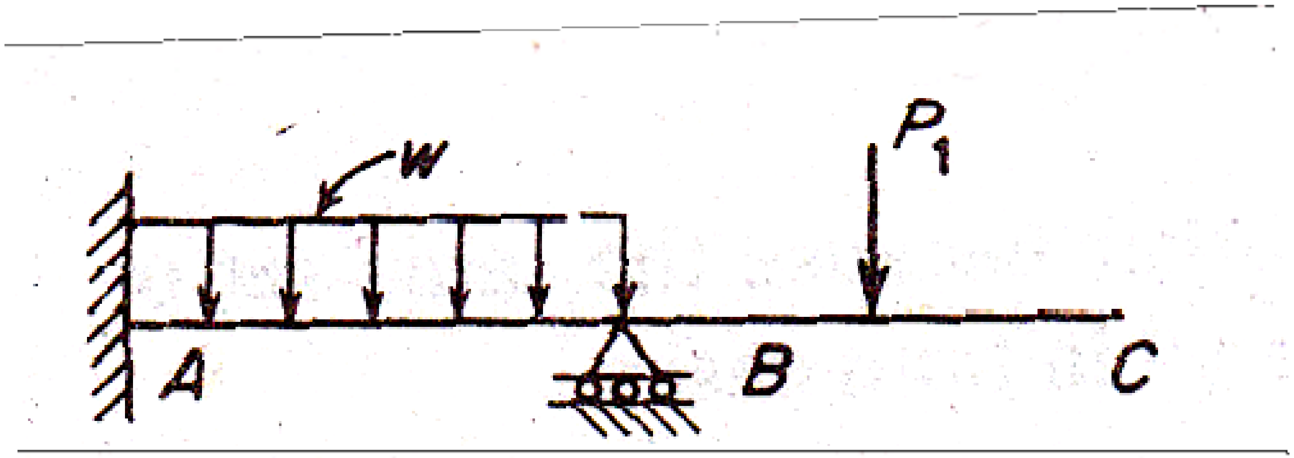
- Example 1



Beam with actual applied loads

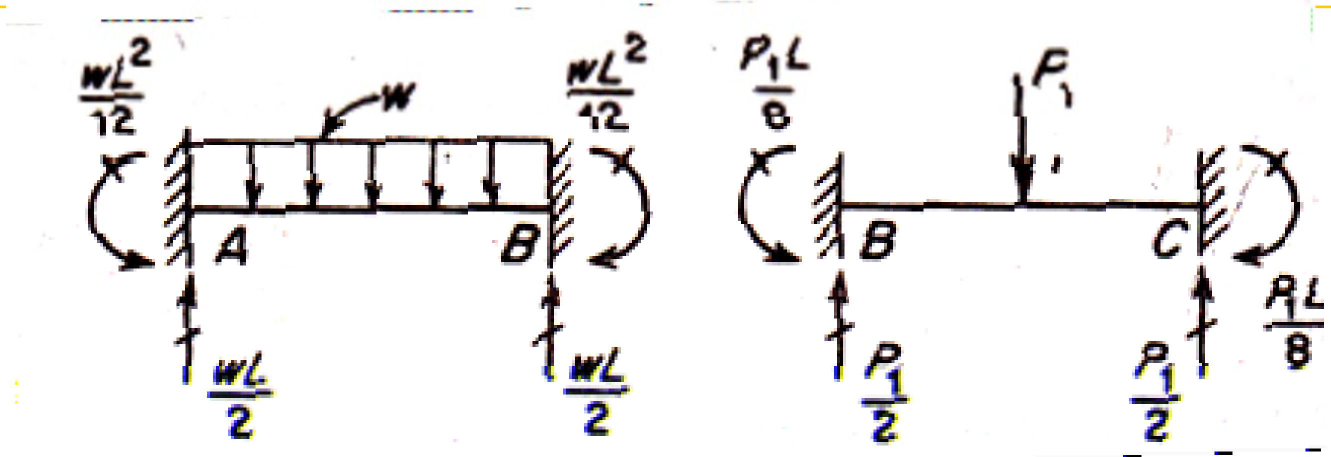


Applied joint loads



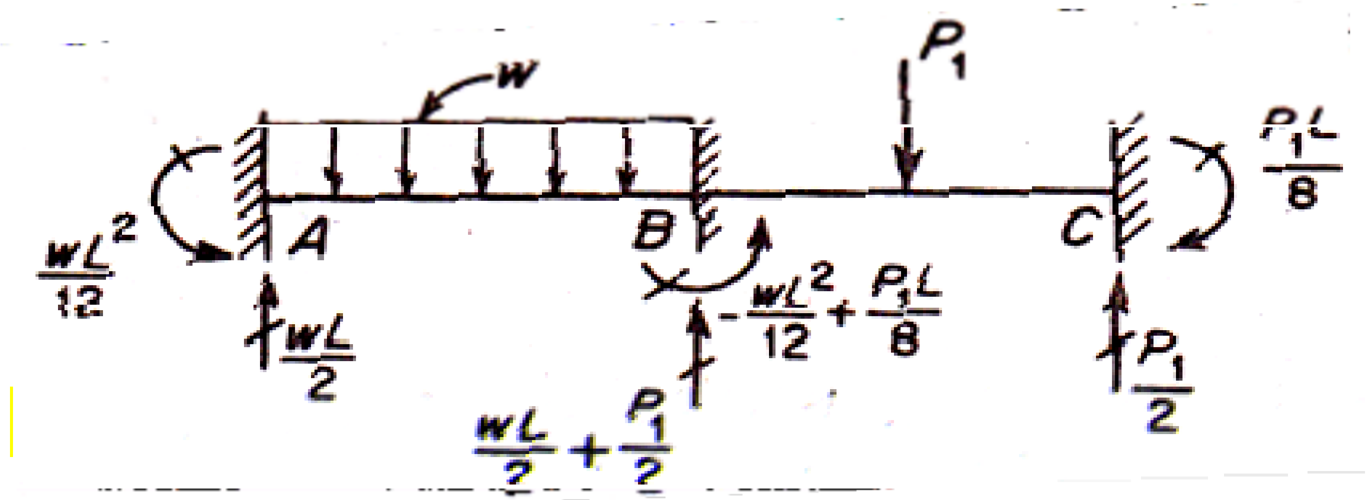
Applied loads other than joint loads

(To be converted to equivalent joint loads)

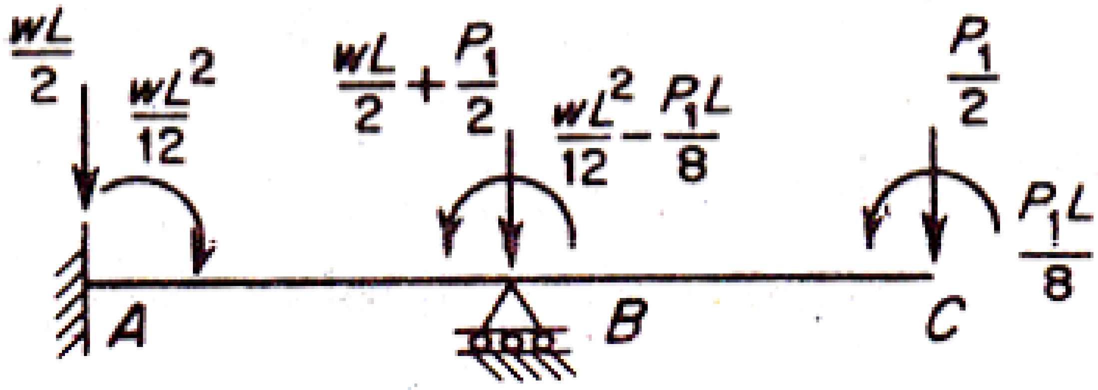


Member fixed end actions

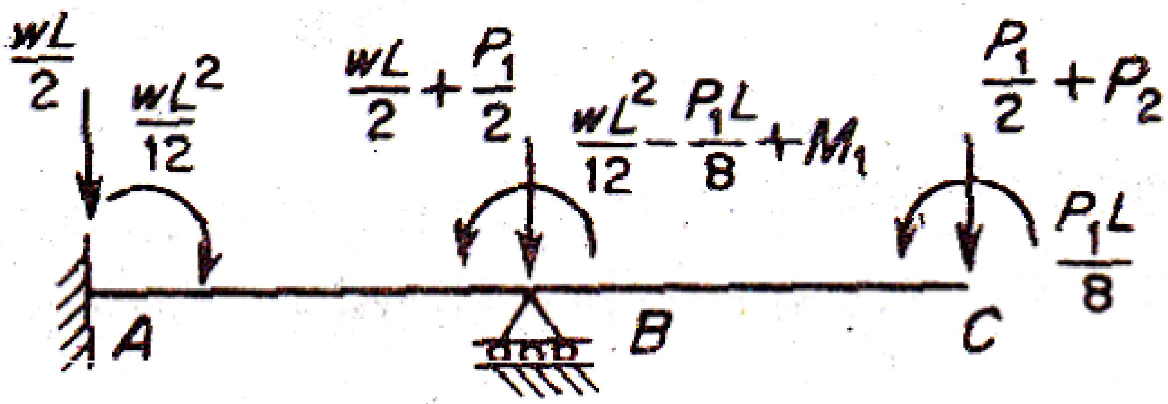
(Due to applied loads other than joint loads)



Fixed end actions for the entire beam

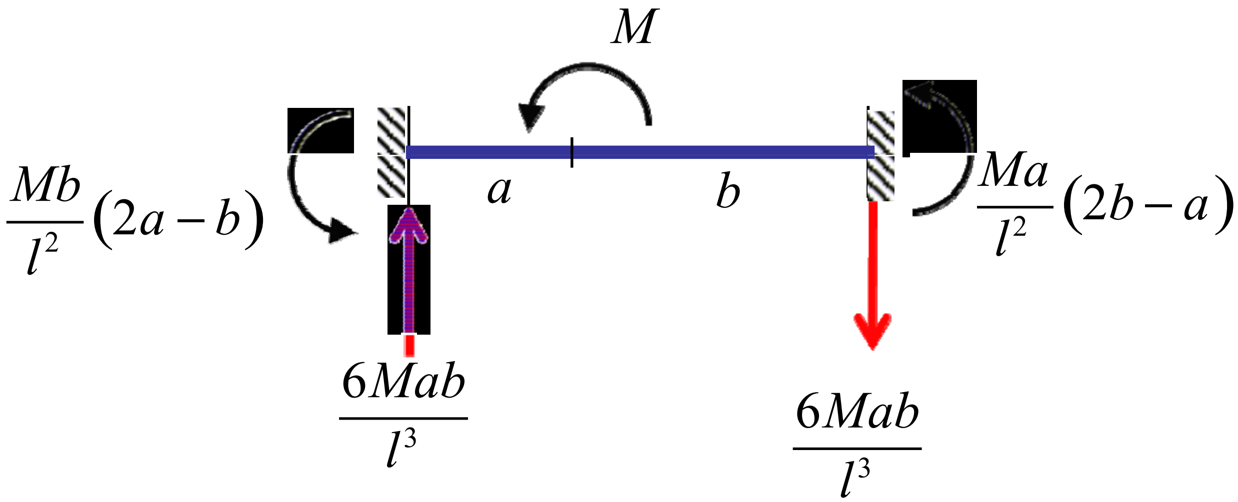


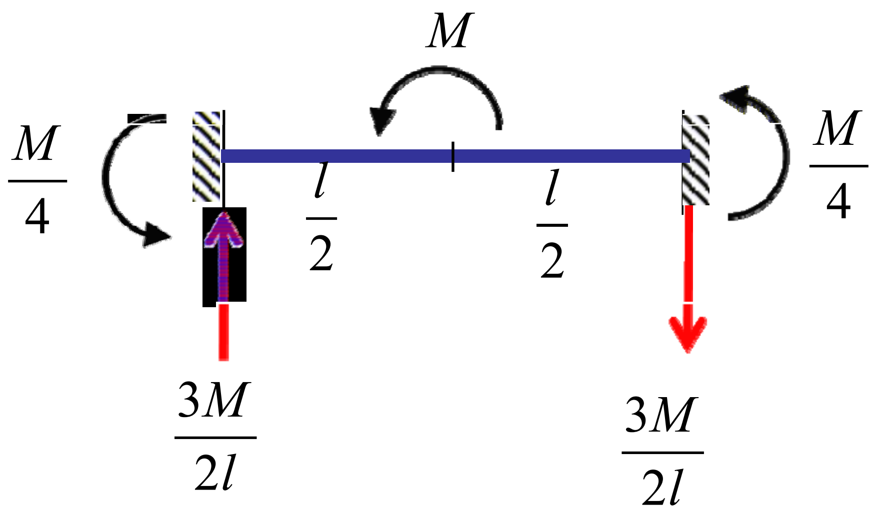
Equivalent joint loads (Negative of fixed end actions)



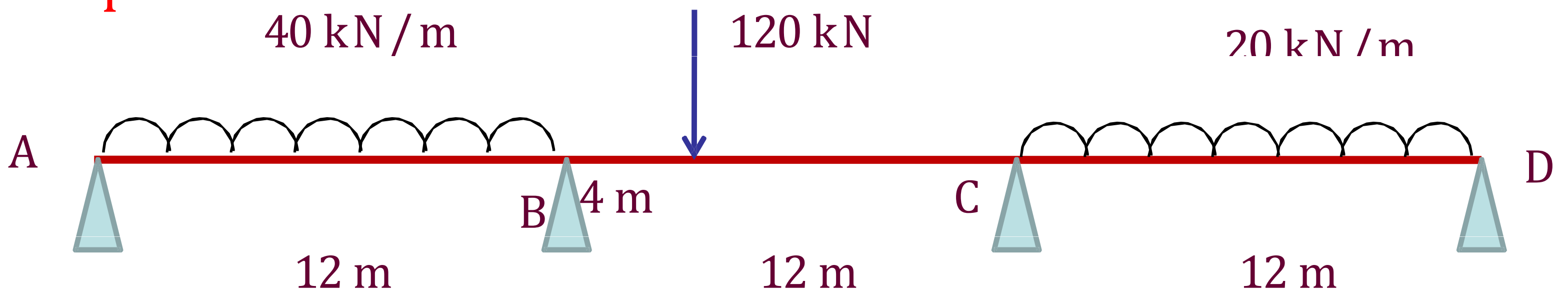
Combined joint loads
(Applied joint loads + Equivalent joint loads)

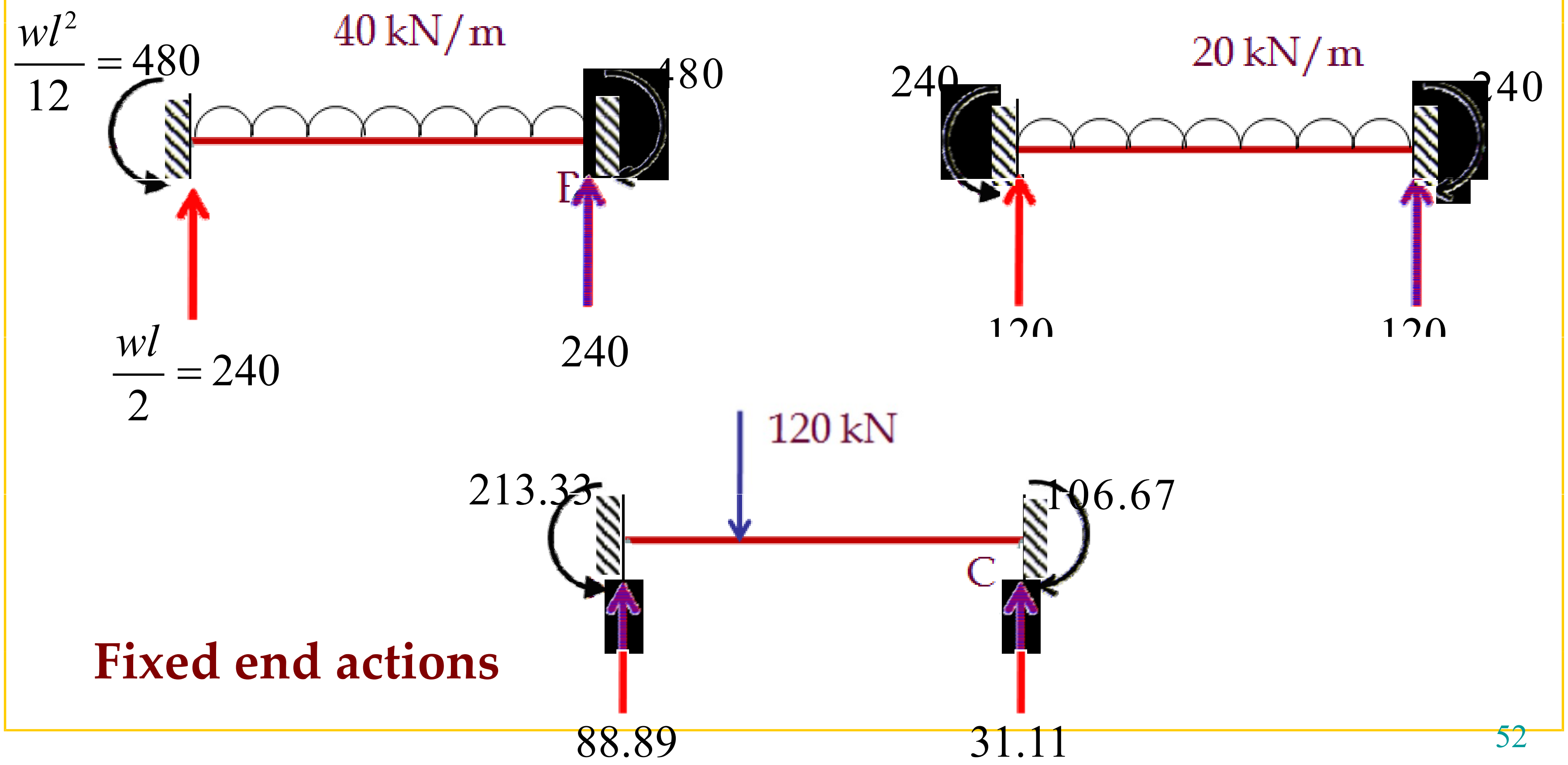
Fixed end actions



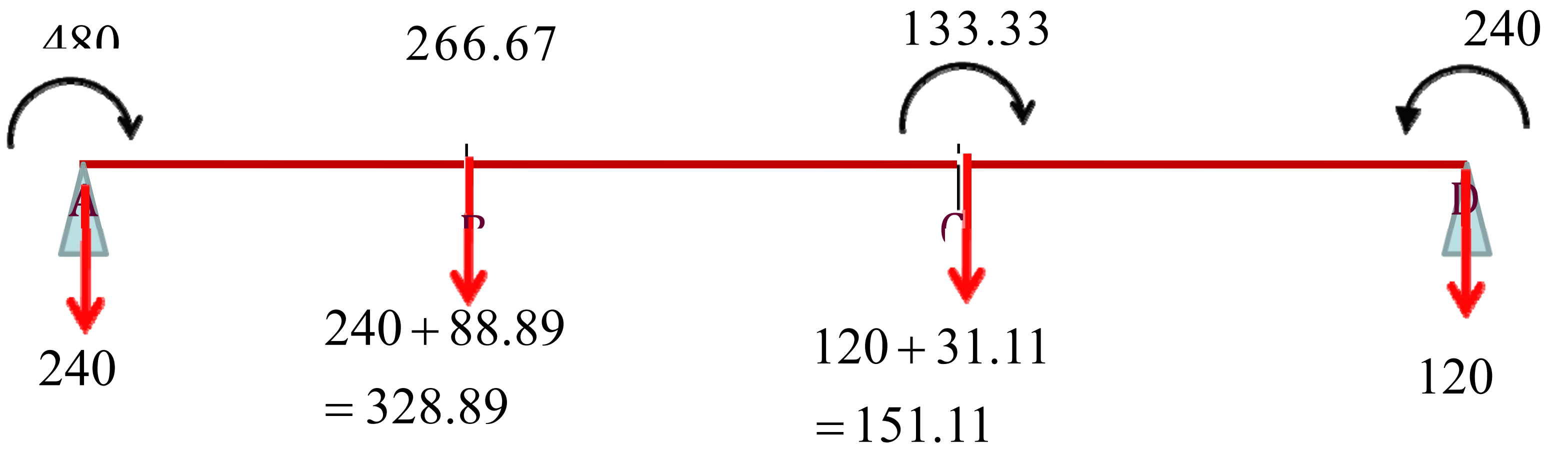


- Example 2





Fixed end actions



Equivalent joint loads
(Opposite of fixed end actions)

Combined joint loads are same as equivalent joint loads here, since there are no loads applied to joints directly

- Superposition of combined joint loads and restraint actions gives the actual loads.

  - Superposition of joint displacements due to the combined joint loads and restraint actions gives the displacements produced by the actual loads.

  - But joint displacements due to restraint actions are zero. Thus, joint displacements due to the combined joint loads give the displacements produced by the actual loads

- But member end actions due to actual loads are obtained by superimposing member end actions due to restraint actions and combined joint loads

Summary

Matrix analysis of structures

- **Definition of flexibility and stiffness influence coefficients – development of flexibility matrices by physical approach & energy principle.**