Homogeneous Function

A function $y = f(x_1, x_2, \cdots, x_n)$ is homogeneous of degree k if, for any number s where $s > 0$

$$s^k Y = f(sx_1, sx_2, \cdots, sx_n)$$

Euler’s Theorem

- Homogeneity of degree 1 is often called **linear homogeneity**.
- An important property of homogeneous functions is given by **Euler’s Theorem**.

Euler's Theorem

For any multivariate function $y = f(x_1, x_2, \cdots, x_n)$

that is homogeneous of degree k,

$$ky = x_1 f_1(x_1, x_2, \cdots, x_n) + \cdots + x_n f_n(x_1, x_2, \cdots, x_n)$$

for any set of values $(x_1, x_2, \cdots, x_n)$, where $f_i(x_1, x_2, \cdots, x_n)$

is the partial derivative of the function with respect to

its ith argument.

Proof Euler's Theorem

Definition homogeneous function $s^K y = f(sx_1, sx_2, \cdots, sx_n)$

Take the partial derivative of the above with respect to s

$ks^{k-1} y = x_1 f_1(sx_1, sx_2, \cdots, sx_n) + \cdots + x_n f_n(sx_1, sx_2, \cdots, sx_n)$

Letting $s = 1$, we get Euler's Theorem

$ky = x_1 f_1(x_1, x_2, \cdots, x_n) + \cdots + x_n f_n(x_1, x_2, \cdots, x_n)$

The converse of this theorem holds. If the above is true, then the original function is homogeneous of degree k.

Arguments of Functions that are Homogeneous degree zero

Any function $f(x_1, x_2, \cdots, x_i, \cdots, x_n)$ that is homogeneous of degree zero can be written as

$$f\left(\frac{x_1}{x_i}, \frac{x_2}{x_i}, \cdots, 1, \cdots, \frac{x_n}{x_i}\right) \text{ for any } i = 1, 2, \ldots, n.$$

Proof : Since the function is homogeneous of degree 0,

$$s^0 f(x_1, x_2, \cdots, x_i, \cdots, x_n) = f(sx_1, sx_2, \cdots, sx_i, \cdots, sx_n)$$

$$\text{Let } s = \frac{1}{x_i}, \text{ then}$$

$$f(x_1, x_2, \cdots, x_i, \cdots, x_n) = f\left(\frac{x_1}{x_i}, \frac{x_2}{x_i}, \cdots, 1, \cdots, \frac{x_n}{x_i}\right)$$

QED

First Partial Derivatives of Homogeneous Functions

If the function, $f\left(x_1, x_2, \cdots, x_n\right)$ is homogeneous of degree k, then each of ists first partial derivatives

$f_i = \dfrac{\partial f\left(x_1, x_2, \cdots, x_n\right)}{\partial x_i}$ for any $\mathrm{i} = 1, 2, \cdots, \mathrm{n}$, is homogeneous of degree k-1.

Proof of previous slide

$$We\ know\ f(sx_1, sx_2, \cdots, sx_n) = s^k f(x_1, x_2, \cdots, x_n)$$

$$\frac{\partial f(sx_1, sx_2, \cdots, sx_n)}{\partial x_i} = \frac{\partial f(sx_1, sx_2, \cdots, sx_n)}{\partial (sx_i)} \cdot \frac{d(sx_i)}{dx_i}$$

$$= sf_i(sx_1, sx_2, \cdots, sx_n) \quad and$$

$$\frac{\partial s^k f(x_1, x_2, \cdots, x_n)}{\partial x_i} = s^k f_i(x_1, x_2, \cdots, x_n) \text{setting the two equal}$$

$$sf_i(sx_1, sx_2, \cdots, sx_n) = s^k f_i(x_1, x_2, \cdots, x_n)\ or$$

$$f_i(sx_1, sx_2, \cdots, sx_n) = s^{k-1} f_i(x_1, x_2, \cdots, x_n)$$

Which implies the derivative is homogeneous of degree k-1.