Differential Calculus

First derivative $\dfrac{dy}{dx} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$,

Second derivative $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$,

...

...

Inductively, $\dfrac{d^n y}{dx^n} = \dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right)$.

They are also denoted by $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$. $y^{(0)}$ denotes y.

n^{th} order derivatives of some standard functions:

1. $y = e^{ax}$

$$y_1 = \frac{dy}{dx} = a\,e^{ax}$$

$$y_2 = \frac{d^2y}{dx^2} = a.a\,e^{ax} = a^2\,e^{ax}$$

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$$y_n = a^n\,e^{ax}.$$

2. $y = a^{mx}$ where m is a positive integer.

$y_1 = D(a^{mx}) = m\, a^{mx} \log a.$

$y_2 = m \log a \,.\, (m\, a^{mx} \log a)$

$= (m \log a)^2\, a^{mx}$

$y_3 = (m \log a)^3\, a^{mx}$

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$y_n = (m \log a)^n\, a^{mx}.$

3. $y = (ax + b)^m$, where m is a positive integer such that $m > n$.

$y_1 = m(ax + b)^{m-1}.a$

$y_2 = m(m - 1)(ax + b)^{m-2}. a^2$

$y_3 = m(m - 1)(m - 2)(ax + b)^{m-3} a^3$

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$y_n = m(m- 1)(m - 2) \ldots [m - (n - 1)](ax + b)^{m-n} a^n$.

4. $y = \frac{1}{ax + b}$

Let us write $y = (ax + b)^{-1}$

$y_1 = -1\ (ax + b)^{-2}.a = (-1)^1\ 1!\ (ax + b)^{-2}.a$

$y_2 = (-1)\ (-2)\ (ax + b)^{-3}\ a^2 = (-1)^2\ 2!\ (ax + b)^{-3}\ .\ a^2$

$y_3 = (-1)\ (-2)\ (-3)\ (ax + b)^{-4}\ a^3 = (-1)^3\ 3!\ (ax + b)^{-4}\ a^3$

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$y_n = (-1)^n\ n!\ (ax + b)^{-(n+1)}\ a^n$

$$y_n = \frac{(-1)^n n! a^n}{(ax + b)^{n+1}}$$

5. $y = \log(ax + b)$

$y_1 = a(ax + b)^{-1}$

$y_2 = a(-1)(ax + b)^{-2}.a = a^2(-1)^1\ 1!\ (ax + b)^{-2}$

$y_3 = a^2(-1)(-2)(ax + b)^{-3}\ .a = a^3(-1)^2\ 2!\ (ax + b)^{-3}$

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$y_n = a^n (-1)^{n-1} (n - 1)!\ (ax + b)^{-n}$

$$y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

6. $y = e^{ax} \sin(bx + c)$

$y_1 = e^{ax}. b \cos(bx + c) + ae^{ax} \sin(bx + c)$,

$= e^{ax} [b \cos(bx + c) + a \sin(bx + c)]$

Put $a = r \cos\theta$, $b = r \sin\theta$

Then $\theta = \tan^{-1}(b/a)$ and

$a^2 + b^2 = r^2 (\cos^2\theta + \sin^2\theta) = r^2$

$y_1 = e^{ax} [r \sin \theta \cos (bx + c) + r \cos \theta \sin (bx + c)]$

Note: $\sin (A + B) = \sin A \cos B + \cos A \sin B$

$y_1 = r e^{ax} \sin (\theta + bx + c)$

Similarly we get,

$y_2 = r^2 e^{ax} \sin (2\theta + bx + c)$,

$y_3 = r^3 e^{ax} \sin (3\theta + bx + c)$

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$y_n = r^n e^{ax} \sin (n\theta + bx + c)$

where $r = \sqrt{a^2 + b^2}$ and $q = \tan^{-1} (b/a)$.

Exercise: If $y = e^{ax} \cos(bx + c)$, $y_n = r^n e^{ax} \cos(n\theta + bx + c)$,

where $r = \sqrt{a^2 + b^2}$ and $q = \tan^{-1}(b/a)$.

Examples:

1. Find the n^{th} derivative of $y = \cos h^2 3x$

Solution: Write $\cos h^2 3x = \frac{1}{4}(e^{6x} + e^{-6x} + 2)$

$$y_n = \frac{1}{4}[6^n e^{6x} + (-6)^n e^{-6x}] .$$

Find the n^{th} derivative of : (1) sin h 2x sin 4x

Solution: $D^n[\sinh 2x \sin 4x]$

$$= \frac{1}{2}(D^n\ [e^{2x} \sin 4x] - D^n\ [e^{-2x}\ \sin 4x])$$

$$= \frac{1}{2} 20^{n/2} \{e^{2x}\ \sin (4x + n \tan^{-1} 2) - e^{-2x}\ \sin (4x - n \tan^{-1} 2)\}$$

(2) $y = \log (4x^2 – 1)$

Solution: Let $y = \log (4x^2 – 1) = \log [(2x + 1) (2x – 1)]$

Therefore $y = \log (2x + 1) + \log (2x – 1)$.

$$y_n = \frac{(-1)^{n-1}(n-1)!2^n}{(2x+1)^n} + \frac{(-1)^{n-1}(n-1)!2^n}{(2x-1)^n}$$

Find the nth derivative of $y = \dfrac{x^2}{(x+2)(2x+3)}$

Solution:

$$y = \frac{x^2}{(x+2)(2x+3)} = \frac{x^2}{2x^2+7x+6}$$

$$= \frac{\frac{1}{2}(2x^2+7x+6) - \frac{1}{2}(7x+6)}{2x^2+7x+6} = \frac{1}{2} - \frac{1}{2}\frac{(7x+6)}{(x+2)(2x+3)}$$

$$= \frac{1}{2}\left[1 - \frac{A}{x+2} + \frac{B}{2x+3}\right] = \frac{1}{2}\left[1 - \frac{8}{x+2} + \frac{9}{2x+3}\right]$$

$$\frac{d^n}{dx^n}\left[\frac{x^2}{(x+2)(2x+3)}\right] = -\frac{8(-1)^n n!}{2(x+2)^{n+1}} + \frac{9(-1)^n n! 2^n}{2(2x+3)^{n+1}}$$

Leibnitz's Theorem:

If u and v are functions of x possessing derivatives of the n^{th} order, then

$(uv)_n$ =

$$^nC_0 uv_n + {}^nC_1 u_1 v_{n-1} + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_{n-1} u_{n-1} v_1 + {}^nC_n u_n v.$$

Proof: The Proof is by the principle of mathematical induction on n.

Step 1: Take n = 1

By direct differentiation, $(uv)_1 = uv_1 + u_1 v$

For $n = 2$, $(uv)_2 = u_2v + u_1v_1 + u_1v_1 + uv_2$

$$= u_2v + {}^2C_1u_1v_1 + {}^2C_2\, uv_2$$

Step 2: We assume that the theorem is true for $n = m$

$(uv)_m =$

$${}^mC_0uv_m + {}^mC_1\, u_1\, v_{m-1} + \ldots + {}^mC_{m-1}\, u_{m-1}v_1 + {}^mC_mu_mv.$$

Differentiating both sides we get

$$(uv)_{m+1} = {}^mC_0u\, v_{m+1} + {}^mC_0\, u_1\, v_m + {}^mC_1\, u_1\, v_m + {}^mC_1\, u_2\, v_{m-1} + \ldots$$

$$\ldots + {}^mC_mu_mv_1 + {}^mC_mu_{m+1}v.$$

Note: (i) ${}^{m}C_{r-1} + {}^{m}C_{r} = {}^{(m+1)}C_{r}$

(ii) $1 + {}^{m}C_{1} = 1 + m = {}^{(m+1)}C_{1}$

(iii) ${}^{m}C_{m} = 1 = {}^{(m+1)}C_{m+1}$

$$(uv)_{m+1} = {}^{m}C_{0}u\ v_{m+1} + ({}^{m}C_{0} + {}^{m}C_{1})u_{1}\ v_{m} + ({}^{m}C_{1} + {}^{m}C_{2})u_{2}\ v_{m-1} + \dots$$
$$\dots + ({}^{m}C_{m-1} + {}^{m}C_{m})u_{m}v_{1} + {}^{m}C_{m}u_{m+1}v.$$

$$(uv)_{m+1} =$$
$${}^{m+1}C_{0}uv_{m+1} + {}^{m+1}C_{1}\ u_{1}\ v_{m} + \dots + {}^{m+1}C_{m}\ u_{m}v_{1} + {}^{m+1}C_{m+1}u_{m+1}v.$$

Therefore the theorem is true for m + 1 and hence by the principle of mathematical induction, the theorem is true for any positive integer n.

Example: If $y = \sin(m \sin^{-1} x)$ then prove that

(i) $(1 - x^2) y_2 - xy_1 + m^2 y = 0$

(ii) $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} + (m^2 - n^2) y_n = 0.$

$$y_1 = \cos(m \sin^{-1} x)\, m \frac{1}{\sqrt{1 - x^2}}$$

$$\sqrt{1 - x^2}\, y_1 = m \cos(m \sin^{-1} x)$$

$$(1 - x^2) y_1^2 = m^2 \cos^2(m \sin^{-1} x)$$

$$= m^2 [1 - \sin^2(m \sin^{-1} x)]$$

$$= m^2 (1 - y^2).$$

Differentiating both sides we get

$$(1 - x^2)2y_1 . y_2 + y_1^2 (-2x) = m^2 (- 2y. y_1)$$

$$(1 - x^2) y_2 - xy_1 + m^2 .y = 0$$

Applying Leibnitz's rule we get

$$[(1 - x^2) y_{n+2} + {}^nc_1 (- 2x) . y_{n+1} + {}^nc_2 (-2) .y_n]$$

$$- [x y_{n+1} + {}^nc_1.1. y_n] + m^2 y_n = 0$$

$$(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} + (m^2 - n^2) y_n = 0.$$