Structural Analysis - II

Plastic Analysis

Module IV

Plastic Theory

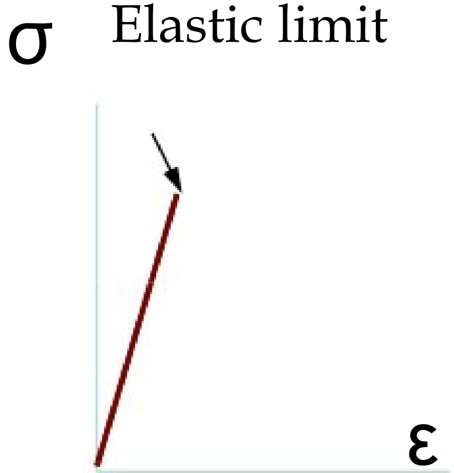
- Introduction-Plastic hinge concept-plastic section modulus-shape factor-redistribution of moments-collapse mechanism-
- Theorems of plastic analysis - Static/lower bound theorem; Kinematic/upper bound theorem-Plastic analysis of beams and portal frames by equilibrium and mechanism methods.

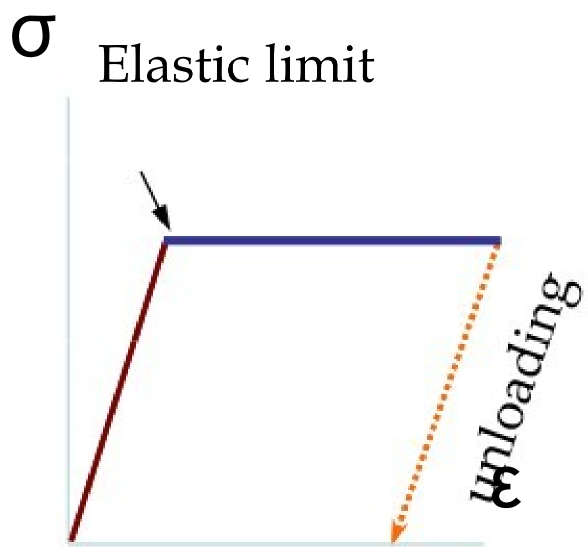
Plastic Analysis -Why? What?

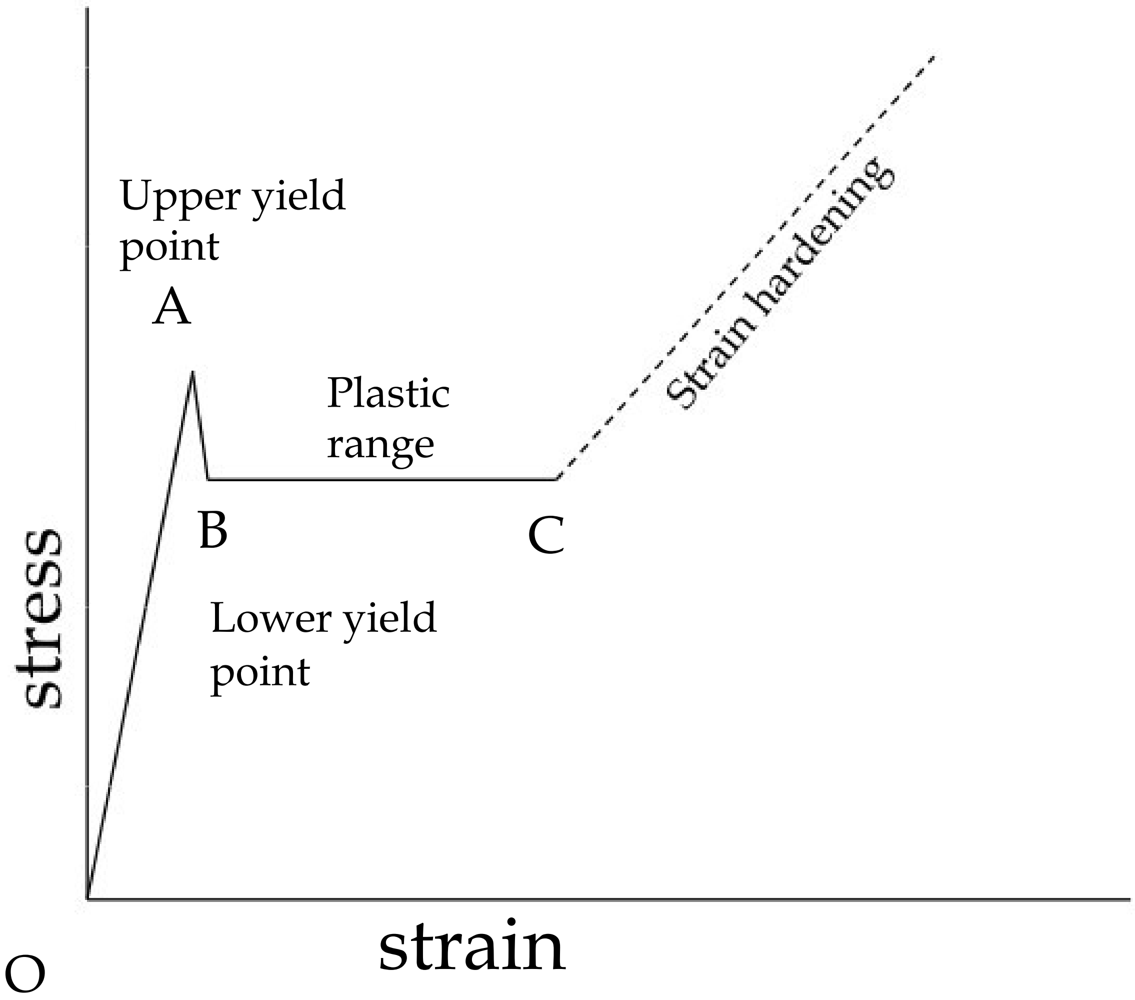
- Behaviour beyond elastic limit?
- Plastic deformation - collapse load
- Safe load – load factor
- Design based on collapse (ultimate) load – limit design
- Economic - Optimum use of material

Materials

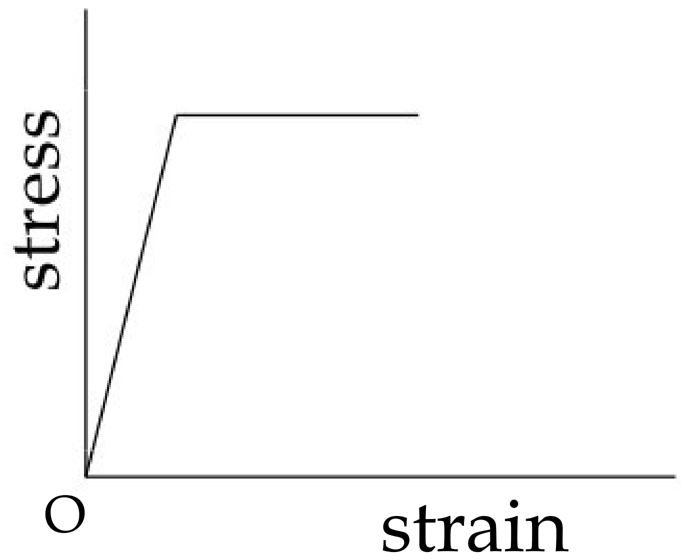
- Elastic
- Elastic-Perfectly plastic







Idealised stress-strain curve of mild steel



Idealised stress-strain curve in plastic theory

- Elastic analysis

  - Material is in the elastic state

  - Performance of structures under service loads

  - Deformation increases with increasing load

- Plastic analysis

  - Material is in the plastic state

  - Performance of structures under ultimate/collapse loads

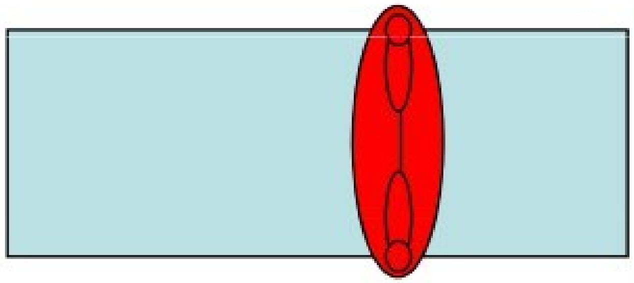
  - Deformation/Curvature increases without an increase in load.

Assumptions

- Plane sections remain plane in plastic condition
- Stress-strain relation is identical both in compression and tension

Process of yielding of a section

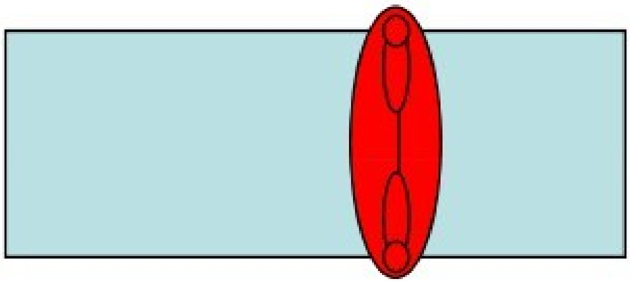
- Let M at a cross-section increases gradually.

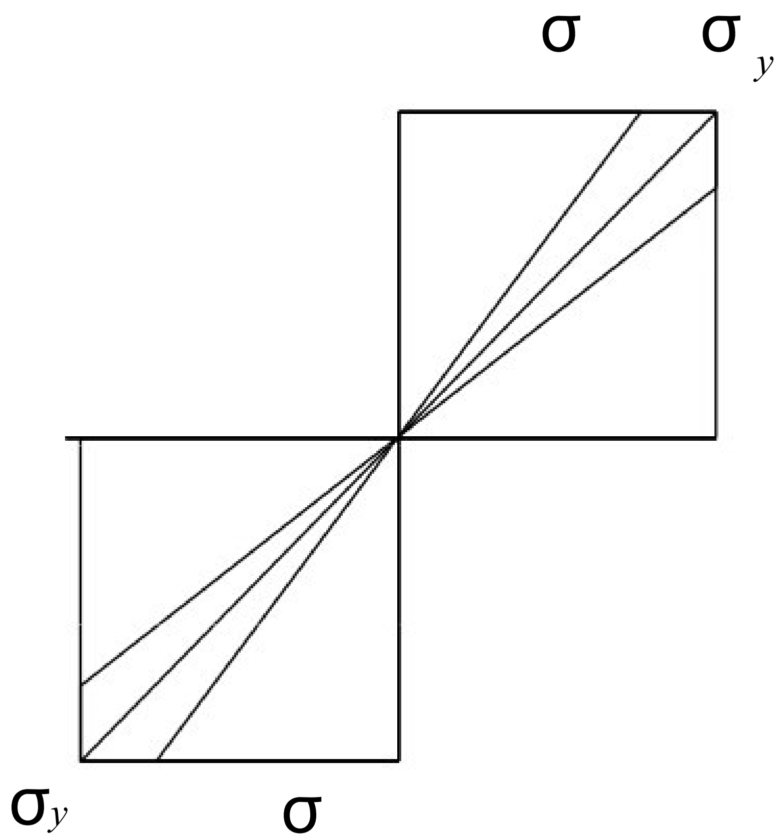


- Within elastic limit, $M = \sigma.Z$
  - **Z is section modulus, I/y**
- Elastic limit - yield stresses reached

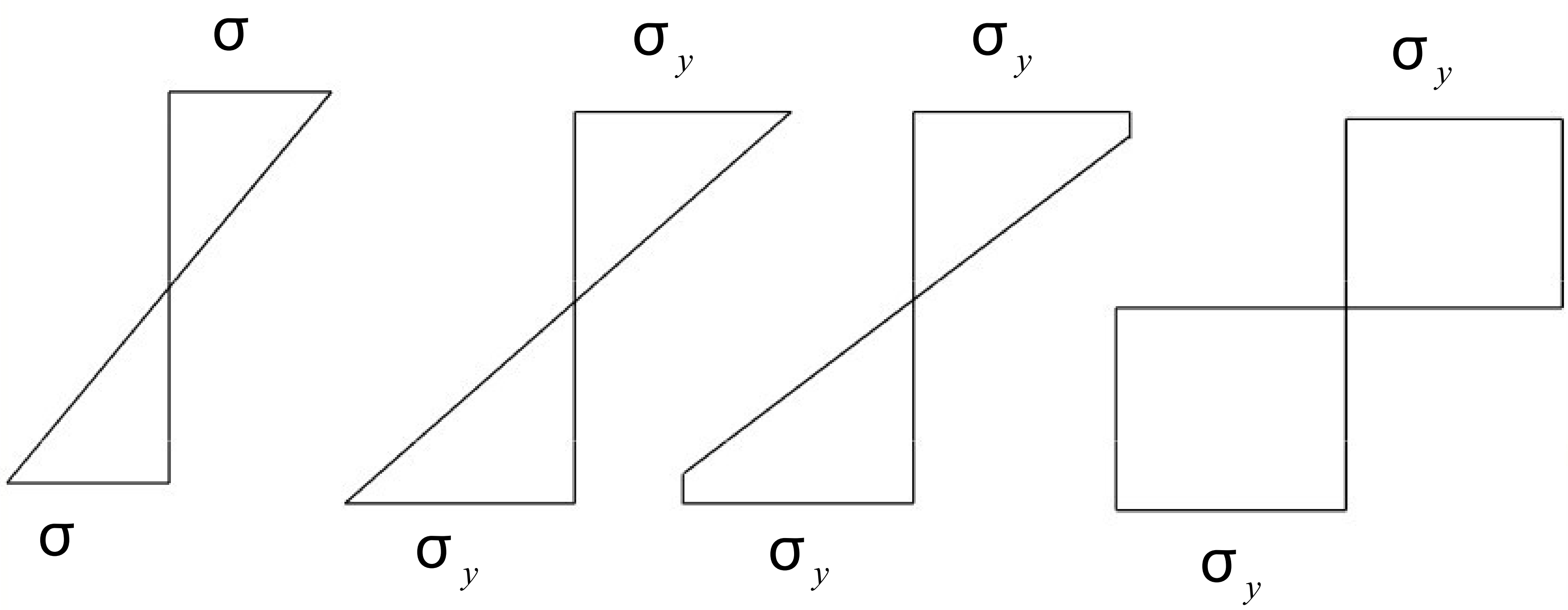
$$M_y = \sigma_y.Z$$

- When moment is increased, yield spreads into inner fibres. Remaining portion still elastic
- Finally, the entire cross-section yields

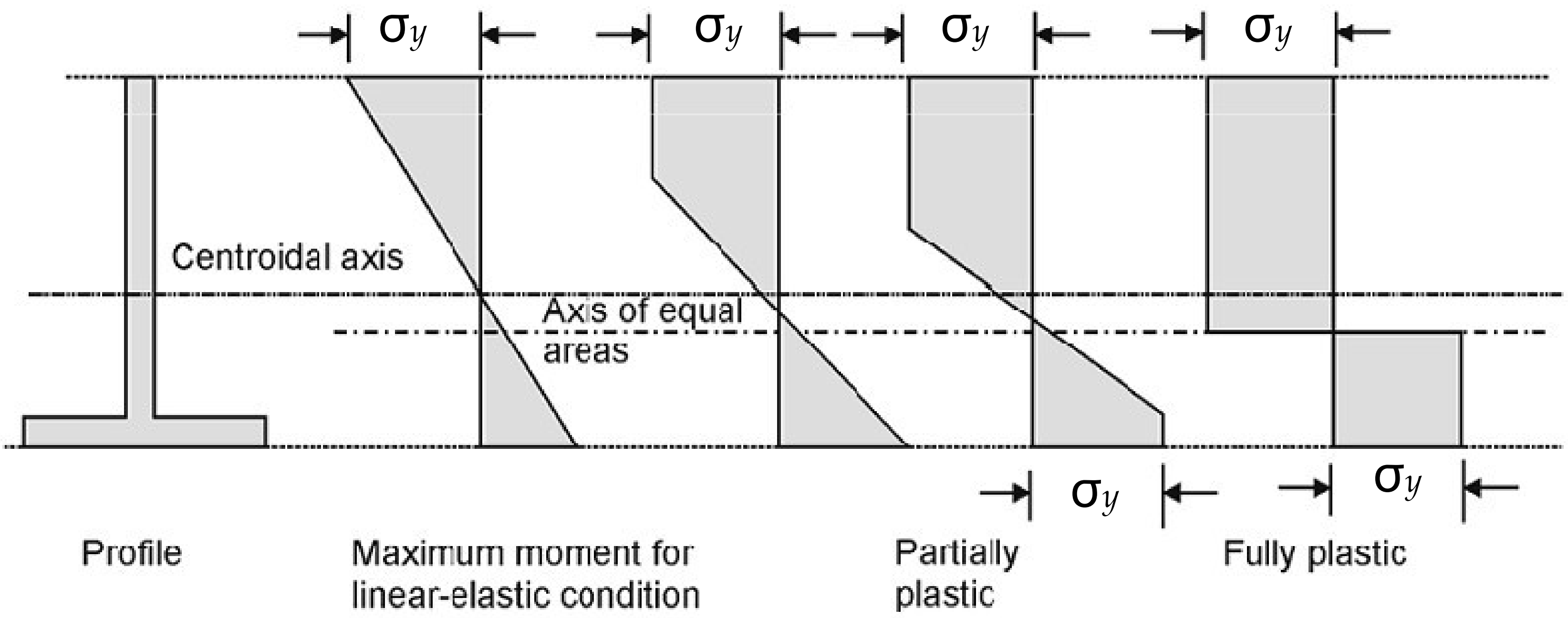



σ
σ y
σy
σ

Change in stress distribution during yielding



Rectangular cross section



Inverted T section

Plastic hinge

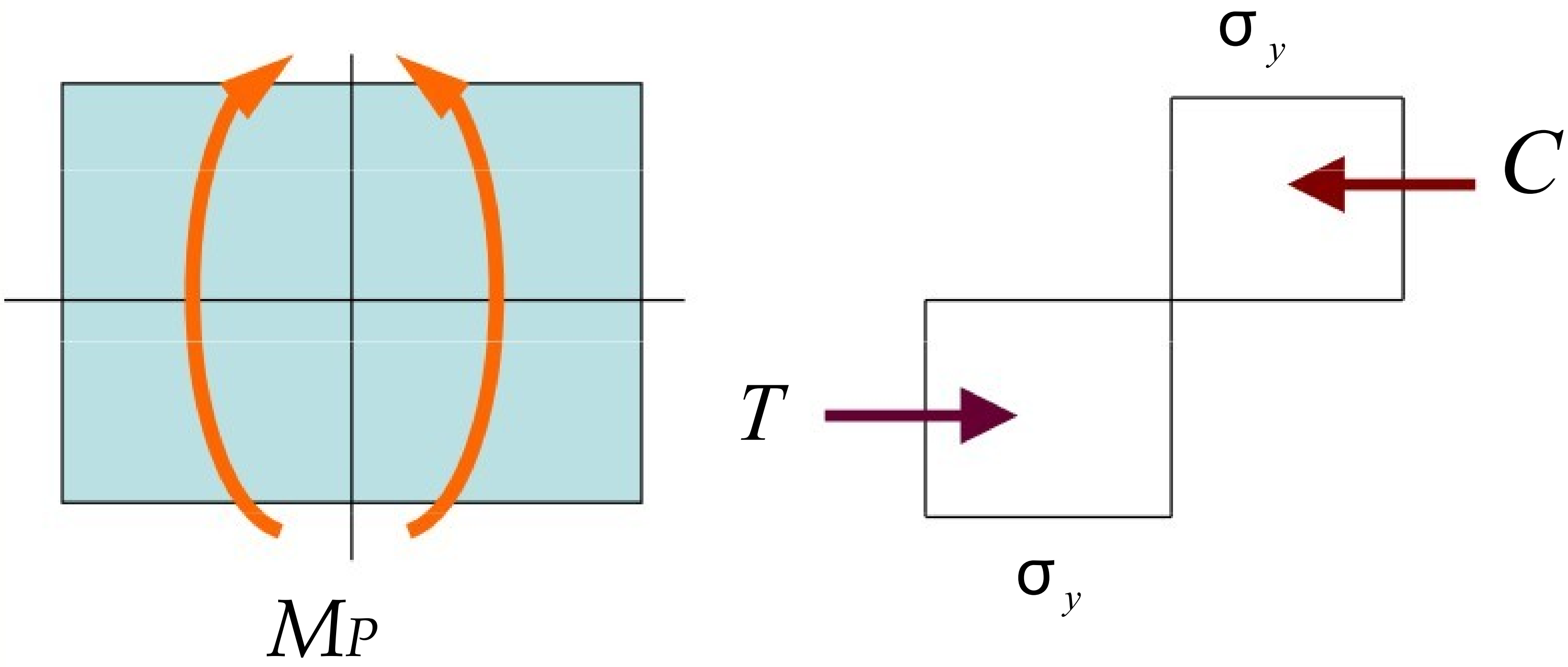
- When the section is completely yielded, the section is fully plastic
- A fully plastic section behaves like a hinge - Plastic hinge

Plastic hinge is defined as an yielded zone due to bending in a structural member, at which large rotations can occur at a section at constant plastic moment, M_P

Mechanical hinge	Plastic hinge
Reality	Concept
Resists zero moment	Resists a constant moment M_P

Mechanical Hinge = Plastic Hinge with $M_P = 0$

- M - Moment corresponding to working load
- M_y - Moment at which the section yields
- M_P - Moment at which entire section is under yield stress



Plastic moment

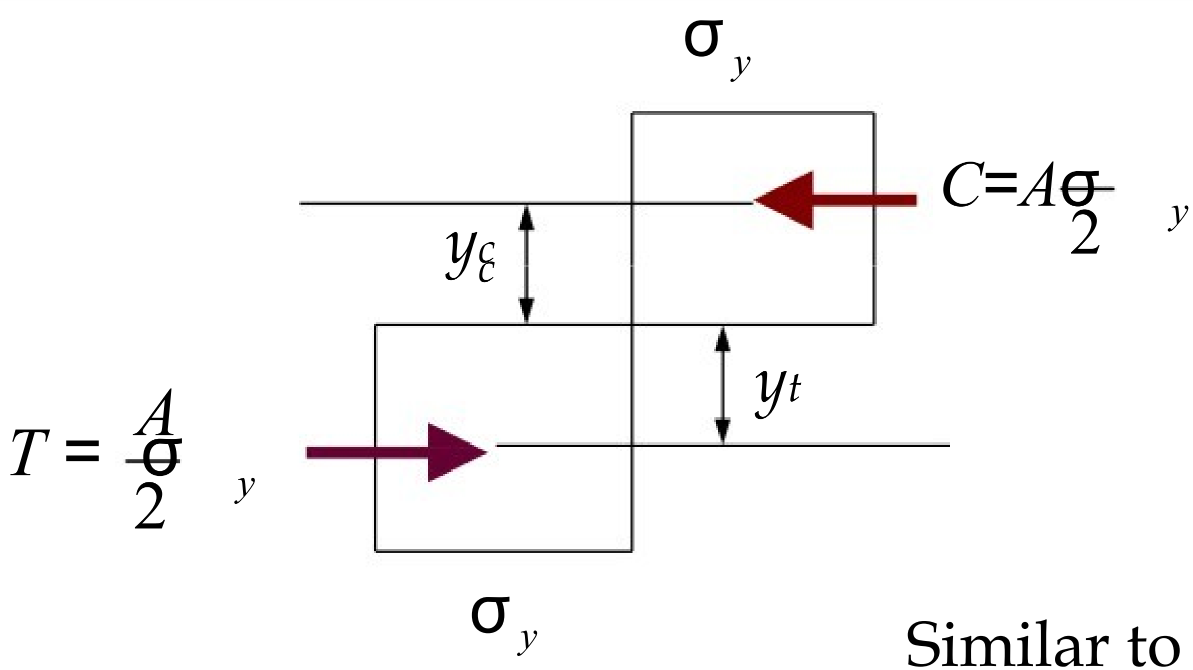
- Moment at which the entire section is under yield stress

$$C = T$$

$$A_c \sigma_y = A_t \sigma_y \Rightarrow A_c = A_t = \frac{A}{2}$$

- NA divides cross-section into 2 equal parts

$$\boxed{C = T = \frac{A}{2}\sigma_y}$$



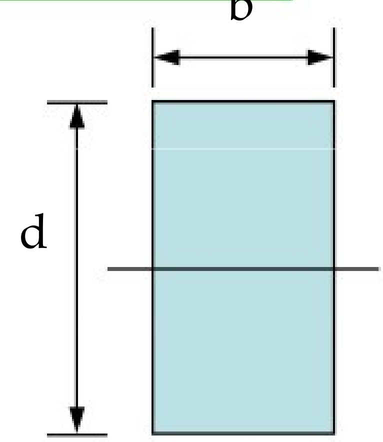
Similar to $\sigma_y Z$

- Couple due to $\frac{A\sigma_y}{2} \Rightarrow \frac{A\sigma_y}{2}(y_c + y_t) = \sigma_y Z_p$

Plastic modulus

$\frac{Z_p}{Z}(>1)$ is the **shape factor**

Shape factor for various cross-sections



Rectangular cross-section:

Section modulus

$$Z = \frac{I}{y} = \frac{\left(bd^3/12\right)}{\left(d/2\right)} = \frac{bd^2}{6}$$

Plastic modulus $$Z_p = \frac{A}{2}\left(y_c + y_t\right) = \frac{bd}{2}\left(\frac{d}{4} + \frac{d}{4}\right) = \frac{bd^2}{4}$$

Shape factor $$\frac{Z_p}{Z} = 1.5$$

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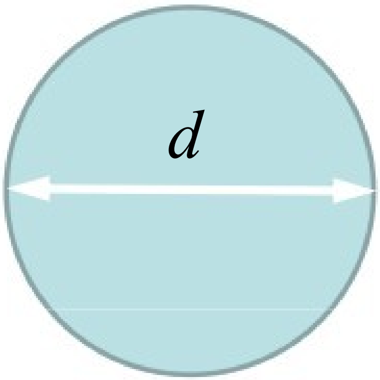
Circular section



$$Z_p = \frac{A}{2}(y_c + y_t)$$

$$= \left(\frac{\pi d^2}{8}\right)\left(\frac{2d}{3\pi} + \frac{2d}{3\pi}\right) = \frac{d^3}{6}$$

$$Z = \frac{(\pi d^4/64)}{d/2} = \frac{\pi d^3}{32}$$

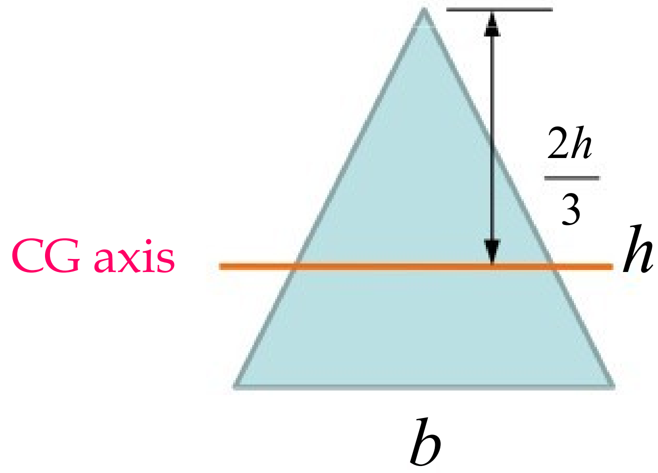
$$S = \frac{Z_p}{Z} = 1.7$$

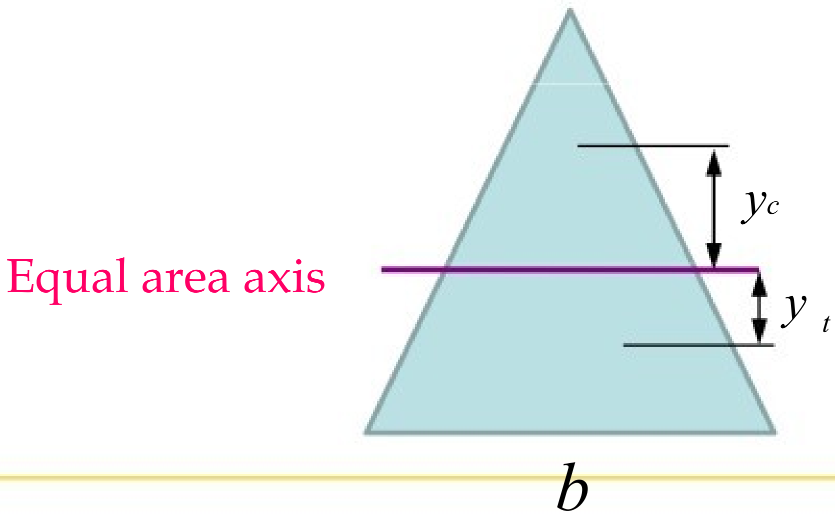
Triangular section

$$Z = \frac{\left(\frac{bh^3}{36}\right)}{2h/3} = \frac{bh^2}{24}$$

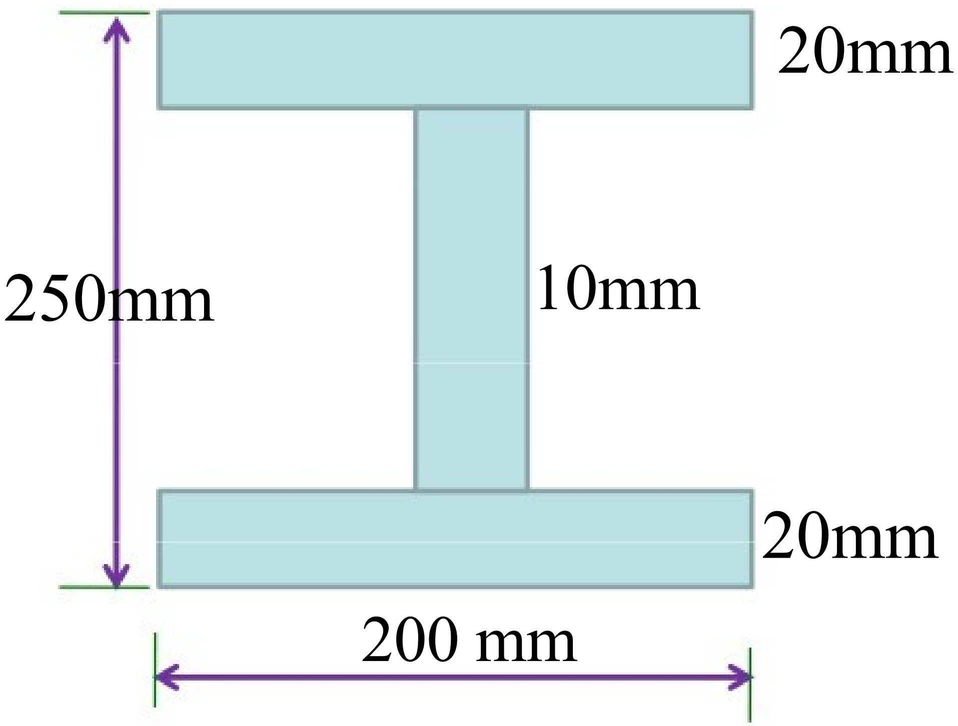
$$Z_p = \frac{A}{2}\left(y_c + y_t\right)$$

S = 2.346





I section



S = 1.132

Mp = 259.6 kNm

Load factor

$$Load\ \ factor = \frac{collapse\ load}{working\ \ load} = \frac{M_P}{M} = \frac{\sigma_y Z_P}{\sigma Z}$$

Rectangular cross-section:

$$M_P = \sigma_y Z_P = \sigma_y \frac{bd^2}{4} \qquad M = \sigma Z = \sigma \frac{bd^2}{6} = \frac{\sigma_y}{1.5} \frac{bd^2}{6}$$

$$\therefore LF = \frac{M_P}{M} = \left(\sigma_y \frac{bd^2}{4}\right) \div \left(\frac{\sigma_y}{1.5} \frac{bd^2}{6}\right) = 2.25$$

Factor of safety

$$\text{Factor of Safety} = \frac{\text{Yield Load}}{\text{Working Load}} = \frac{W_y}{W}$$

$$= \frac{\text{Yield Stress}}{\text{WorkingStress}} = \frac{\sigma_y}{\sigma}$$

$$= \frac{\sigma_y}{(\sigma_y/1.5)} = 1.5$$

Elastic Analysis - Factor of Safety

Plastic Analysis - Load Factor

Mechanisms of failure

- A statically determinate beam will collapse if one plastic hinge is developed
- Consider a simply supported beam with constant cross section loaded with a point load P at midspan
- If P is increased until a plastic hinge is developed at the point of maximum moment (just underneath P) an unstable structure will be created.
- Any further increase in load will cause collapse

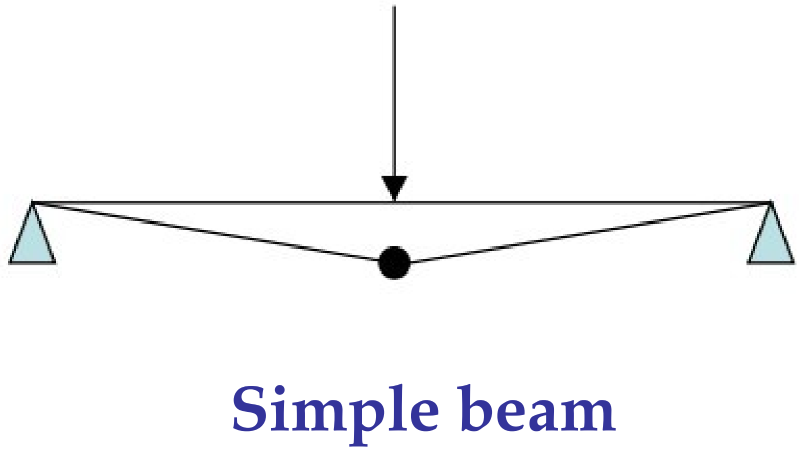
- For a statically indeterminate beam to collapse, more than one plastic hinge should be developed

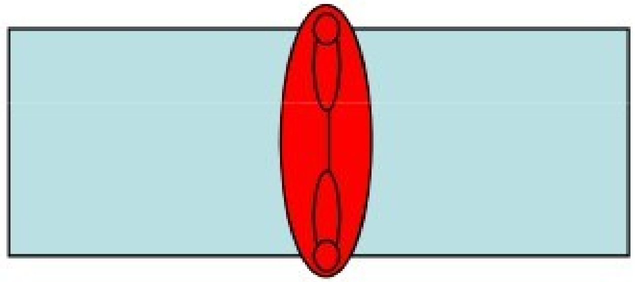
- The plastic hinge will act as real hinge for further increase of load (until sufficient plastic hinges are developed for collapse.)

- As the load is increased, there is a redistribution of moment, as the plastic hinge cannot carry any additional moment.

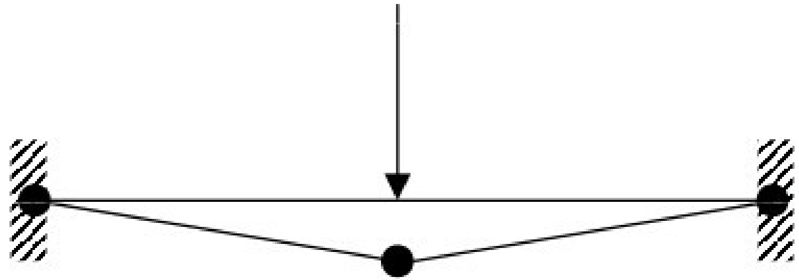
Beam mechanisms

Determinate beams & frames: Collapse after first plastic hinge





Indeterminate beams & frames: More than one plastic hinge to develop mechanism



Fixed beam

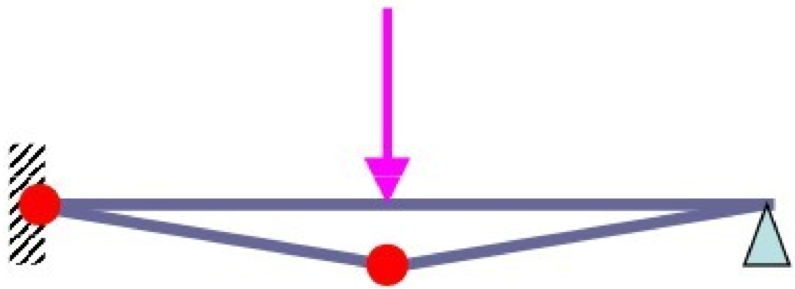
Plastic hinges develop at the ends first

Beam becomes a simple beam

Plastic hinge develops at the centre

Beam collapses

Indeterminate beam: More than one plastic hinge to develop mechanism



Propped cantilever

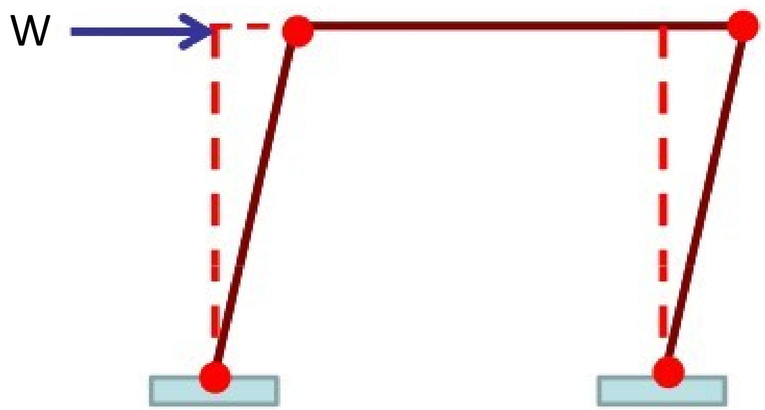
Plastic hinge develops at the fixed support first

Beam becomes a simple beam

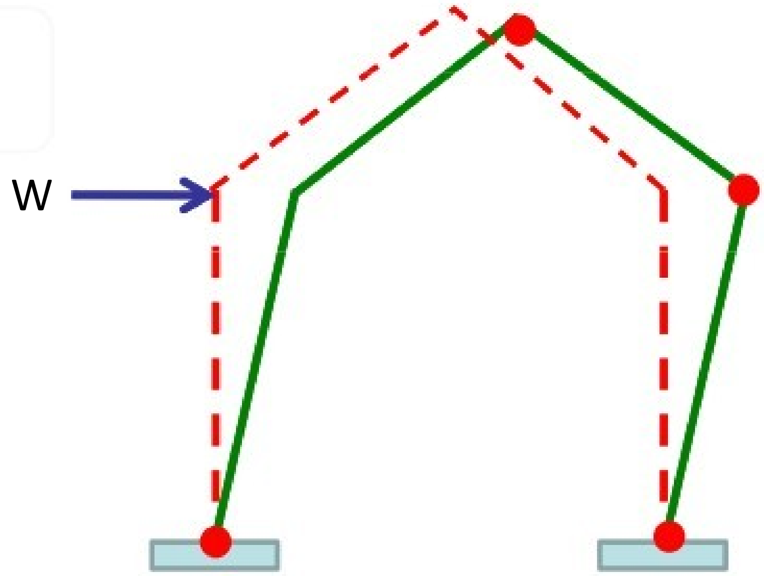
Plastic hinge develops at the centre

Beam collapses

Panel mechanism/sway mechanism



Gable Mechanism



Composite (combined) Mechanism

- Combination of the above

Methods of Plastic Analysis

- **Static method** *or* **Equilibrium method**

  - Lower bound: A load computed on the basis of an assumed equilibrium BM diagram in which the moments are not greater than M_P is always less than (or at the worst equal to) the true ultimate load.

- **Kinematic method** *or* **Mechanism method** *or* **Virtual work method**

  - Work performed by the external loads is equated to the internal work absorbed by plastic hinges

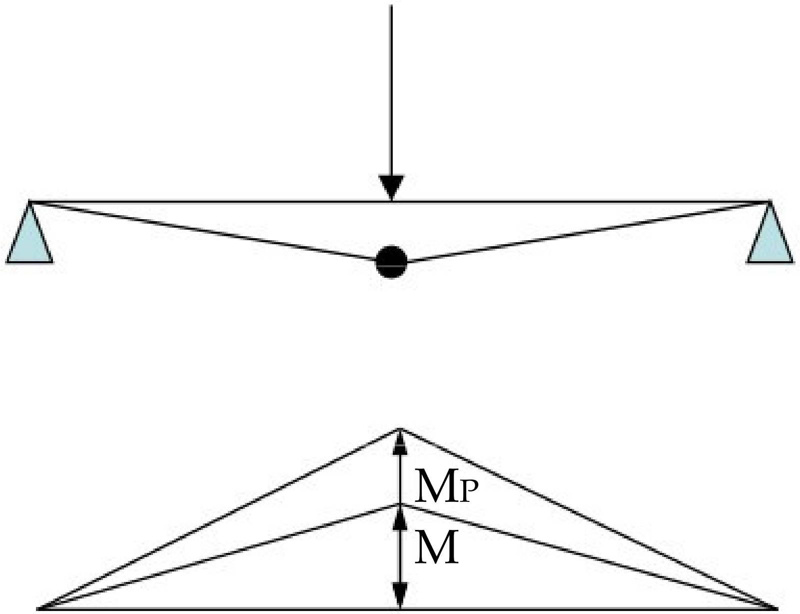
  - Upper bound: A load computed on the basis of an assumed mechanism is always greater than (or at the best equal to) the true ultimate load.

- Collapse load (W_c): Minimum load at which collapse will occur - Least value

- Fully plastic moment (M_P): Maximum moment capacity for design - Highest value

Determination of collapse load

1. Simple beam



Equilibrium method:

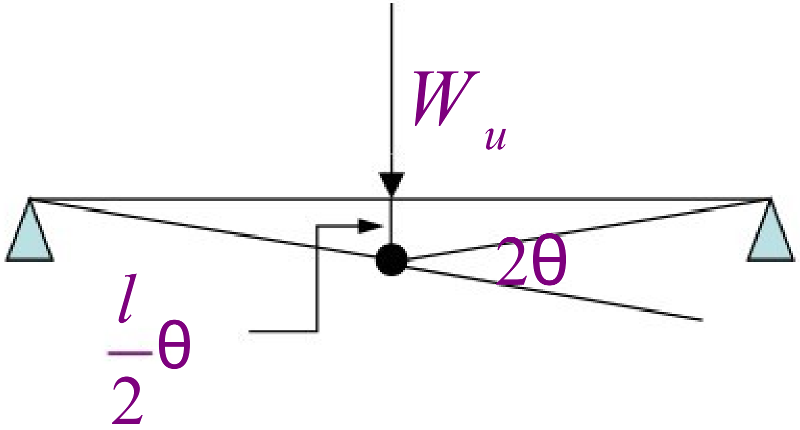
$$M_P = \frac{W_u . l}{4}$$

$$\therefore W_u = \frac{4M_P}{l}$$

Virtual work method:

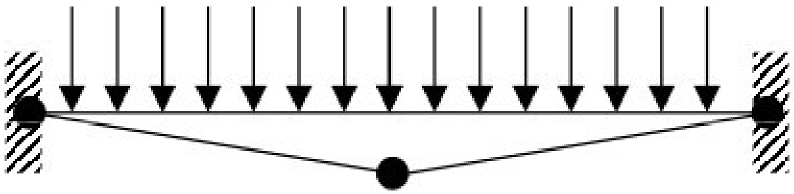
$$W_E = W_I$$

$$W_u\left(\frac{l}{2}\theta\right) = M_P.2\theta$$



$$\therefore W_u = \frac{4M_P}{l}$$

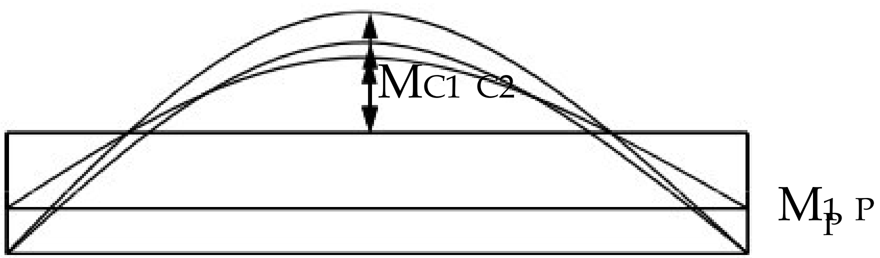
2. Fixed beam with UDL

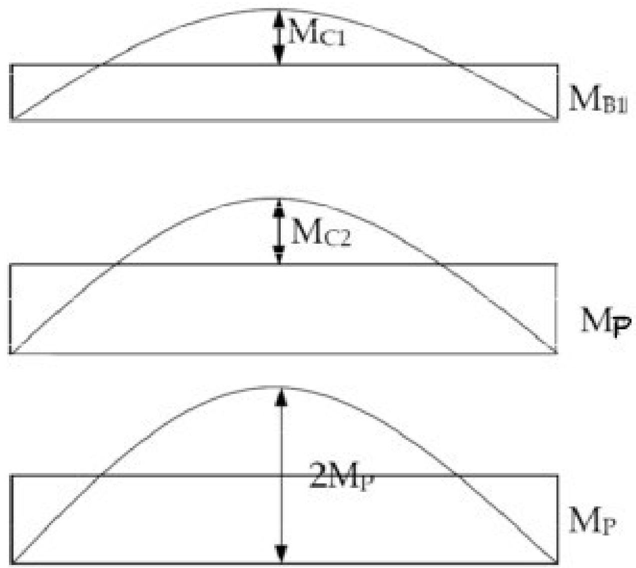


$$M_{CENTRE} = \frac{wl^2}{24},$$

$$M_{ENDS} = \frac{wl^2}{12} > M_{CENTRE}$$

Hence plastic hinges will develop at the ends first.

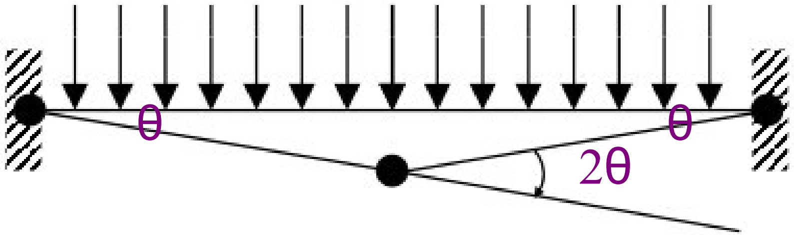




Equilibrium:

$$2M_P = \frac{w_u l^2}{8}$$

$$\therefore w_u = \frac{16M_P}{l^2}$$

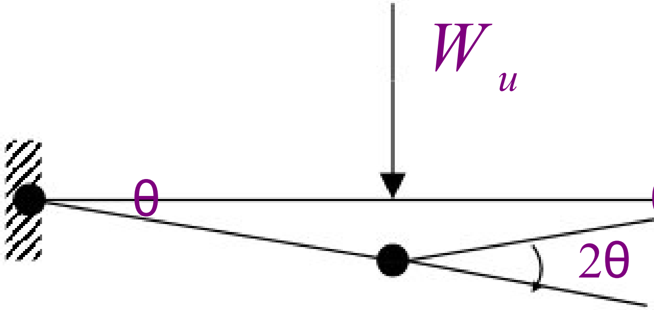


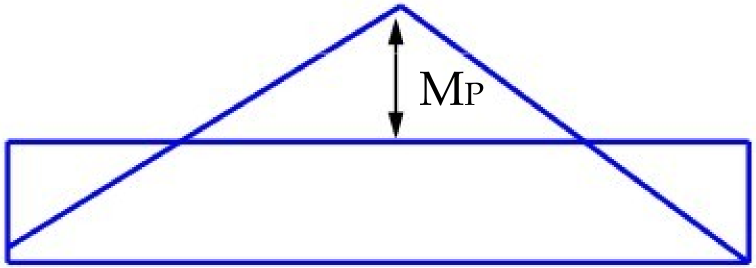
Virtual work: $W_E = W_I$

$$2\left(w_u \frac{l}{2}\right)\left(\frac{0 + \frac{l}{2}\theta}{2}\right) = M_P(\theta + 2\theta + \theta)$$

$$\therefore w_u = \frac{16M_P}{l^2}$$

3. Fixed beam with point load





Virtual work:

$$W_u\left(\frac{l}{2}\theta\right) = M_P(\theta + 2\theta + \theta)$$

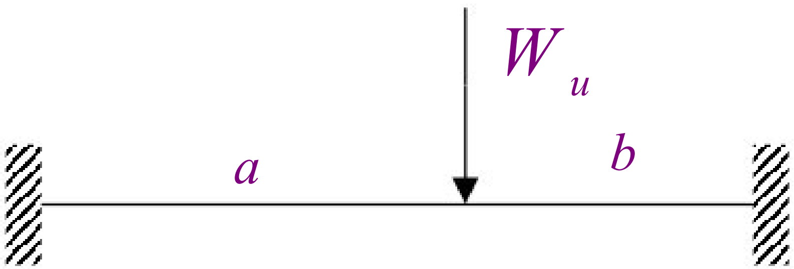
$$\therefore W_u = \frac{8M_P}{l}$$

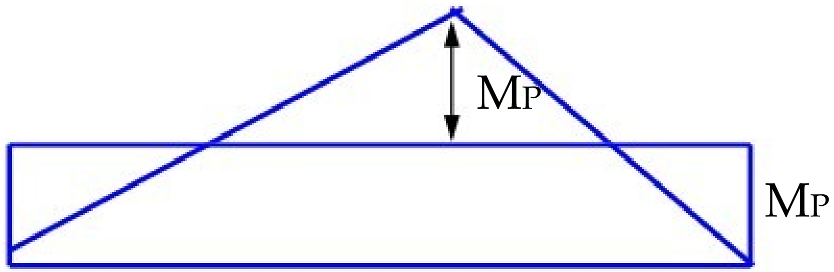
Equilibrium:

$$2M_P = W_u\frac{l}{4}$$

$$\therefore W_u = \frac{8M_P}{l}$$

4. Fixed beam with eccentric point load



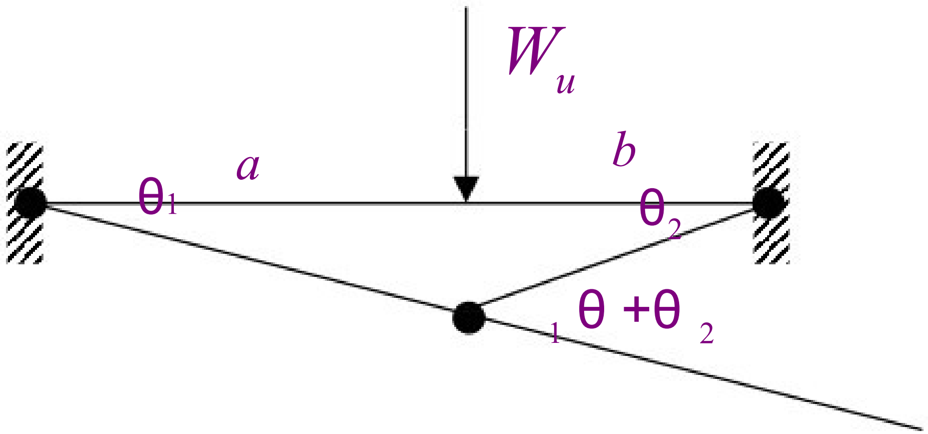


Equilibrium:

$$2M_P = W_u \frac{ab}{l}$$

$$\therefore W_u = \frac{2M_P l}{ab}$$

Virtual work:



$$a\theta_1 = b\theta_2$$

$$\Rightarrow \theta_1 = \frac{b}{a}\theta_2$$

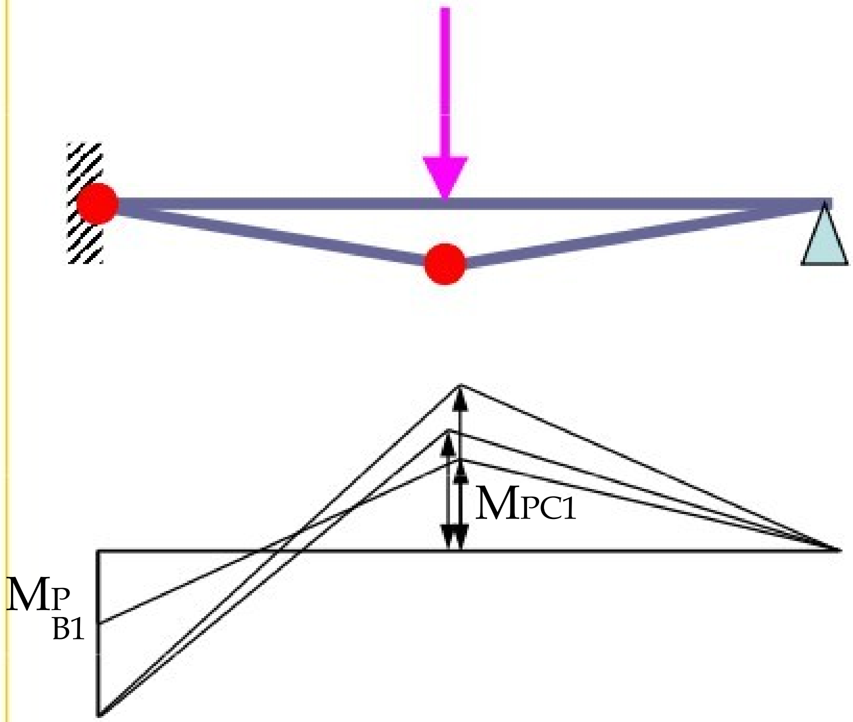
$$W_u(a\theta_1) = M_P\left[\theta_1 + (\theta_1+\theta_2) + \theta_2\right]$$

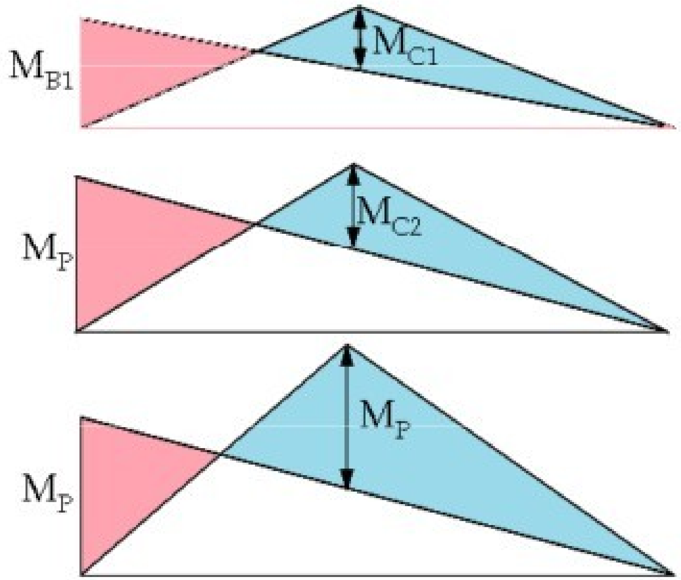
$$W_u(b\theta_2) = M_P\left[2\frac{b}{a}\theta_2 + 2\theta_2\right]$$

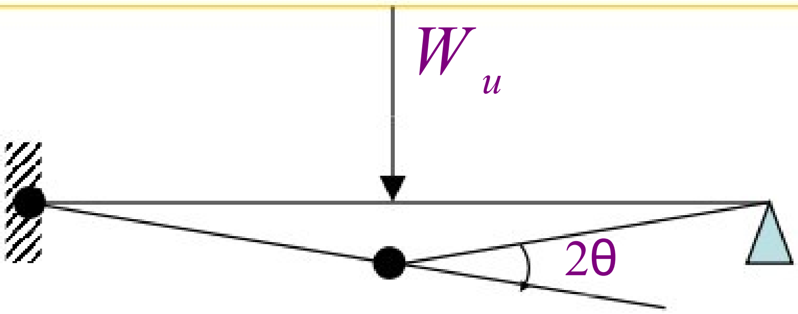
$$\therefore W_u = \frac{M_P}{b\theta_2}\left[2\frac{b}{a}\theta_2 + 2\theta_2\right] = \frac{2M_P(a+b)}{ab}$$

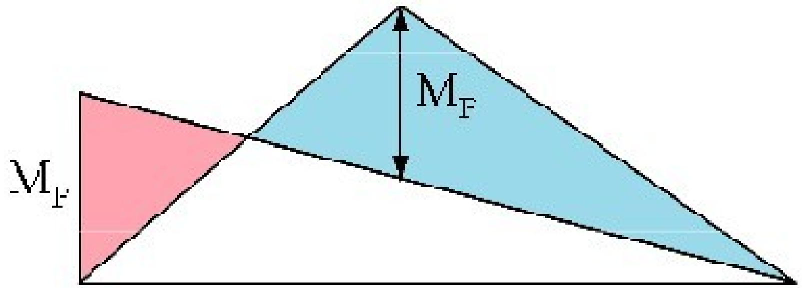
$$W_u = \frac{2M_P l}{ab}$$

5. Propped cantilever with point load at midspan









Virtual work:

$$W_E = W_I$$

$$(W_u)\left(\frac{l}{2}\right)\theta = M_P(\theta + 2\theta)$$

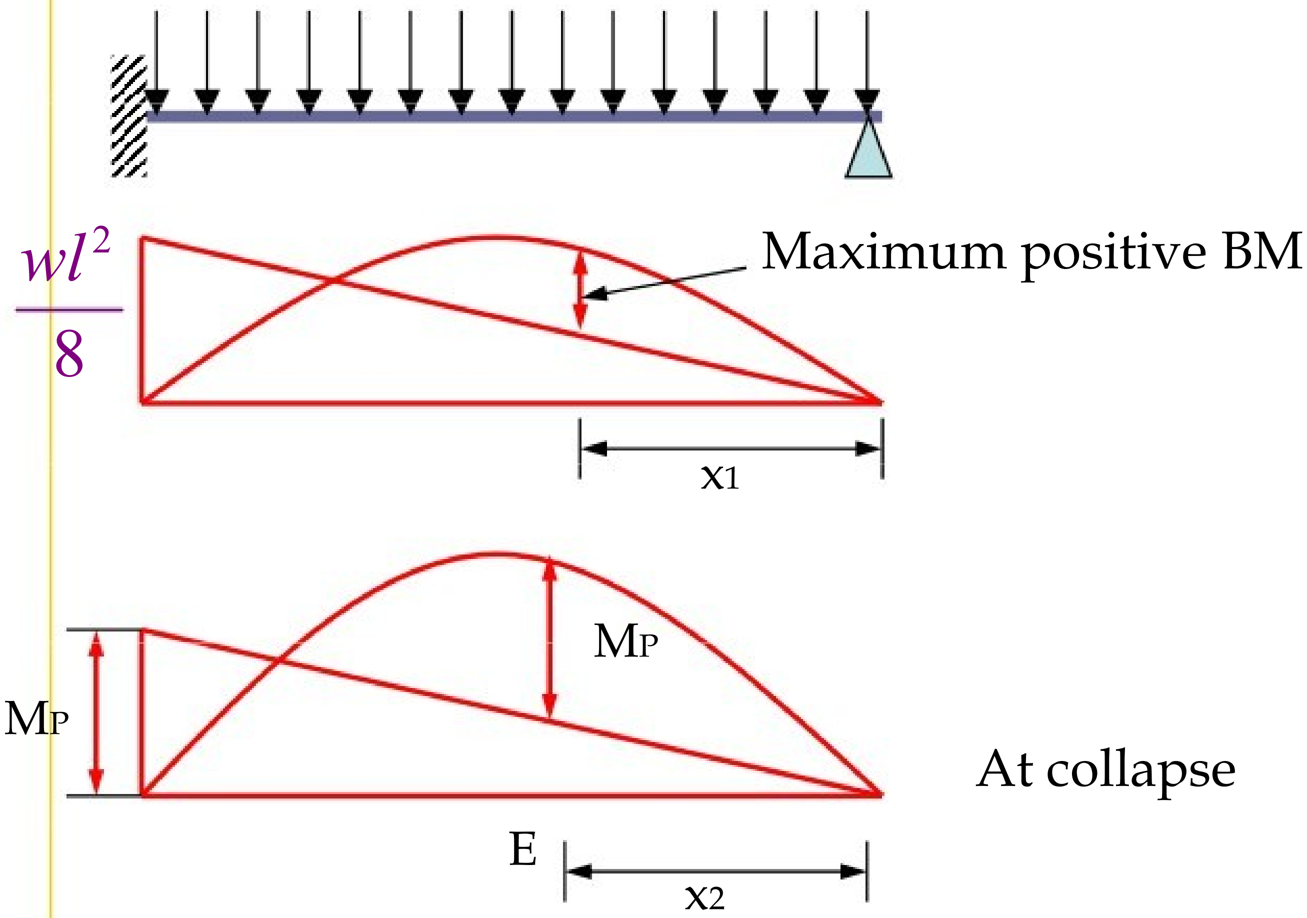
$$\therefore W_u = \frac{6M_P}{l}$$

Equilibrium:

$$M_P + 0.5M_P = \frac{W_u l}{4}$$

$$\therefore W_u = \frac{6M_P}{l}$$

6. Propped cantilever with UDL



At collapse

Required to locate ***E***

$$M_E = \frac{w_u l x_2}{2} - \frac{w_u x_2^2}{2} - M_P \left(\frac{x_2}{l} \right) = M_P \qquad \text{———} \qquad (1)$$

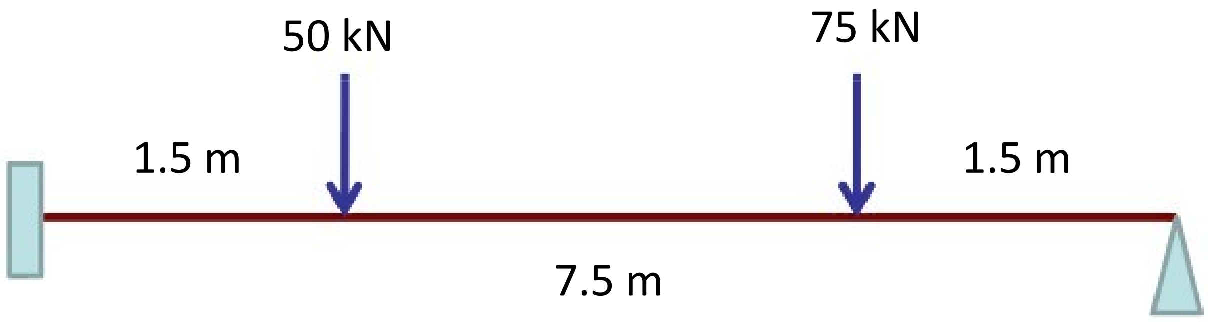
For maximum, $\dfrac{dM_E}{dx_2} = 0$

$$\frac{w_u l}{2} - w_u x_2 - \frac{M_P}{l} = 0 \qquad \text{———} \qquad (2)$$

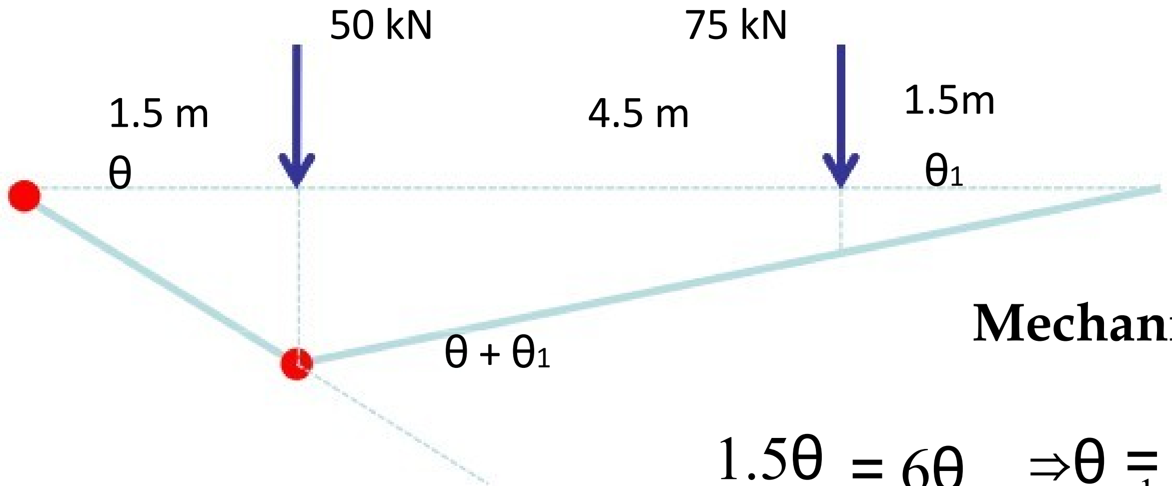
From (1) and (2), $x_2 = 0.414l$

From (2), $w_u = 11.656 \dfrac{M_P}{l^2}$

Problem 1: For the beam, determine the design plastic moment capacity.



- Degree of Indeterminacy, N = 3 - 2 = 1
- No. of hinges, n = 3
- No. of independent mechanisms ,r = n - N = 2

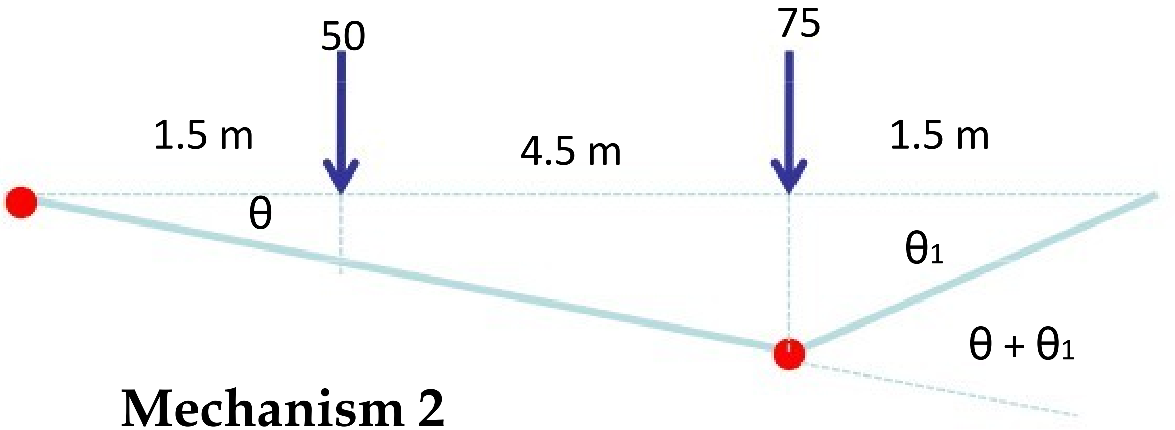


Mechanism 1

$$1.5\theta = 6\theta_1 \Rightarrow \theta_1 = \theta\frac{1.5}{6}$$

$$50(1.5\theta)+75\left(1.5\times\frac{1.5}{6}\theta\right)=M_p\left(\theta+\theta+\frac{1.5}{6}\theta\right)$$

$$\therefore M_p = 4$$



Mechanism 2

$$6\theta = 1.5\theta_1$$

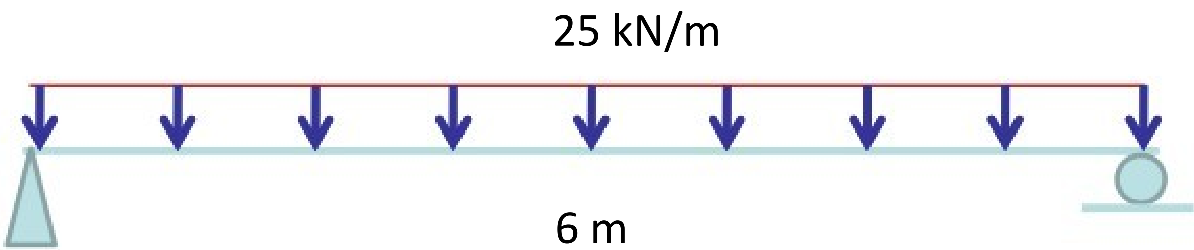
$$\Rightarrow \theta = \frac{1.5\theta_1}{6}$$

$$50\left(1.5 \times \frac{1.5}{6}\theta_1\right) + 75(1.5\theta_1) = M_p\left(\frac{1.5}{6}\theta_1 + \frac{1.5}{6}\theta_1 + \theta_1\right)$$

$$\therefore M_p = 87.5 kNm$$

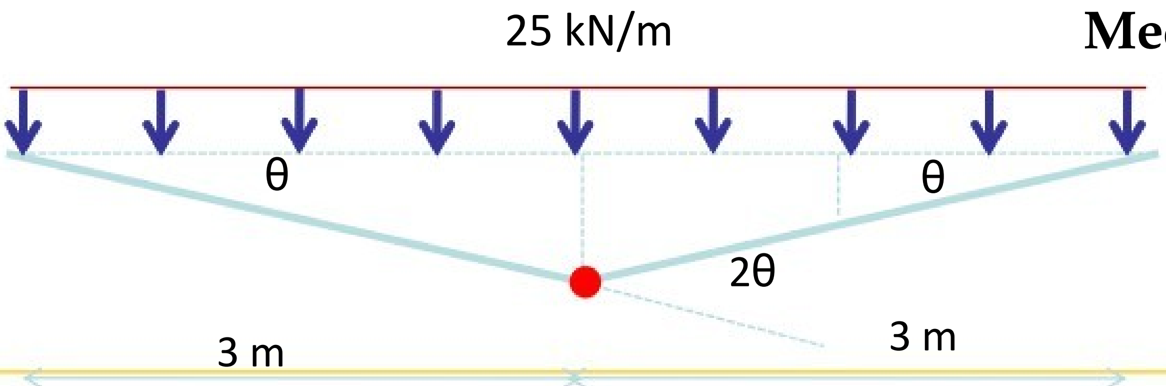
Design plastic moment (Highest of the above) $= 87.5\ kNm$

Problem 2: A beam of span 6 m is to be designed for an ultimate UDL of 25 kN/m. The beam is simply supported at the ends. Design a suitable I section using plastic theory, assuming σ_y= 250 MPa.



- Degree of Indeterminacy, N = 2 - 2 = 0
- No. of hinges, n = 1
- No. of independent mechanisms, r = n-N = 1

Mechanism



Internal work done $W_I = 0 + M_p \times 2\theta + 0 = 2M_p\theta$

External work done $W_E = 2 \times 25 \times \left(3 \times \frac{0+3\theta}{2}\right) = 225\theta$

$W_I = W_E \Rightarrow 2M_p\theta = 225\theta$ $\therefore M_p = 112.5 kNm$

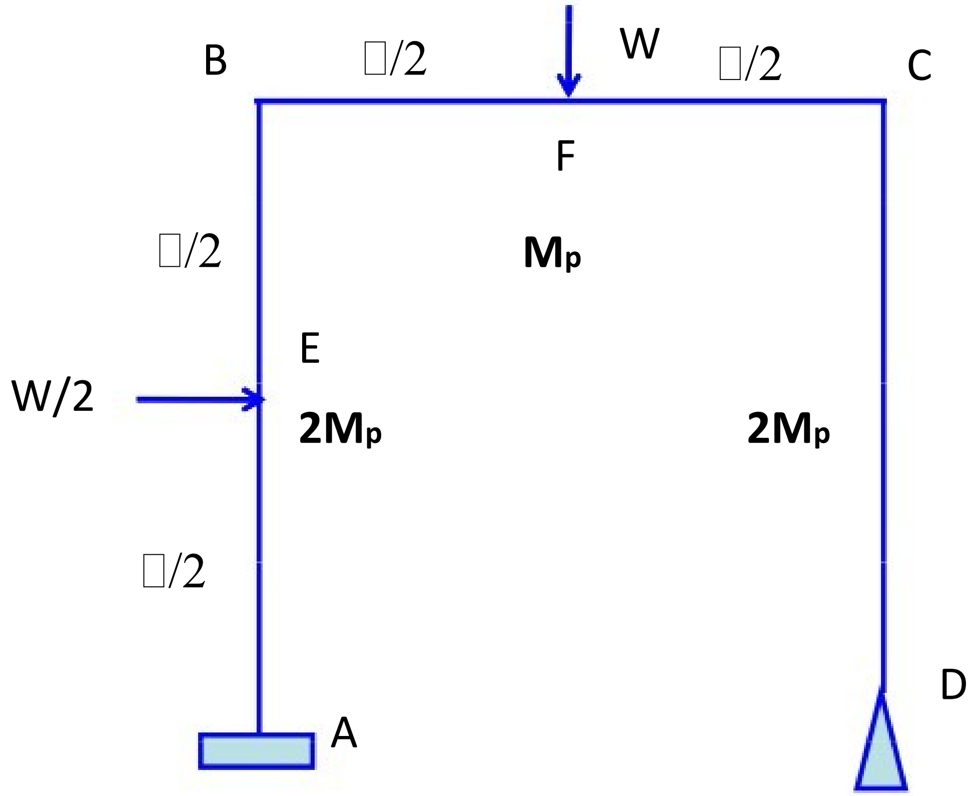
Plastic modulus $Z_P = \frac{M_P}{\sigma_y} = \frac{112.5 \times 10^6}{250} = 4.5 \times 10^5 \; mm^3$

Section modulus $Z = \frac{Z_P}{S} = \frac{4.5 \times 10^5}{1.15} = 3.913 \times 10^5 \; mm^3$

Assuming shape factor S = 1.15

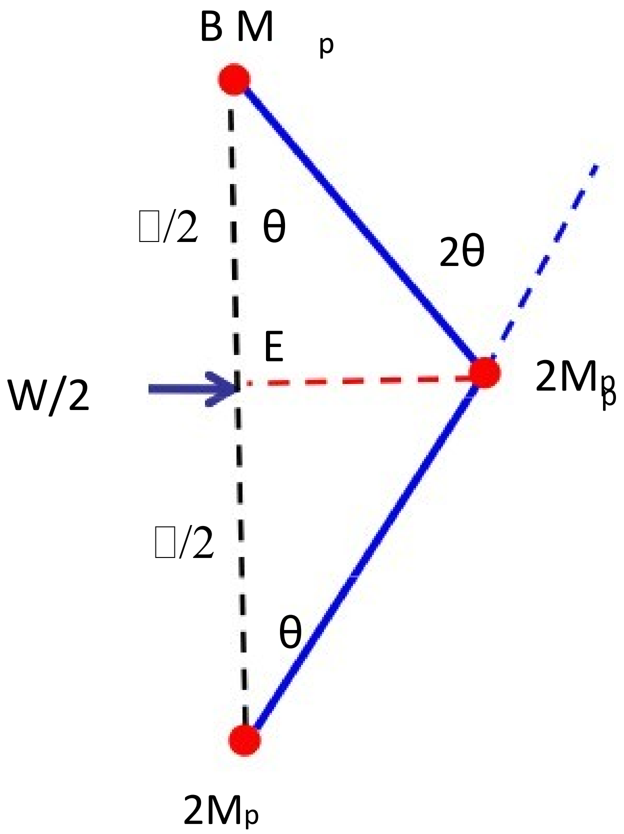
Adopt ISLB 275@330 N/m (from Steel Tables - SP 6)

Problem 3: Find the collapse load for the frame shown.



- Degree of Indeterminacy, N = 5 - 3 = 2
- No. of hinges, n = 5 (at A, B, C, E & F)
- No. of *independent* mechanisms ,r = n - N = 3
    - Beam Mechanisms for members AB & BC
    - Panel Mechanism

Beam Mechanism for AB



$$W_I = M_p\theta + 2M_p(2\theta) + M_p\theta = M_p\theta \quad p$$

$$W_E = \frac{W□}{22}\theta$$

$$W_E = W_I \Rightarrow W_c = \frac{28M_p}{□}$$

Beam Mechanism for BC

W

B □/2 F □/2 C

θ θ

Mp $\frac{□}{2}$θ Mp Mp

2θ

Mp

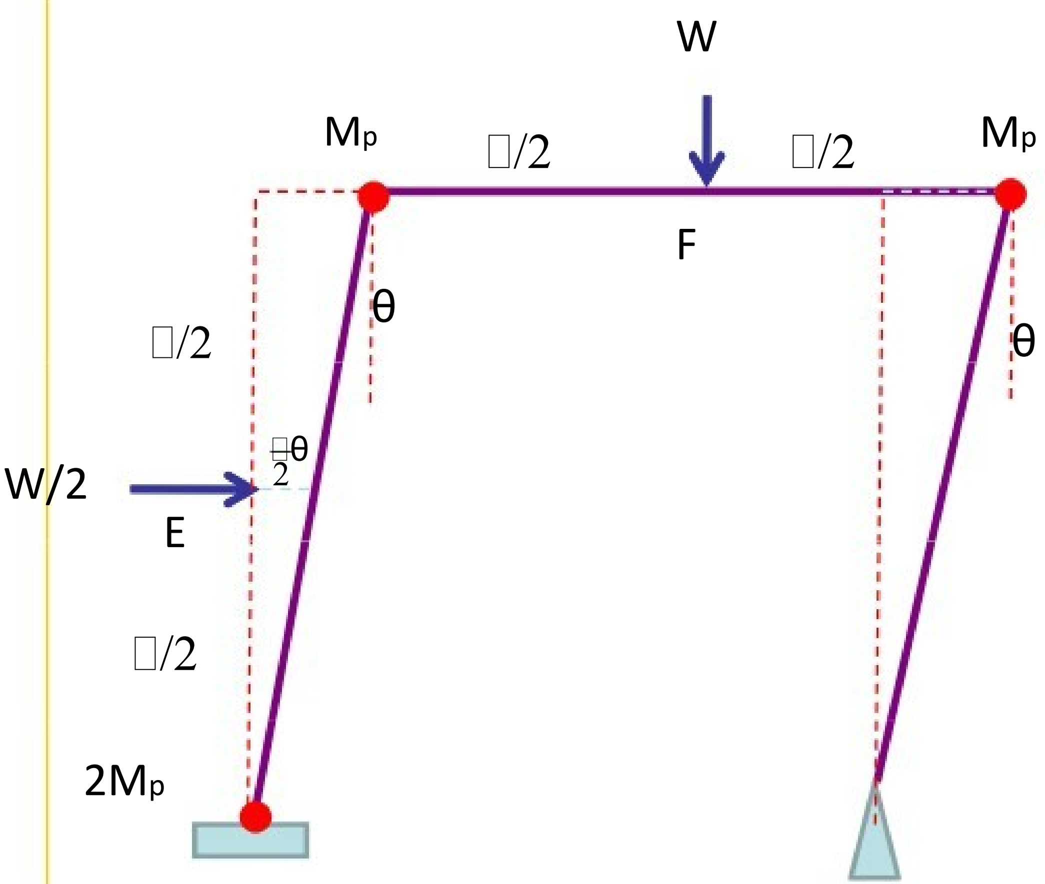
$$W_I = M_p\theta + M_p(2\theta) + M_p\theta = 4M_p\theta$$

$$W_E = W\frac{□}{2}\theta$$

$$W_E = W_I \Rightarrow W_c = \frac{8M_p}{□}$$

Panel Mechanism

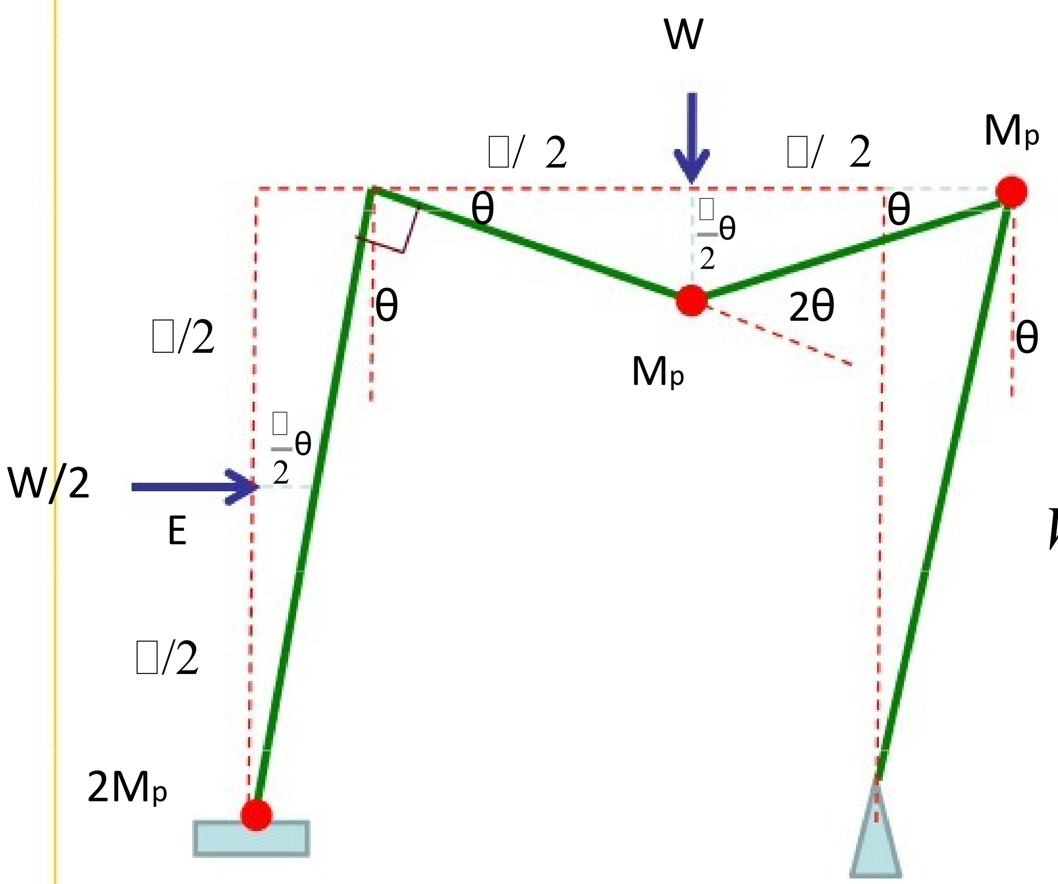
$$W_I = 2M_p\theta + M_p\theta + M_p\theta = 4M_p\theta$$



$$W_E = \frac{W\square}{22}\theta$$

$$W_E = W_I \Rightarrow W_c = \frac{16M_p}{\square}$$

Combined Mechanism



$$W_I = 2M_p(\theta) + M_p(2\theta) + M_p(\theta + \theta)$$

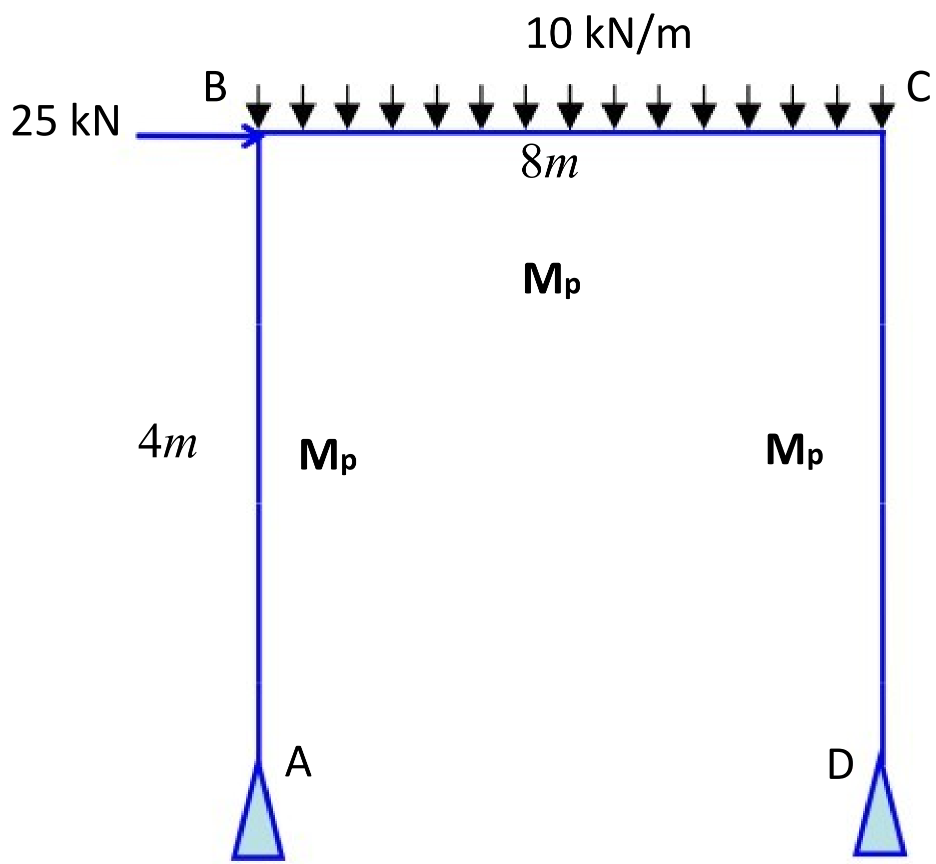
$$= 6M_p\theta$$

$$W_E = \frac{W□}{2}\frac{\theta}{2} + W\frac{□}{2}\theta = \frac{3}{4}W□\theta$$

$$W_E = W_I \Rightarrow W_c = \frac{8M_p}{□}$$

True Collapse Load, (*Lowest of the above,*) $W_c = \frac{8M_p}{□}$

Problem 4: A portal frame is loaded upto collapse. Find the plastic moment capacity required if the frame is of uniform section throughout.



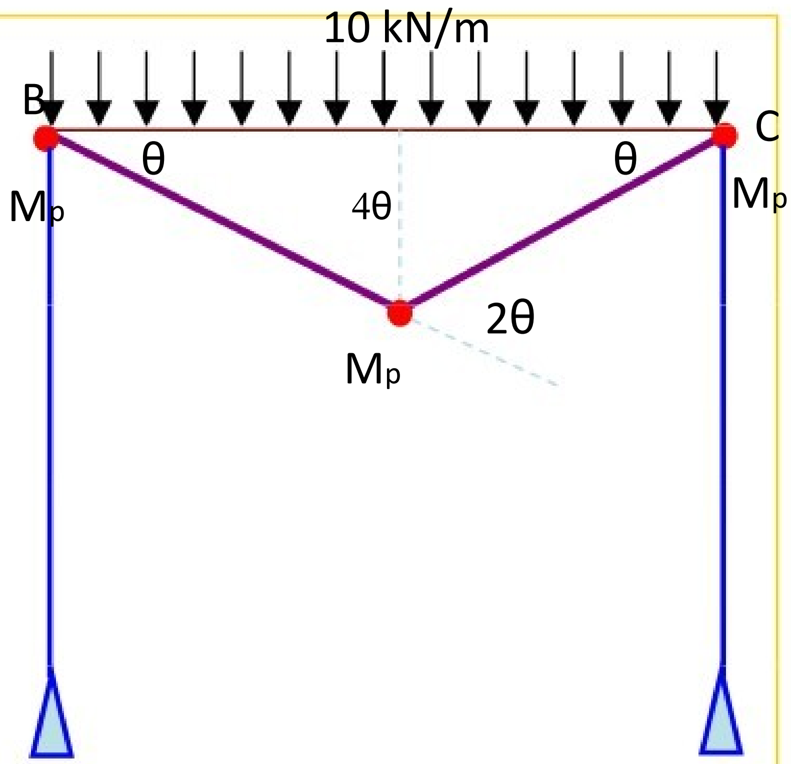
- Degree of Indeterminacy, N = 4 - 3 = 1
- No. of possible plastic hinges, n = 3
  (at B, C and between B&C)
- No. of *independent* mechanisms ,r = n - N = 2
  - Beam Mechanism for BC
  - Panel Mechanism

Beam Mechanism for BC

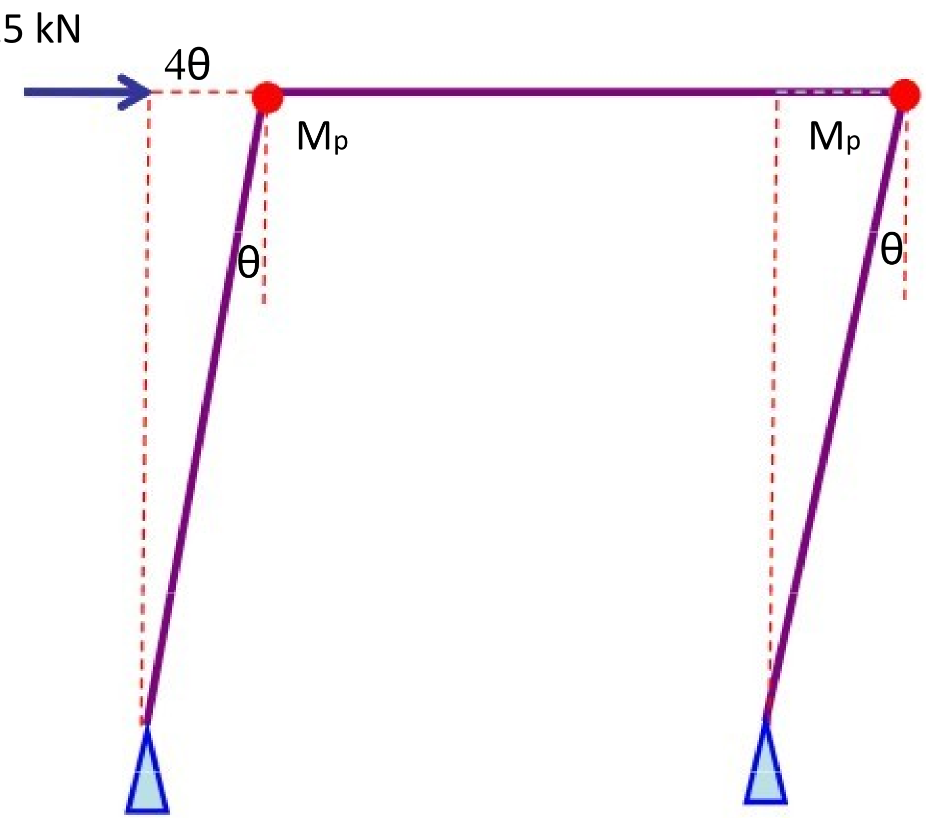
$$W_E = 2\times 10\times \left(4\times \frac{0+4\theta}{2}\right) = 160\theta$$

$$W_I = M_p(\theta+2\theta+\theta) = 4M_p\theta$$

$$\therefore M_p = 40kNm$$



Panel Mechanism

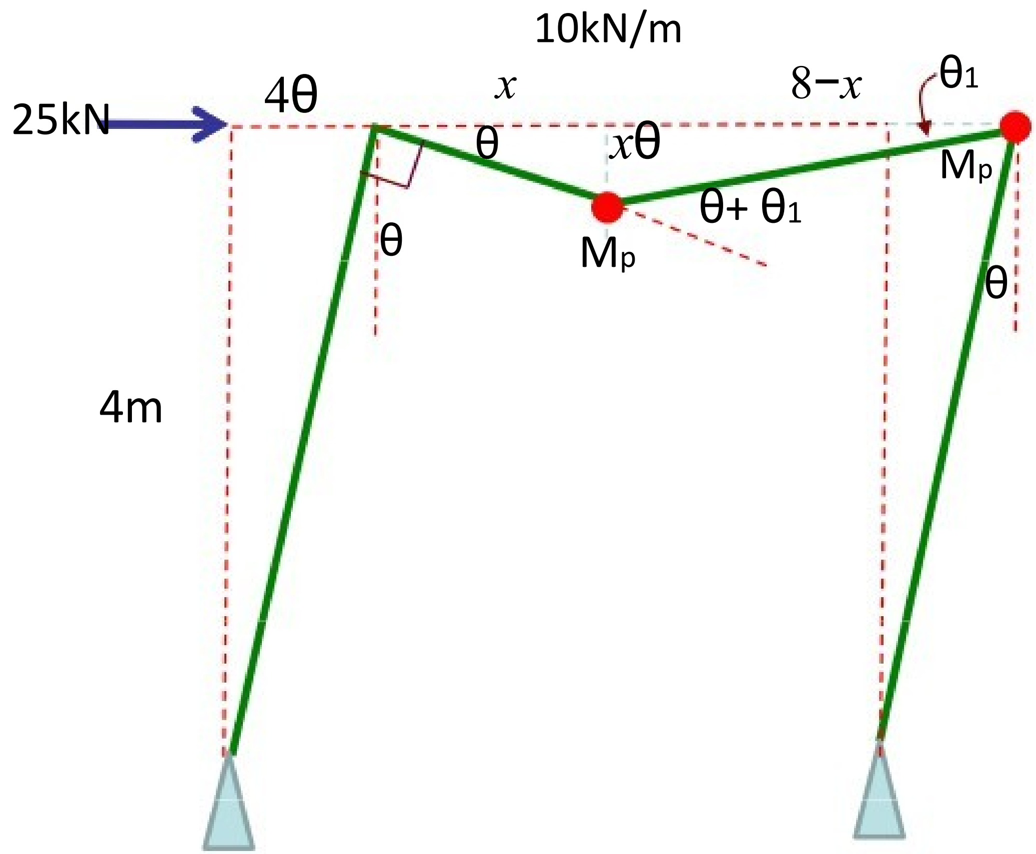


$$W_E = W_I$$

$$\Rightarrow M_p(\theta+\theta) = 25 \times 4\theta$$

$$\Rightarrow M_p = 50 kNm$$

Combined Mechanism



It is required to locate the plastic hinge between B & C

Assume plastic hinge is formed at *x* from B

$$x\theta=(8-x)\theta_1$$

$$W_E = 25\times4\theta + 10x\times\left(\frac{x\theta}{2}\right)+10\times(8-x)\times\left(\frac{(8-x)\theta_1}{2}\right)$$

$$W_I = M_p\left[(\theta_1+\theta)+\theta_1+\theta\right] = 2M_p\left[\frac{x}{8-x}\theta+\theta\right]$$

$$W_E = W_I \Rightarrow M_p = \frac{5(5+2x)(8-x)}{4}$$

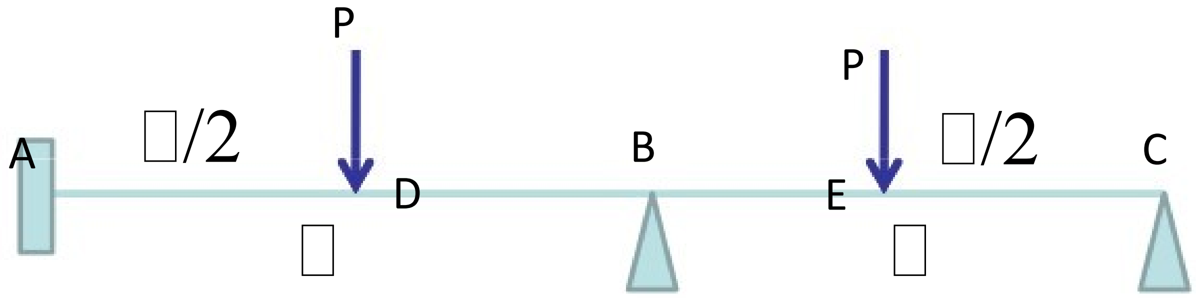
For maximum, $\dfrac{dM_P}{dx} = 0$

$$\Rightarrow x = 2.75\ m$$

$$\therefore M_p = \frac{5(5+2x)(8-x)}{4} = 68.91\,kNm$$

Design plastic moment of resistance,(largest of the above), $M_p = 68.91\,kNm$

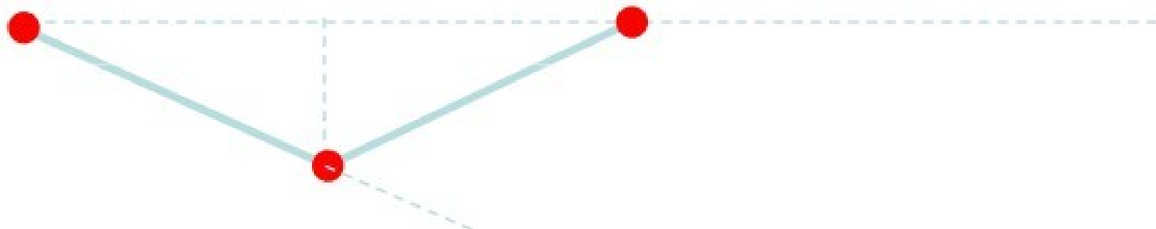
Problem 5: Determine the Collapse load of the continuous beam.



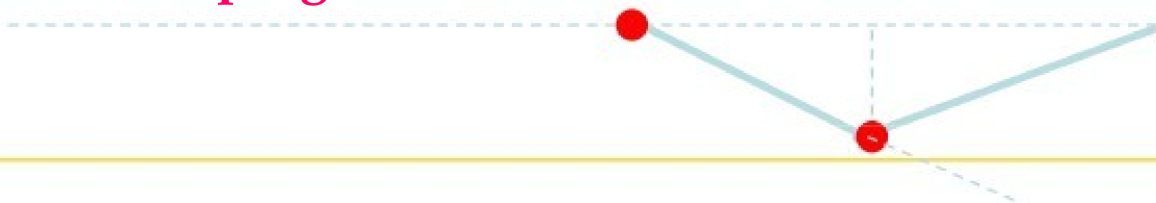
$SI = 4 - 2 = 2$

A collapse can happen in two ways:

1. Due to hinges developing at A, B and D

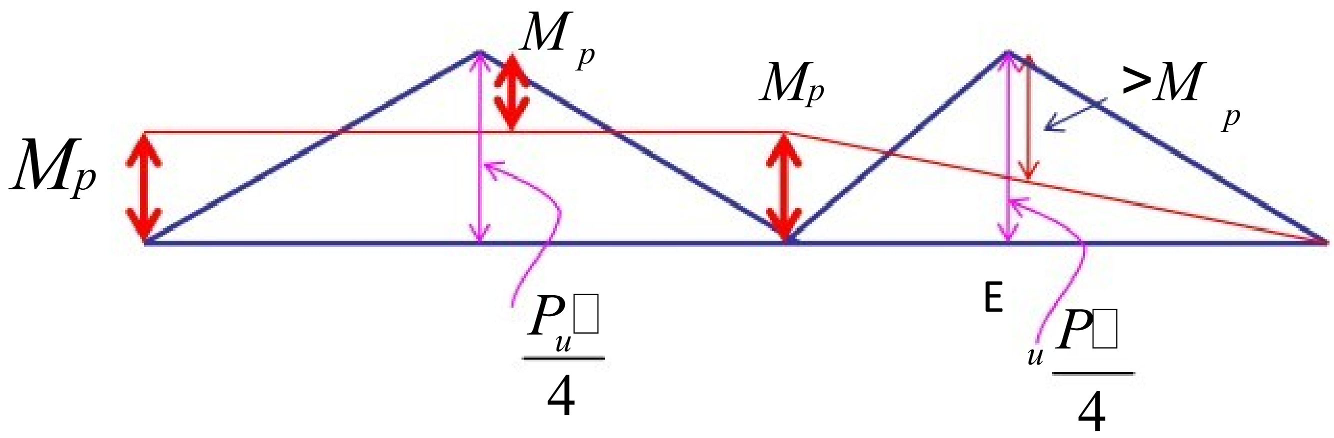


2. Due to hinges developing at B and E



Equilibrium:

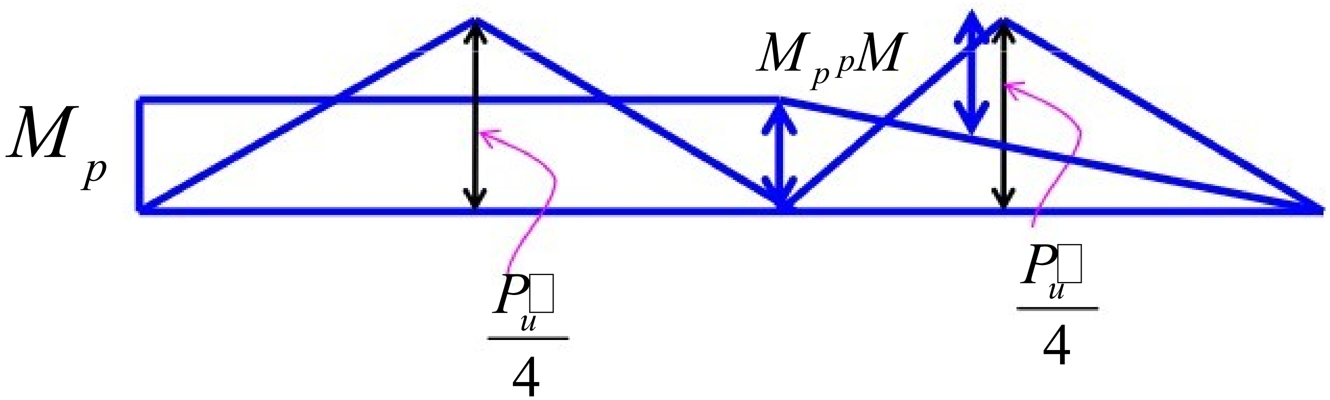
Hinges at A, B and D



$$\frac{P_u \square}{4} = M_p + M_p \Rightarrow P_u = \frac{8M_p}{\square}$$

Moment at *E* is greater than ***Mp***. Hence this mechanism is not possible.

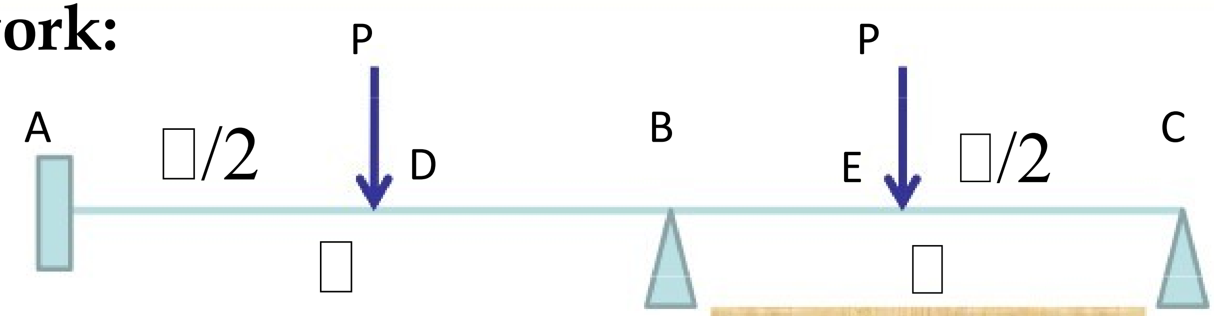
Hinges at B and E



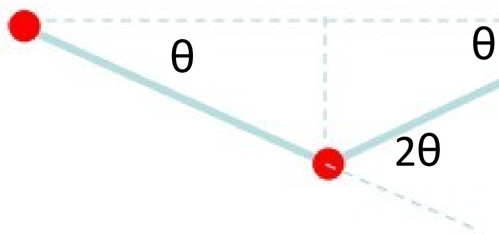
$$\frac{P_u\square}{4} = M_p + \frac{M_p}{2} \Rightarrow P_u = \frac{6M_p}{\square}$$

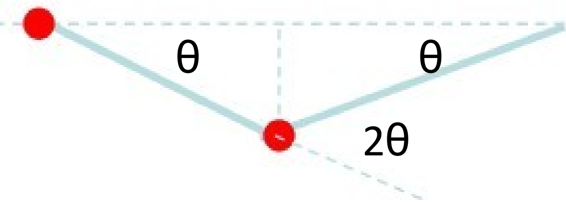
True Collapse Load, $P_u = \frac{6M_p}{\square}$

Virtual work:



$SI = 4 - 2 = 2$





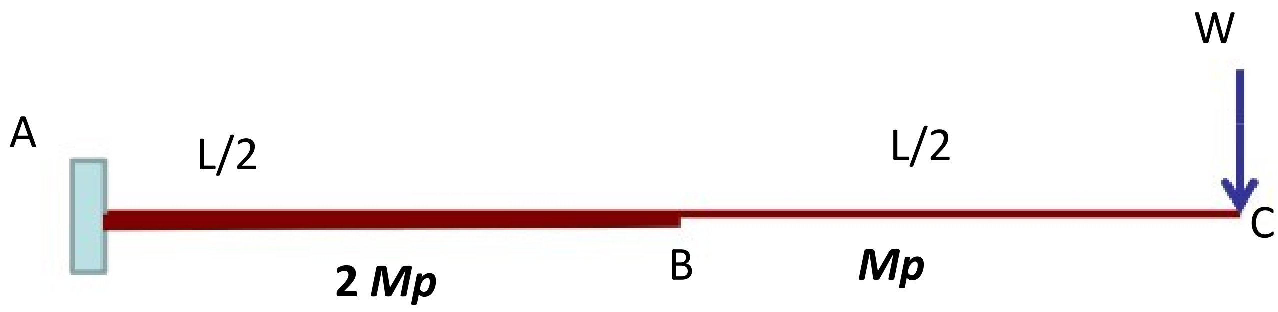
Hinges at A, B and D

$$P_u\left(\frac{□}{2}\theta\right) = M_p(\theta + 2\theta + \theta) \Rightarrow P_u = \frac{8M_p}{□}$$

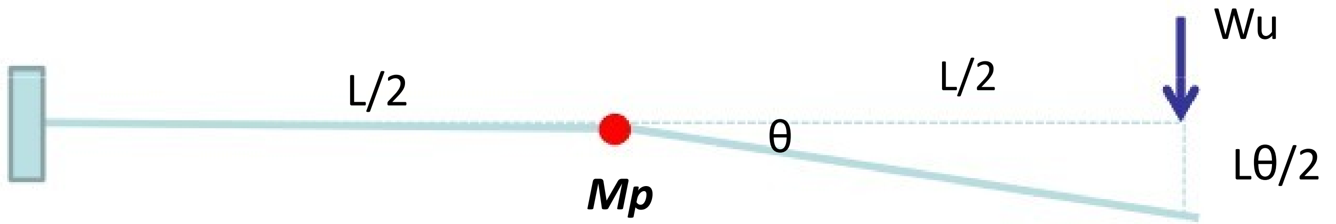
Hinges at B and E

$$P_u\left(\frac{□}{2}\theta\right) = M_p(\theta + 2\theta) \Rightarrow P_u = \frac{6M_p}{□}$$

Problem 6: For the cantilever, determine the collapse load.

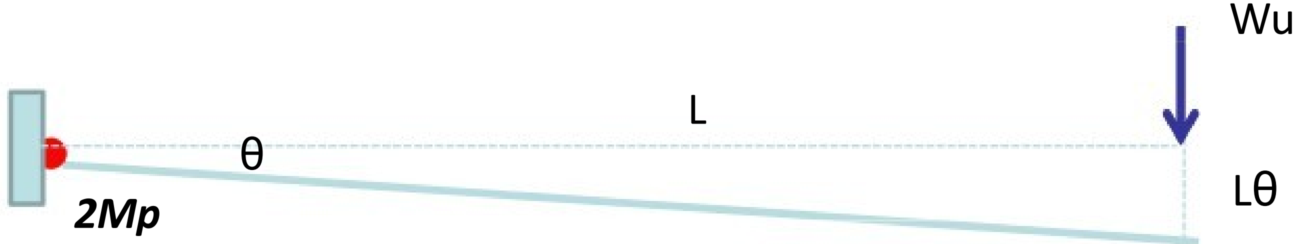


- Degree of Indeterminacy, N = 0
- No. of possible plastic hinges, n = 2 (at A&B)
- No. of *independent* mechanisms ,r = n - N = 2



Mechanism 1

$$W_u \times \frac{L}{2}\theta = M_p\theta \qquad \therefore W_u = \frac{2M_p}{L}$$

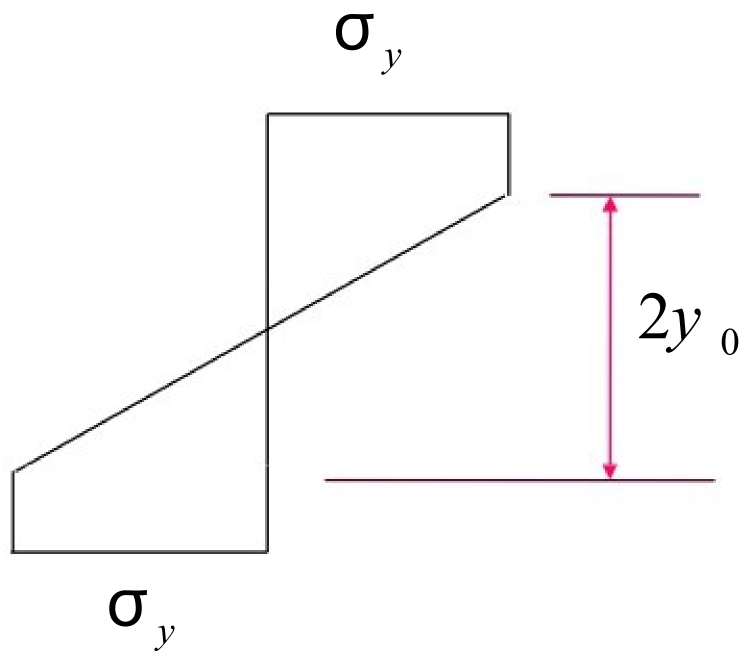


Mechanism 2

$$W_u \times L\theta = 2M_p\theta \qquad \therefore W_u = \frac{2M_p}{L}$$

True Collapse Load, (*Lowest of the above,*) $W_c = \frac{2M_p}{L}$

Problem 7: A beam of rectangular section $b \times d$ is subjected to a bending moment of **0.9 *Mp***. Find out the depth of elastic core.



Let the elastic core be of depth $2y_0$

External bending moment must be resisted by the internal couple.

Distance of CG from NA,

$$y' = \frac{b\left(\frac{d}{2} - y_0\right) \times \sigma_y \times \left[y_0 + \frac{1}{2}\left(\frac{d}{2} - y_0\right)\right] + b y_0 \frac{\sigma_y}{2} \cdot \frac{2}{3} y_0}{b\left(\frac{d}{2} - y_0\right)\sigma_y + b y_0 \frac{\sigma_y}{2}} = \frac{3d^2 - 4y_0^2}{12(d - y_0)}$$

Internal couple (moment of resistance)

$$= 2\times\left\{b\left(\frac{d}{2}-y_0\right)\sigma_y + by_0\frac{\sigma_y}{2}\right\}\times\frac{3d^2-4y_0^2}{12(d-y_0)}$$

$$= \frac{3d^2-4y^2}{12}b\sigma_y$$

External bending moment = $0.9M_p = 0.9\times Z_p\sigma_y = 0.9\times\frac{bd^2}{4}\sigma_y$

Equating the above, $\frac{3d^2-4y_0^2}{12}b\sigma_y = 0.9\times\frac{bd^2}{4}\sigma_y$

$$\Rightarrow y_0 = 0.274d$$

Hence, depth of elastic core $= 2y_0 = 0.548d$

Summary

Plastic Theory

- Introduction-Plastic hinge concept-plastic section modulus-shape factor-redistribution of moments-collapse mechanism-
- Theorems of plastic analysis - Static/lower bound theorem; Kinematic/upper bound theorem-Plastic analysis of beams and portal frames by equilibrium and mechanism methods.